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HYPERGRAPH ASSOCIATED WITH LIE ALGEBRA OF UPPER TRIANGULAR MATRICES

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ABSTRACT. For an associated combinatorial structure with Lie algebra $\mathbf{g}_n$ of upper triangular matrices, an allowable, forbidden, and the graphs that are not associated with $\mathbf{g}_n$ of any three vertices are determined. This work also introduces a neoteric association of hypergraph with Lie algebra of upper triangular matrix $\mathcal{G}_n$ for an element of Lie algebra $\mathbf{g}_n$. The properties of this structure are analyzed, characterized and have been presented as an algorithm for finite order.

1. INTRODUCTION

Establishing relationships between different mathematical fields is always an important goal in mathematical research that paves the way with different techniques for better solution of real-world applications. A rapidly emerging theory called Lie Theory has sparked renewed interest not only of its theoretic approach but also moving towards various applications in physics, engineering, and many more. At this point, our primary goal is to establish a relationship between the Lie algebra of upper triangular matrices and hypergraph.

Lie algebra of upper triangular matrices $\mathbf{g}_n$ is the subalgebra of $gl(n, \mathbb{R})$, the general linear Lie algebra consisting of all $n \times n$ matrices over $\mathbb{R}$ with the commutator: [x,y]=xy-yx, for $x, \ y \in gl(n, \ \mathbb{R})$. E_{ij} denote the matrix whose sole nonzero entry is 1 in the $(i, \ j)$ position forms the basis of upper triangular matrix Lie algebra. Many

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$\mathbf{g}$	n-dimensional Lie algebra.
$\mathbf{g}_n$	Lie algebra of $n \times n$ upper triangular matrices
$\mathcal{B}$	Basis of $\mathbf{g}$
$\mathcal{B}_n$	Basis of $\mathbf{g}_n$
G	Graph associated with $\mathbf{g}_n$
$\mathcal{G}_n$	Hypergraph associated with one of the elements of $\mathbf{g}_n$ containing exactly $\frac{n(n+1)}{2}$
$\mathcal{H}_s$	Hypergraph associated with one of the elements of $\mathbf{g}_n$ containing s number of vertices.

TABLE 1. Notation

algebraists associates graph with rings and modules [9, 11, 12, 17]. Carriazo [5] initiated an association of graph (combinatorial structure of dimension 2) with finite dimensional Lie algebra. In [6] Ceballos, defined the association of graph structure with Lie algebra of upper triangular matrices. Also, triangular association is given in [10].

In the literature, Lie algebra have not been associated with a combinatorial structure like hypergraph. Hypergraph is the generalisation of graph and representing structures by hypergraph-based methods has been recently increasing because of its n-ary relations [1] that is, it allows vertices to be multiply connected by hyperedges. For this reason, various practical problems like image processing [18], DNA sequencing [16], networking [2] and so on use this representation.

The construction of a hypergraph for the Lie algebra of upper triangular matrices is carried out by defining a new type of commutator with Lie triple as the primary tool introduced in Section 4. The rest of the paper is organized as follows. Section 2 reviews some preliminaries and notations. In Section 3, we have found forbidden configuration (Theorem 3.2), allowable configuration (Theorem 3.5) and necessary and sufficient condition (Theorem 3.7) of a graph associated with $\mathbf{g}_n$. Section 4 introduces the hypergraph association with $\mathbf{g}_n$ and its properties are established.

2. Preliminaries

For the descriptions on Lie algebra, digraph, hypergraph and so on, one can refer [3, 6, 8].

Definition 2.1. A Lie algebra $\mathbf{g}$ is a vector space with a second bilinear inner composition law $([.,.])$ called the bracket product or Lie bracket, which satisfies $[\theta, \theta] = 0$, for all $\theta \in \mathbf{g}$ and

$J(\theta,\gamma,\omega)=0$, for all $\theta,\gamma,\omega \in \mathbf{g}$ where J is the Jacobiator defined as, $J(\theta,\gamma,\omega)=[[\theta,\gamma],\omega]+[[\gamma,\omega],\theta]+[[\omega,\theta],\gamma]$ known as Jacobi identity.

Definition 2.2. Given a digraph $G=(V,E)$, a vertex $v \in V$ is a going-in (resp. a going-out) if all the edges incident with v are oriented towards v (resp. oriented from v) (Ref [6] Figure 1).

Definition 2.3. A graph is said to be well oriented if all of its vertices are either going-in or going-out.

Definition 2.4. A hypergraph on $X=\{x_1,x_2,\dots,x_p\}$ is a family, $H=(E_1,E_2,\dots,E_q)$ of subsets of X such that $E_i \neq \emptyset \quad i=1,2,\dots,q$ and $\bigcup_{i=1}^{q} E_i = X$. The elements $x_1,x_2,\dots,x_p$, of X are called vertices, and the sets $E_1,E_2,\dots,E_q$ are the edges of the hypergraph.

Definition 2.5. The rank and anti-rank of a hypergraph H is $r(H)=\max_j |E_j|$ and $s(H)=\min_j |E_j|$ respectively and hypergraph is said to be uniform if rank is equal to anti-rank i.e., $r(H)=s(H)$.

Definition 2.6. Given $n \in N$, the Lie algebra $\mathbf{g_n}$ is the matrix algebra consisting of all $n\times n$ upper triangular matrices. This algebra is solvable [13] and of dimension $\frac{n(n+1)}{2}$. Its vectors are expressed as,

$$\mathbf{g_n}(y_{r,s})=\begin{pmatrix} y_{11} & y_{12} & \dots & y_{1n} \\ 0 & y_{22} & \dots & y_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & y_{nn} \end{pmatrix}, \qquad y_{rs}\in \mathbf{R}. \tag{2.1}$$

The basis of $\mathbf{g_n}$ is $\mathcal{B}_n=\{Y_{i,j}=\mathbf{g_n}(y_{r,s})\}_{1\le i\le j\le n}$, where

$$y_{r,s}=\begin{cases} 1, & if\ (r,s)=(i,j); \\ 0, & if\ (r,s)\neq(i,j). \end{cases}$$

The law with respect to the basis $\mathcal{B}_n$ is,

$$\begin{aligned} [Y_{i,j},Y_{j,k}] &= Y_{i,k},\ \text{for } 1\le i<j<k\le n \qquad (Type\ 1); \\ [Y_{i,i},Y_{i,j}] &= Y_{i,j},\ \text{for } 1\le i<j\le n \qquad (Type\ 2); \\ [Y_{i,j},Y_{j,j}] &= Y_{i,j},\ \text{for } 1\le i<j\le n \qquad (Type\ 3). \end{aligned} \tag{2.2}$$

2.1. **Associating combinatorial structures with Lie algebras.** Given an n-dimensional Lie algebra $\mathbf{g}$ with basis $\mathcal{B}=\{v_i\}_{i=1}^{n}$, recall the method introduced in [5, 10] for associating a combinatorial structure with $\mathbf{g}$. If $[v_x,v_y]=\sum_{z=1}^{n} f_{x,y}^{z} v_z$, a combinatorial structure can be associated with $\mathbf{g}$ as follows:

a) Draw vertex x for each $v_x \in \mathcal{B}$.
b) Given three vertices $x < y < z$, draw the full triangle xyz if and only if $(f^z_{x,y}, f^x_{y,z}, f^y_{x,z}) \neq (0,0,0)$. Edges xy, yz and xz have weight $f^z_{x,y}$, $f^x_{y,z}$ and $f^y_{x,z}$ respectively.
  b1) Use a discontinuous line (named a ghost edge) for edges with weight zero.
  b2) If two triangles xyz and xyl satisfy that $f^z_{x,y} = f^l_{x,y}$, draw only one edge between vertices x and q shared by the two triangles.
c) Given two vertices $x < y$, draw a directed edge from y to x if $f^x_{x,y} \neq 0$ or a directed edge from x to y if $f^y_{x,y} \neq 0$.

Ceballos [6] have defined an order for associating each vertex with a vector from the basis $\mathcal{B}_n$ of $\mathbf{g}_n$. More concretely, the order is the one of the elements of each row of matrix $g_n(y_{r,s})$ in equation 2.1 as follows,

$$\begin{array}{rcl} \{Y_{1,1}, Y_{1,2}, \dots, Y_{1,n}\} & \text{with} & \{v_1, v_2, \dots, v_n\}, \\ \{Y_{2,2}, Y_{2,3}, \dots, Y_{2,n}\} & \text{with} & \{v_{n+1}, v_{n+2}, \dots, v_{2n-1}\}, \\ & \vdots & \\ \{Y_{n,n}\} & \text{with} & \{v_{\frac{n(n+1)}{2}}\}. \end{array}$$

3. Graphs associated with Lie algebra of upper triangular matrices

In this section, we intend to characterize the type of the graphs that are associated with upper triangular matrix Lie algebra. Theorem 3.2 and 3.5 given below can be proved by checking its Jacobi identity using Note 3.1.

Note 3.1. For any three vertices i, j and k that corresponds to v_i, v_j and v_k respectively, which in turn corresponds to one of the elements of $\{Y_{p,i}\}_{i=p}^n$ and v_k corresponds to one of the elements of $\{Y_{(p+l),k}\}_{k=(p+l)}^n$ where $p = \{1, 2, \dots, n-1\}$ and $l = \{0, 1, \dots, n-p\}$, Jacobi identity is,

$$\begin{aligned} J(Y_{p,i}, Y_{p,j}, Y_{(p+l),k}) &= [[Y_{p,i}, Y_{p,j}], Y_{(p+l),k}] + [[Y_{p,j}, Y_{(p+l),k}], Y_{p,i}] \\ &\quad + [[Y_{(p+l),k}, Y_{p,i}], Y_{p,j}]. \end{aligned}$$

In general, the possible value of i is p and j is $(p+l)$, hence Jacboi identity becomes
$J(Y_{p,p}, Y_{p,(p+l)}, Y_{(p+l),k}) = [Y_{p,(p+l)}, Y_{(p+l),k}] + [Y_{p,k}, Y_{p,p}] + [[Y_{(p+l),k}, Y_{p,p}], Y_{p,(p+l)}]$.

Here, $k = (p+l)$ but l takes a value 0. So the first commutator of a last component cannot be defined by any of the law given in equation 2.2.

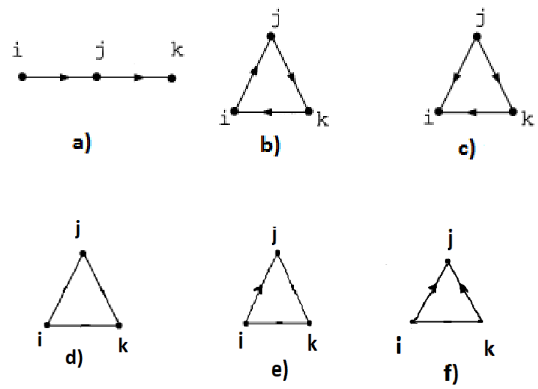

FIGURE 1. Forbidden Configuration

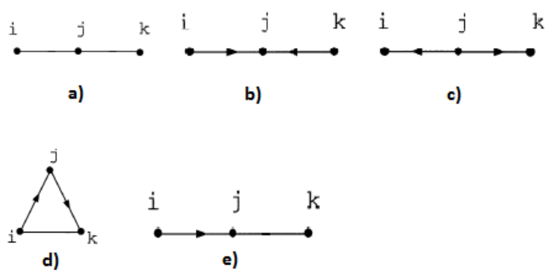

FIGURE 2. Allowable configuration

Theorem 3.2. *Let G be a graph associated with matrix Lie algebra $\mathbf{g_n}$. Then the configurations given in Figure 1 are forbidden in G for any three distinct vertices i, j and k.*

Note 3.3. The following are the different cases of vertices, that belong to one of the partitions given in Subsection 2.1.
Case 1: v_i corresponds to one of the elements of $\{Y_{p,i}\}_{i=p}^{n}$, v_j corresponds to one of the elements of $\{Y_{(p+l),j}\}_{j=(p+l)}^{n}$ and v_k corresponds to one of the elements of $\{Y_{(p+t),k}\}_{k=(p+t)}^{n}$.
Case 2: v_i, v_k corresponds to one of the elements of $\{Y_{(p+l),j}\}_{j=(p+l)}^{n}$ and v_j corresponds to one of the elements of $\{Y_{p,i}\}_{i=p}^{n}$.
Case 3: v_i corresponds to one of the elements of $\{Y_{p,i}\}_{i=p}^{n}$, v_j corresponds to one of the elements of $\{Y_{(p+t),j}\}_{j=(p+t)}^{n}$ and v_k corresponds to one of the elements of $\{Y_{(p+l),k}\}_{k=(p+l)}^{n}$.
Case 4: v_i, v_j corresponds to one of the elements of $\{Y_{p,i}\}_{i=p}^{n}$ and v_k

corresponds to one of the elements of$\{Y_{(p+l),j}\}_{j=(p+l)}^{n}$.
Case 5: v_i corresponds to one of the elements of $\{Y_{p,i}\}_{i=p}^{n}$ and v_j, v_k corresponds to one of the elements of $\{Y_{(p+l),j}\}_{j=(p+l)}^{n}$.
Case 6: v_i, v_j and v_k corresponds to one of the elements of $\{Y_{p,i}\}_{i=p}^{n}$.
Case 7: v_i corresponds to one of the elements of $\{Y_{(p+t),i}\}_{i=(p+t)}^{n}$, v_j corresponds to one of the elements of $\{Y_{(p+l),j}\}_{j=(p+l)}^{n}$ and v_k corresponds to one of the elements of $\{Y_{p,k}\}_{k=p}^{n}$.

Lemma 3.4. *The configurations given in Figure 2 are forbidden for,*

1) *$i > j > k$, all the configuration except Figure 2 (e).*
2) *Figure 2 (a), $i < j < k$, $j < i$; $j < k$ and $i < j$; $k < j$ except the Case 1, Case 2, and Case 3 respectively in Note 3.3.*
3) *Figure 2 (b) except the Case 4 of $i < j < k$.*
4) *$i < j < k$ and $j < i$; $j < k$, except Case 5 and 6 in Note 3.3 for Figure 2 (c).*
5) *Fig 2 (d), except $i < j < k$.*
6) *Figure 2 (e), except the Case 4 and Case 7 of $i < j < k$ and $k < j < i$ in Note 3.3.*

Theorem 3.5. *Given any three distinct vertices i, j and k in a graph G which is associated with a matrix Lie algebra $\mathbf{g}_n$. The configurations given in Figure 2 are allowable except the cases discussed in Lemma 3.4.*

Proof of Theorem 3.2 and Theorem 3.5 are provided in Annexure.

Note 3.6. Remaining possible graphs excluding the configuration in Figure 1 and 2 with three vertices are not associated with matrix Lie algebra $\mathbf{g}_n$, because for any two vertices say i and j there must exist an edge from either i to j or j to i by equation 2.2.

Theorem 3.7 has much deviation from Theorem 3.2 in [6], since the graph associated with $\mathbf{g}_n$ has no double edge, and the proof can be derived easily.

Theorem 3.7. *Let G be a graph with $n(n+1)/2$ vertices and without 3-cycles. Then G is associated with matrix Lie algebra $\mathbf{g}_n$ if and only if G is well oriented.*

Proof. Let G be a graph without 3-cycles with $n(n+1)/2$ vertices associated with matrix Lie algebra. Suppose G is not well oriented, then forbidden configuration a) of Theorem 3.2 would appear. So, G must be well oriented.

Conversely, let G be a graph without 3-cycles, that is well oriented, with vertices $1, 2, \ldots, \frac{n(n+1)}{2}$. Let V be $\frac{n(n+1)}{2}$ dimensional vector space and $\{X_{1,1}, X_{1,2}, \ldots, X_{n,n}\}$ be any basis of V.

The sets $\{X_{1,i}\}_{i=1}^{n}$ corresponds to $\{e_i\}_{i=1}^{n}$, $\{X_{2,j}\}_{j=2}^{n}$ corresponds to $\{e_i\}_{i=n+1}^{2n-1}$, ..., $\{X_{n,n}\}$ corresponds to $e_{\frac{n(n+1)}{2}}$. Let us define Lie bracket as follows,

1) If vertices i and j are not adjacent in G, then associated e_i and e_j corresponds to $X_{i,j}$ and $X_{k,l}$ respectively. Then,

$$[X_{i,j}, X_{k,l}] = 0.$$

2) If i is going-in vertex and
   i) if $i < j$ then associated e_i and e_j corresponds to $X_{i,j}$ and $X_{j,j}$ respectively. Then,

$$[X_{i,j}, X_{j,j}] = X_{i,j}.$$

   ii) if $i > j$ then associated e_i and e_j corresponds to $X_{i,i}$ and $X_{i,j}$ respectively. Then,

$$[X_{i,i}, X_{i,j}] = X_{i,j}.$$

3) If i is going-out vertex and if $i < j$ then associated e_i and e_j corresponds to $X_{i,i}$ and $X_{i,j}$ respectively. Then,

$$[X_{i,i}, X_{i,j}] = X_{i,j}.$$

Since G is well oriented all its vertices are either going-in or going-out. This provide a product in V by above definition and linear extension.

For any two adjacent vertices, one of them is going-in and other is going-out vertex so, this product is skew-symmetric and also it satisfies the Jacobi identity.

For any three vertices i, j and k corresponds to e_i, e_j and e_k respectively, we have the following cases for $i < j < k$, if

- they are not adjacent one to each other, $J(e_i, e_j, e_k) = 0$.
- two of them are adjacent, but the third one is not adjacent to any of the others, it is easy to show that $J(e_i, e_j, e_k) = 0$.
- one of them, say j, is adjacent to the other two, and if j is a going-in vertex, the three vertices present configuration b) of Fig 2, then

$$[X_{i,i}, X_{i,j}] = X_{i,j},$$
$$[X_{i,j}, X_{j,j}] = X_{i,j},$$
$$[X_{i,i}, X_{j,j}] = 0,$$

and, consequently;

$$\begin{aligned}
[[X_{i,i}, X_{i,j}], X_{j,j}] &= [X_{i,j}, X_{j,j}] = X_{i,j},\\
[[X_{i,j}, X_{j,j}], X_{i,i}] &= [X_{i,j}, X_{i,i}] = -X_{i,j},\\
[[X_{j,j}, X_{i,j}], X_{i,i}] &= 0,
\end{aligned}$$

and thus, $J(e_i, e_j, e_k) = 0$. The proof is analogous when j is a going-out vertex. This means that the product is Lie bracket which gives a Lie algebra structure to V, associated to G.

□

4. Hypergraph association with Lie algebra of upper triangular matrices

In this section, association of hypergraph (combinatorial structure of dimension > 2) to a Lie algebra of upper triangular matrix is presented. For the hypergraph association, we have defined Type 4 commutator by combining Type 2 and Type 3 and used Type 1 commutator given in equation 2.2 with respect to the basis $\mathcal{B}_n$ as,

$$\begin{aligned}
[Y_{i,j}, Y_{j,k}] &= Y_{i,k}, \text{ for } 1 \leq i < j < k \leq n \qquad (\textit{Type } 1),\\
[[Y_{i,i}, Y_{i,j}], Y_{j,j}] &= Y_{i,j}, \text{ for } 1 \leq i < j \leq n \qquad (\textit{Type } 4).
\end{aligned}$$

If G be a graph associated with Lie algebra $\mathbf{g}_n$ defined by the Lie brackets of Type 1 and Type 4, then number of full triangles is $\binom{n}{3} + \binom{n}{2}$. Since the graph G has only full triangles the number of edges equals $3 * \left(\binom{n}{3} + \binom{n}{2}\right)$. Also, if the graph G' is associated with Lie algebra by the Lie brackets defined in equation 2.2, then any pair of vertices is not reachable and the number of edges is $2 * \binom{n}{2} + 3 * \binom{n}{3}$, though the length of a path between any pair of vertices is at most two if a path exists. Due to this hindrance, hypergraph is used, which is an effective representation. Let us denote a hypergraph $\mathcal{G}_n$ which is associated with one of the elements of $\mathbf{g}_n$ with exactly $\frac{n(n+1)}{2}$ vertices.

Construction 4.1. *Hypergraph associated with $\mathbf{g}_n$ is as follows:*

(a) *Draw a hyperedge for the vertices corresponding to the diagonal elements, by Type 4 each diagonal element is connected with rest of the diagonal elements.*
(b) *Given three vertices i, j and k draw a hyperedge ijk if and only if corresponding basis elements persuade Type 1 or Type 4.*

Hypergraphs for an element of 2 and 3 dimensional matrix Lie algebra are shown in Figure 3.

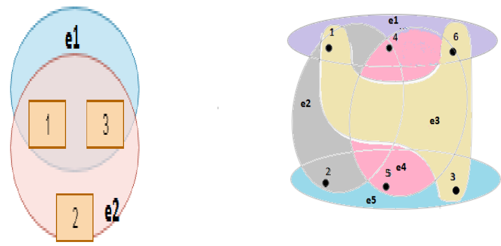

FIGURE 3. $\mathcal{G}_2$ and $\mathcal{G}_3$

Remark 4.2. *In general, the rank of hypergraph $\mathcal{G}_n$ of n dimensional matrix Lie algebra is always equal to n and anti-rank is 3 for any dimension n.*

Remark 4.3. *$\mathcal{G}_3$ is the only uniform hypergraph which is associated with $\mathbf{g}_3$.*

The following are some of the general results concerning to the hypergraph associated to an upper triangular matrix Lie algebra.

Theorem 4.4. *If $\mathcal{G}_n$ is a hypergraph associated with one of the elements of matrix Lie algebra $\mathbf{g_n}$, then the vertices corresponding to diagonal elements have degree n, and the remaining vertices have degree $n-1$.*

Proof. Let $\mathcal{G}_n$ be a hypergraph associated with an element of $\mathbf{g}_n$ consisting of all $n \times n$ upper triangular matrices. The possibility of combining diagonal elements by Type 4 is $\binom{n}{2}$, and it is evident that each element occurs $n-1$ times, and by the hyperedge e_d constructed as defined in Construction 4.1 (a) contains all the diagonal elements in $n \times n$ matrices of $\mathbf{g}_n$. Hence the degree of vertices corresponding to diagonal elements is $n-1+1=n$.

There are $\binom{n}{3}$ ways of aggregating non-diagonal elements of one of the $n \times n$ matrices of $\mathbf{g}_n$, in which each element occurs $n-2$ times, and by basis elements $Y_{i,i}$ of Type 4, vertices corresponding to every non-diagonal elements is already incident with a hyperedge, yields the desired result. □

Lemma 4.5. *The sum of the degrees of vertices in hypergraph $\mathcal{G}_n$ of an element in n-dimensional matrix Lie algebra is equal to $\frac{n(n^2+1)}{2}$, for $n \geq 2$.*

Proof. Let an element of Lie algebra $\mathbf{g}_n$ be associated with hypergraph $\mathcal{G}_n$. As explained in Theorem 4.4, there are $\binom{n}{3}$ ways of linking Type 1 and $\binom{n}{2}$ ways of Type 4. By hypergraph association defined each of the above possibilities generates a single hyperedge which is incident with three vertices, and also hyperedge e_d constructed as defined in Construction 4.1 (a), is incident with all the diagonal vertices. Accordingly, the sum of the degrees of vertices of a hypergraph is

$$\begin{aligned}
&\binom{n}{3} * 3 + \binom{n}{2} * 3 + n = \frac{n!}{3!(n-3)!} * 3 + \frac{n!}{2!(n-2)!} * 3 + n \\
&= \frac{n(n-1)(n-2)(n-3)!}{3 * 2 * (n-3)!} * 3 + \frac{n(n-1)(n-2)!}{2 * (n-2)!} * 3 + n \\
&= \frac{n(n-1)(n-2)}{2} + \frac{3n(n-1)}{2} + n \\
&= n[\frac{(n-1)(n+1)+2}{2}] \\
&= \frac{n(n^2+1)}{2}.
\end{aligned}$$

□

4.1. **Generalization for s-vertices.** A Hypergraph $\mathcal{G}_n$ for an element of $\mathbf{g}_n$ has $\frac{n(n+1)}{2}$ vertices and in $\mathcal{G}_{n-1}$, $\frac{n(n-1)}{2}$ vertices, by subtracting k elements in the last column from the top of $\mathbf{g}_n$ yields transitional number of vertices, that is $\frac{n(n+1)}{2} - k$ where $1 \leq k \leq n-1$. Denote a hypergraph for any s with $s \geq 2$ by $\mathcal{H}_s$.

Theorem 4.6. *If $\mathcal{H}_s$ is a hypergraph for an element in $\mathbf{g}_n$ with $n \geq 2$, then the number of hyperedges is,* $\binom{n}{3} + \binom{n}{2} + 1$ *if* $s = \frac{n(n+1)}{2}$ *and* $\left(\binom{n}{3} - \left\{\sum_{i=1}^{n_0}(n-(i+1))\right\}\right) + \left(\binom{n}{2} - n_0\right) + 1$ *if* $\frac{n(n-1)}{2} < s < \frac{n(n+1)}{2}$.

Proof. If $s = \frac{n(n+1)}{2}$, according to the method expounded in Section 3, $\mathcal{H}_s$ associated with an element of $\mathbf{g}_n$ that has $\frac{n(n+1)}{2}$ vertices. In virtue of Theorem 4.4, n vertices have degree n, and remaining vertices have degree $n-1$ then the number of hyperedges is given by

$$\begin{aligned}
&\frac{n(n-1)}{3}+\frac{n*1}{n}+\frac{\left(\frac{n(n+1)}{2}-n\right)*(n-2)}{3}+\frac{\left(\frac{n(n+1)}{2}-n\right)*1}{3}\\
&=\frac{n(n-1)}{3}+1+\frac{n(n-1)(n-2)}{6}+\frac{n(n-1)}{6}\\
&=\frac{n(n-1)}{3}(1+\frac{1}{2})+1+\frac{n(n-1)(n-2)}{6}\\
&=\frac{n(n-1)}{2}+\frac{n(n-1)(n-2)}{6}+1\\
&=\binom{n}{2}+\binom{n}{3}+1.
\end{aligned}$$

Now, to prove for the case $\frac{n(n-1)}{2}<s<\frac{n(n+1)}{2}$. Let us suppose that $n_0=\frac{n(n+1)}{2}-s$, if $n_0=1$ then $y_{1n}=0$, $n_0=2$ then $y_{1n}=0$ and $y_{2n}=0,\ldots,n_0=n-1$ then $y_{1n}=0,\ y_{2n}=0,\ldots,y_{(n-1),n}=0$. It remains that there are $\binom{n}{2}$ ways of fusing the Type 4 and $\binom{n}{3}$ ways of Type 1.

If $y_{1n}=0$ then $(n-2)$ hyperedges does not exist. Similarly for $y_{2n}=0,\ldots,y_{(n-2),n}=0,\ y_{(n-1),n}=0$ then $(n-3),\ldots,1,\ 1$ hyperedges are non exant respectively. Now, this totally accounts $\sum_{i=1}^{n_0}(n-(i+1))$ which proceeds to $\binom{n}{3}-\sum_{i=1}^{n_0}(n-(i+1))$. $\binom{n}{2}$ hyperedges gets diminished depending on n_0. If $n_0=1$ then single hyperedge is less. Hence it equals $\binom{n}{2}-n_0$ and by the algorithm all diagonal elements are made as a single hyperedge. Therefore, the number of hyperedges is $\left(\binom{n}{3}-\left\{\sum_{i=1}^{n_0}(n-(i+1))\right\}\right)+\left(\binom{n}{2}-n_0\right)+1.$ □

Theorem 4.7. *If $\mathcal{H}_s$ is a hypergraph for an element of $\mathbf{g}_n$ with $s\geq 2$, then the path between any two vertices corresponding to Y_{ij} and Y_{kl} has length at most 2.*

Proof. Let $\mathcal{H}_s$ be a hypergraph of s vertices, the length of the path depends on the $i,\ j,\ k$ and l values.
In Case 1) if $s=\frac{n(n+1)}{2}$, by non-zero brackets of Type 1 and Type 4 the following possibilities have path length 2,

- $i=j,\ k\neq l,\ i\neq k$ *and* $j\neq l$,
- $i\neq j,\ k\neq l,\ i\neq k$ *and* $j\neq l$.

In Case 2) if $\frac{n(n-1)}{2} < s < \frac{n(n+1)}{2}$, in addition to above possibilities, if $i \neq j,\ k \leq l,\ j = k$ and il is not an element of $n \times n$ matrix, the length of the path is 2.

All other possibilities of $i,\ j,\ k$ and l the length of a path between any two vertices corresponding to Y_{ij} and Y_{kl} is 1. □

Algorithm 1 is used to construct the hypergraph for an element of $\mathbf{g}_n$. It is implemented using Python 3 on Google Colabatory.

Algorithm 1: GFsVERTICES

```
INPUT: s
n = ⌈(−1+⌈√(8s+1)⌉)/2⌉
t=n(n+1)/2
if s == t then
    Algorithm 2 (n)
else
    v=t-s
    for i = 1 to n do
        for j = i to n do
            DIAGHYPEREDGE
            for k = 1 to n do
                for l = k to n do
                    if i == j and k == l and i ≠ k then
                        if i <= v and r == l then
                            CONTINUE
                        else
                            Add (ij, kl, jk) as hyperedge
                        end if
                    end if
                    if i ≠ j and k ≠ l and j == k then
                        if i <= v and r == l then
                            CONTINUE
                        else
                            Add (ij, kl, il) as hyperedge
                        end if
                    end if
                end for
            end for
        end for
    end for
end if
```

Algorithm 2: HGUTMLA (n)

```
for i = 1 to n do
    for j = i to n do
        if i == j then
            Append ij to hyperedge e_d
        end if
        for k = 1 to n do
            for l = k to n do
                if i == j and k == l and i ≠ k then
                    Add (ij, kl, jk) as hyperedge
                end if
                if i ≠ j and k ≠ l and j == k then
                    Add (ij, kl, il) as hyperedge
                end if
            end for
        end for
    end for
end for
```

Table 2 presents the analysis of construction of hypergraph using Lie commutator with compilation time. This association can be applied to various routing problems in networks.

Table 2. Observations of Algorithm

Number of vertices	Number of Hyperedges	Compilation time in Secs	Dimension of Lie algebra $\mathbf{g}_n$
6	5	0.00404047966003418	3
250	1712	0.03202700614929199	22
755	9231	0.19795751571655273	39
125250	20833251	3581.855294942856	500

Conclusion and Future work

In this work we have identified combinatorial structures associated with Lie algebra of upper triangular matrix for three vertices. We have introduced hypergraph theory association with Lie algebra of upper triangular matrix and entrenched its properties. Also, hypergraph construction using Lie commutator is algorithmized and the outputs are summarized in Table 2. Since hypergraph is an effective way to represent higher order relationship, it opens a wide range to solve many problems with methods and results obtained here. In future, the work can be extended to construct a hypergraph for subalgebra of $gl(n, \mathbb{R})$ and this methodology may offer useful insights for many engineering applications.

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References

1. K. Akimoto, T. Hiraoka, K. Sadamasa, and M. Niepert Akimoto, *Cross-Sentence N-ary Relation Extraction using Lower-Arity Universal Schemas*, Proceedings of the 2019 Conference on Empirical Methods in Natural Language Processing and the 9th International Joint Conference on Natural Language Processing (EMNLP-IJCNLP), (2019), 6226–6232.
2. B. Bai, L. Wang, Z. Han, W. Chen, and T. Svensson, *Caching based socially-aware D2D communications in wireless content delivery networks: A hypergraph framework*, IEEE Wireless Communications, (4) **23** (2016), 74–81.
3. C. Berge, *Hypergraphs North Holland Mathematical Library*, Elsevier Science Publishers BV, 1989.
4. J. Cáceres, M. Ceballos, J. Núñez, M. L. Puertas, and Á. F. Tenorio, *Graph operations and Lie algebras*, Int. J. Comput. Math, (10) **90** (2013), 2092-2104.
5. A. Carriazo, A and L. M. Fernández,and J. Núñez, *Combinatorial structures associated with Lie algebras of finite dimension*, Linear Algebra Appl., **389** (2014), 43–61.
6. M. Ceballos, and J. Núñez, and A. F. Tenorio, *Combinatorial structures and Lie algebras of upper triangular matrices*, Appl. Math. Lett., (3) **25** (2012), 514–519.
7. M. Ceballos,J. Nunez, and A. F. Tenorio, *Triangular configurations and Lie algebras of strictly upper-triangular matrices*, Appl. Comput. Math., (1) **13** (2014), 62-70.
8. M. Ceballos, J. Núñez, and A. F. Tenorio, *Relations between combinatorial structures and Lie algebras: centers and derived Lie algebras*, Bull. Malays. Math. Sci. Soc., (2) **38** (2015), 529-541.
9. F. Esmaeili Khalil Saraei, *The annihilator graph of modules over commutative rings*, J. Algebra Relat. Topics, (1) **9** (2021), 93–108.
10. L. M. Fernández and L. Martín-Martínez, *Lie algebras associated with triangular configurations*, Linear Algebra Appl., **407** (2005), 43–63.
11. M. Karimi, *A graph associated to spectrum of a commutative ring*, J. Algebra Relat. Topics, (2) **2** (2014), 11-23.
12. P.T. Lalchandani, *Exact annihilating-ideal graph of commutative rings*, J. Algebra Relat. Topics, (1) **5** (2017), 27–23.
13. A. Malcev, *On solvable Lie algebras*, Russian Academy of Sciences, Steklov Mathematical Institute of Russian, (5) **9** (1945), 329–356.

14. M. Primc, *Basic representations for classical affine Lie algebras*, J. Algebra, (1) **228** (2000), 1–50.
15. V. S. Varadarajan, *Lie groups, Lie algebras, and their representations*, Springer Science & Business Media, **102**, 2013.
16. S. Venkatraman, G. Rajaram, and K. Krithivasan, *Unimodular hypergraph for DNA sequencing: A polynomial time algorithm*, Proc. Nat. Acad. Sci. India Sect. A, (1) **90**(2020), 49–56.
17. S. Visweswaran and J. Parejiya, *A Note on a graph associated to a commutative ring*, J. Algebra Relat. Topics, (1) **5** (2017), 61–82.
18. J. Yu, D. Tao, and M. Wang, *Adaptive hypergraph learning and its application in image classification*, IEEE Trans. Image Process., (7) **21** (2012), 3262–3272.

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