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INTEGRAL CLOSURE OF A FILTRATION RELATIVE TO A NOETHERIAN MODULE

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ABSTRACT. Let M be a Noetherian $R-$module. In this paper we will introduce the integral closure of a filtration $\mathcal{F} = \{I_n\}_{n\geq 0}$ relative to the Noetherian module M and prove some related results. The integral closure of a filtration $\mathcal{F} = \{I_n\}_{n\geq 0}$ relative to M is a filtration and it has an interesting relationship with the integral closure of the filtration $\widetilde{\mathcal{F}} = \{\widetilde{I}_n\}_{n\geq 0}$, where $\widetilde{I}_n$ is the image of I_n under the natural ring homomorphism $R \to R/(Ann_R(M))$ for every $n \geq 0$.

1. Introduction

Throughout this paper R denotes a commutative ring with identity. Further $\mathbf{N}$ and $\mathbf{N}_0$ will denote the set of natural integers and non-negative integers respectively. Also $\mathbf{Z}$ will denote the set of integer numbers.

The ideas of reduction and integral closure of an ideal in a commutative Noetherian ring A (with identity) were introduced by Northcott and Rees in [2]. It is appropriate for us to recall these definitions.

Let I and J be ideals of a commutative Noetherian ring A. The ideal I is a reduction of the ideal J if $I \subseteq J$ and there exists an integer $n \in \mathbf{N}$ such that $IJ^n = J^{n+1}$. Also an element x of A is said to be integrally dependent on I if there exist a positive integer n and elements $c_k \in I^k$, $k = 1, ..., n$, such that

$$x^n + c_1x^{n-1} + \cdots + c_{n-1}x + c_n = 0.$$

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We know from [2], $x \in A$ is integrally dependent on I if and only if I is a reduction of the ideal $I + Rx$. Further, we know that the set of all elements of A which are integrally dependent on I is an ideal of A. This ideal is called the integral closure of I and denoted by I^{-}.

Now let M be a Noetherian $R-$module. In [5], Sharp, Tiraş and Yassi introduced concepts of reduction and integral closure of an ideal I of a commutative ring R relative to a Noetherian $R-$module M.

Let I and J be ideals of R. The ideal I is said to be a reduction of the ideal J relative to M, if $I \subseteq J$ and there exists an integer $n \in \mathbf{N}$ such that $IJ^nM = J^{n+1}M$. Also an element x of R is said to be integrally dependent on I relative to a Noetherian $R-$module M, if there exists a positive integer n such that

$$x^n M \subseteq \sum_{i=1}^{n} x^{n-i} I^i M.$$

We know from [5], an element x of R is integrally dependent on I relative to a Noetherian $R-$module M, if and only if I is a reduction of the ideal $I + Rx$ relative to M. Moreover in [5], it is shown that the set of all elements of R which are integrally dependent on I relative to M is an ideal of R. This is denoted by $I^{-[M]}$ and is called the integral closure of I relative to M.

Here, we give some definitions and notations which will be helpful for us in the rest of the paper.

A filtration $\mathcal{F} = \{I_n\}_{n\geq 0}$ on R is a descending sequence of ideals I_n of R such that $I_0 = R$ and $I_nI_m \subseteq I_{n+m}$ for all $n, m \in \mathbf{N}_0$. Let $\mathcal{F} = \{I_n\}_{n\geq 0}$ and $\mathcal{G} = \{J_n\}_{n\geq 0}$ be two filtrations. We say $\mathcal{F} \leq \mathcal{G}$ if $I_n \subseteq J_n$ for all n. Also two filtrations $\{\sum_{i=0}^{n} I_{n-i}J_i\}_{n\geq 0}$ and $\{I_nJ_n\}_{n\geq 0}$ are denoted by $\mathcal{F} + \mathcal{G}$ and $\mathcal{F}\mathcal{G}$ respectively.

The integral closure of a filtration $\mathcal{F} = \{I_n\}_{n\geq 0}$ is defined in [3]. For every $n \geq 0$, let J_n be the set of all $x \in R$ such that x satisfies an equation

$$x^m + a_1x^{m-1} + \cdots + a_{m-1}x + a_m = 0$$

for a positive integer m and elements $a_i \in I_{ni}$. Then $\mathcal{F}^{-} = \{J_n\}_{n\geq 0}$ is a filtration such that $\mathcal{F} \leq \mathcal{F}^{-}$. In fact, the integral closure of $\bigoplus_{n\geq 0} I_nt^n$ in $R[t]$ is the $\mathbf{N}_0-$graded ring, $\bigoplus_{n\geq 0} J_nt^n$.

In this paper we will introduce the integral closure of a filtration relative to a Noetherian module and study some related topics.

2. Reduction of a filtration relative to a Noetherian module

In this section we introduce the reduction of a filtration relative to a Noetherian module and prove some of its properties.

Definition 2.1. (See [4, 2.1.3].) Let $\mathcal{F} = \{I_n\}_{n\geq 0}$ and $\mathcal{G} = \{J_n\}_{n\geq 0}$ be filtrations on R. $\mathcal{F}$ is said to be a reduction of $\mathcal{G}$ if $\mathcal{F} \leq \mathcal{G}$ and there exists a positive integer d such that

$$J_n = \sum_{i=0}^{d} I_{n-i}J_i \quad \textit{for every } n \geq 1.$$

Here, and throughout this paper, $I_i = R$ if $i \leq 0$.

Let $\mathcal{F} = \{I_n\}_{n\geq 0}$ be a filtration on R. We know from [4], the graded subring $R[t^{-1}, I_1t, I_2t^2, ...]$ of $R[t, t^{-1}]$ is called the Rees ring of R with respect to the filtration $\mathcal{F} = \{I_n\}_{n\geq 0}$ and denoted by $\mathcal{R}(R, \mathcal{F})$. For an $R-$module M the set

$$M[t^{-1}, I_1t, I_2t^2, ...] = \{\sum_{j=r}^{s} m_jt^j \in M[t, t^{-1}] : m_j \in I_jM, r, s \in \mathbf{Z}\}$$

is shown by $\mathcal{R}(M, \mathcal{F})$. We know $\mathcal{R}(M, \mathcal{F})$ is a graded $\mathcal{R}(R, \mathcal{F})-$module by the following scalar multiplication

$$(\sum_{i=n}^{m} a_it^i)(\sum_{j=r}^{s} m_jt^j) = \sum_{i=n}^{m}\sum_{j=r}^{s} a_im_jt^{i+j}$$

for every $\sum\limits_{i=n}^{m} a_it^i \in \mathcal{R}(R, \mathcal{F})$ and $\sum\limits_{j=r}^{s} m_jt^j \in \mathcal{R}(M, \mathcal{F})$.

Now let $\mathcal{F} = \{I_n\}_{n\geq 0}$ and $\mathcal{G} = \{J_n\}_{n\geq 0}$ be filtrations on R such that every ideal J_n is finitely generated. We know from [4, 2.3], $\mathcal{F}$ is a reduction of $\mathcal{G}$ if and only if $\mathcal{R}(R, \mathcal{G})$ is a finitely generated $\mathcal{R}(R, \mathcal{F})-$module.

Definition 2.2. (See [4, 2.1.4].) Let R be a Noetherian ring and $\mathcal{F} = \{I_n\}_{n\geq 0}$ be a filtration on R. If there exists a positive integer d such that

$$I_n = \sum_{i=1}^{d} I_{n-i}I_i \quad \textit{for every } n \geq 1$$

then the filtration $\mathcal{F} = \{I_n\}_{n\geq 0}$ is said a Noetherian filtration.

We know from [4, 2.2.1], a filtration $\mathcal{F} = \{I_n\}_{n\geq 0}$ on R is a Noetherian filtration if and only if $\mathcal{R}(R, \mathcal{F})$ is a Noetherian ring.

Definition 2.3. Let $\mathcal{F} = \{I_n\}_{n\geq 0}$ and $\mathcal{G} = \{J_n\}_{n\geq 0}$ be filtrations on R and let M be a Noetherian $R - module$. Then $\mathcal{F}$ is said to be a reduction of $\mathcal{G}$ relative to M if $\mathcal{F} \leq \mathcal{G}$ and there exists a positive integer d such that

$$J_n M = \sum_{i=0}^{d} I_{n-i} J_i M \quad \quad for\ every\ n \geq 1.$$

Theorem 2.4. *Let $\mathcal{F} = \{I_n\}_{n\geq 0}$ and $\mathcal{G} = \{J_n\}_{n\geq 0}$ be filtrations on R and let $\mathcal{F} \leq \mathcal{G}$. Let M be a Noetherian $R - module$. Then $\mathcal{F}$ is a reduction of $\mathcal{G}$ relative to M if and only if $\mathcal{R}(M, \mathcal{G})$ is a finitely generated $\mathcal{R}(R, \mathcal{F})-$module.*

Proof. ($\Rightarrow$) Since $\mathcal{F}$ is a reduction of $\mathcal{G}$ relative to M there exists a positive integer d such that

$$J_n M = \sum_{i=0}^{d} I_{n-i} J_i M \quad for\ every\ n \geq 1.$$

Then $\mathcal{R}(R, \mathcal{F})-$module $\mathcal{R}(M, \mathcal{G})$ is generated by $J_0M, J_1M, \ldots, J_dM$. Since M is a Noetherian $R-$module, $J_0M, J_1M, \ldots, J_dM$ are finitely generated as an $R-module$. This shows $\mathcal{R}(M, \mathcal{G})$ is a finitely generated $\mathcal{R}(R, \mathcal{F})-$module.

($\Leftarrow$) Let $\{\alpha_1, \ldots, \alpha_s\}$ be a finitely generator for $\mathcal{R}(R, \mathcal{F})-$module $\mathcal{R}(M, \mathcal{G})$. By adding the appropriate zero homogeneous component, we can assume that

$$\alpha_t = a_{t1} m_{t1} + \cdots + a_{tk} m_{tk}$$

where $a_{t1} \in J_1, \ldots, a_{tk} \in J_k$ and $m_{t1}, \ldots, m_{tk} \in M$ for every $1 \leq t \leq s$. This is clear that $\mathcal{R}(R, \mathcal{F})-$module $\mathcal{R}(M, \mathcal{G})$ can be generated by all homogeneous components $a_{ti} m_{ti}$ for every $1 \leq t \leq s$ and $1 \leq i \leq k$. Now let $x \in J_n M$. Then we can see, $x = \sum_{t=1}^{s} \sum_{i=1}^{k} r_{ti} a_{ti} m_{ti}$ where $r_{ti} \in I_{n-i}$. This shows that $J_n M \subseteq \sum_{i=0}^{k} I_{n-i} J_i M$. Now the proof is completed because the inverse inclusion is clear. □

Corollary 2.5. *Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$, and $\mathcal{K}$ be filtrations on a Noetherian ring R and let M be a Noetherian $R - module$.*

(a) *If $\mathcal{F}$ is a reduction of $\mathcal{H}$ relative to M and $\mathcal{G}$ is a reduction of $\mathcal{K}$ relative to M then $\mathcal{F}+\mathcal{G}$ is a reduction of $\mathcal{H}+\mathcal{K}$ relative to M.*

(b) *If $\mathcal{F}$ is a reduction of $\mathcal{G}$ relative to M and also a reduction of $\mathcal{K}$ relative to M then $\mathcal{F}$ is a reduction of $\mathcal{G}+\mathcal{K}$ relative to M.*

(c) *If $\mathcal{F}$ is a reduction of $\mathcal{H}$ relative to M and $\mathcal{F}\leq\mathcal{G}\leq\mathcal{H}$ then $\mathcal{G}$ is a reduction of $\mathcal{H}$ relative to M. Further if $\mathcal{F}$ is a Noetherian filtration then $\mathcal{F}$ is a reduction of $\mathcal{G}$ relative to M.*

Proof. (a) Let $\mathcal{F}=\{I_n\}_{n\geq 0},\mathcal{G}=\{J_n\}_{n\geq 0},\mathcal{H}=\{H_n\}_{n\geq 0}$, and $\mathcal{K}=\{K_n\}_{n\geq 0}$. We can see

$$\mathcal{R}(R,\mathcal{F}+\mathcal{G})=R[u,I_1t,J_1t,I_2t^2,J_2t^2,...]$$

and

$$\mathcal{R}(M,\mathcal{H}+\mathcal{K})=R[u,H_1t,K_1t,H_2t^2,K_2t^2,...].$$

Since $\mathcal{F}$ is a reduction of $\mathcal{H}$ relative to M, $\mathcal{R}(M,\mathcal{H})$ is a finitely generated $\mathcal{R}(R,\mathcal{F})-$module by 2.4. Similarly $\mathcal{R}(M,\mathcal{K})$ is a finitely generated $\mathcal{R}(R,\mathcal{G})-$module . Now we can see $\mathcal{R}(M,\mathcal{H}+\mathcal{K})$ is a finitely generated $\mathcal{R}(R,\mathcal{F}+\mathcal{G})-$module and so the claim follows from 2.4.

(b) It is clear by (a) since $\mathcal{F}+\mathcal{F}=\mathcal{F}$.

(c) The first part follows from 2.4. Now let $\mathcal{F}$ be a Noetherian filtration. Then by [4, 2.2.1], $\mathcal{R}(R,\mathcal{F})$ is Noetherian. Since $\mathcal{F}$ is a reduction of $\mathcal{H}$ relative to M, $\mathcal{R}(M,\mathcal{H})$ is a finitely generated module over the Noetherian ring $\mathcal{R}(R,\mathcal{F})$. Therefore $\mathcal{R}(M,\mathcal{G})$ is a finitely generated $\mathcal{R}(R,\mathcal{F})-$module. Then $\mathcal{F}$ is a reduction of $\mathcal{G}$ relative to M by 2.4. □

Remark 2.6. Let $\mathcal{F}=\{I_n\}_{n\geq 0}$ and $\mathcal{G}=\{J_n\}_{n\geq 0}$ be filtrations on R and let M be an $R-$module. Let $\mathcal{F}$ be a reduction of $\mathcal{G}$ relative to M. Then there exists a positive integer d such that

$$J_nM=\sum_{i=0}^{d}I_{n-i}J_iM \quad for\ every\ n\geq 1.$$

Let $d<d'$. Since $\sum\limits_{i=d+1}^{d'}I_{n-i}J_iM\subseteq J_nM$, we have

$$J_nM=\sum_{i=0}^{d}I_{n-i}J_iM+\sum_{i=d+1}^{d'}I_{n-i}J_iM=\sum_{i=0}^{d'}I_{n-i}J_iM.$$

Lemma 2.7. *Let $\mathcal{F}=\{I_n\}_{n\geq 0},\mathcal{G}=\{J_n\}_{n\geq 0},\mathcal{K}=\{K_n\}_{n\geq 0}$, and $\mathcal{H}=\{H_n\}_{n\geq 0}$ be filtrations on R and let M be an $R-$ module.*

(a) *If $\mathcal{F}$ is a reduction of $\mathcal{G}$ relative to M and $\mathcal{K}$ is a reduction of $\mathcal{H}$ relative to M then $\mathcal{F}\mathcal{K}$ is a reduction of $\mathcal{G}\mathcal{H}$ relative to M.*

(b) *If $\mathcal{F}$ is a reduction of $\mathcal{G}$ relative to M and $\mathcal{G}$ is a reduction of $\mathcal{K}$ relative to M then $\mathcal{F}$ is a reduction of $\mathcal{K}$ relative to M.*

Proof. (a) Let $\mathcal{F}$ be a reduction of $\mathcal{G}$ relative to M and also $\mathcal{K}$ be a reduction of $\mathcal{H}$ relative to M. By 2.6, we can choose a positive integer d such that

$$J_nM = \sum_{i=0}^{d} I_{n-i}J_iM \quad and \quad H_nM = \sum_{i=0}^{d} K_{n-i}H_iM$$

for every $n \geq 1$. Now we have

$$J_nH_nM = \sum_{i=0}^{d} I_{n-i}J_iH_nM = \sum_{i=0}^{d} I_{n-i}J_i(\sum_{i=0}^{d} K_{n-i}H_iM).$$

Since for every $1 < t < d$ we have

$$I_{n-t}J_t \subseteq I_{n-t-1}J_{t-1} \subseteq \cdots \subseteq I_{n-1}J_1$$

and

$$K_{n-t}H_t \subseteq K_{n-t-1}H_{t-1} \subseteq \cdots \subseteq K_{n-1}H_1,$$

we can see that

$$J_nH_nM = \sum_{i=0}^{d} I_{n-i}J_i(\sum_{i=0}^{d} K_{n-i}H_iM) \subseteq \sum_{i=0}^{d} I_{n-i}K_{n-i}J_iH_iM$$

for every $n \geq 1$. Now (a) is clear because the inverse inclusion is clear.

(b) Since $\mathcal{G}$ is a reduction of $\mathcal{K}$ relative to M, there exists a positive integer d such that

$$K_nM = \sum_{i=0}^{d} J_{n-i}K_iM \quad for\ every\ n \geq 1.$$

Now since $\mathcal{F}$ is a reduction of $\mathcal{G}$ relative to M, there exists a positive integer d' such that for every $n - i$ we have $J_{n-i}M = \sum_{t=0}^{d'} I_{n-i-t}J_tM$. This shows that

$$\begin{aligned} K_nM &= \sum_{i=0}^{d} K_i \sum_{t=0}^{d'} I_{n-i-t}J_tM \\ &\subseteq \sum_{i=0}^{d} K_i \sum_{t=0}^{d'} I_{n-i-t}K_tM \\ &\subseteq \sum_{i=0}^{d} \sum_{t=0}^{d'} I_{n-i-t}K_{i+t}M \end{aligned}$$

and this shows that

$$K_n M \subseteq \sum_{i=0}^{d+d'} I_{n-i} K_i M \quad \textit{for every } n \geq 1.$$

Now the proof is completed because the inverse inclusion is clear. □

3. Integral closure of a filtration relative to a Noetherian module

In this section we define the integral closure of a filtration relative to a Noetherian module and prove some of its properties. For this, we introduce a useful notation.

Remark 3.1. Let M be a Noetherian $R-$module. In the remainder of this paper, as shown in [5], the commutative Noetherian ring $R/Ann_R(M)$ is denoted by $\widetilde{R}$. Further for every ideal I of R, the ideal $I + Ann_R(M)/Ann_R(M)$ of $\widetilde{R}$ is denoted by $\widetilde{I}$. Also an element $x + Ann_R(M) \in R/Ann_R(M)$ is denoted by $\widetilde{x}$. If $\mathcal{F} = \{I_n\}_{n\geq 0}$ is a filtration of ideals of R then the filtration $\{\widetilde{I}_n\}_{n\geq 0}$ of ideals of $\widetilde{R}$ is denoted by $\widetilde{\mathcal{F}}$.

Theorem 3.2. *Let $\mathcal{F} = \{I_n\}_{n\geq 0}$ be a filtration on R and let M be a Noetherian $R-$module. For every $n \geq 0$, we assume that J_n contains all $x \in R$ such that*

$$x^k M \subseteq \sum_{i=1}^{k} x^{k-i} I_{ni} M$$

for a positive integer k. Further we assume that the integral closure of filtration $\widetilde{\mathcal{F}}$ on $\widetilde{R}$ be $(\widetilde{\mathcal{F}})^{-} = \{\widetilde{K}_n\}_{n\geq 0}$. Then $x \in J_n$ if and only if $\widetilde{x} \in \widetilde{K}_n$.

Proof. ($\Leftarrow$) Let $\widetilde{x} \in \widetilde{K}_{n\geq 0}$. Then there exist a positive integer k and elements $\widetilde{a}_i \in \widetilde{I}_{ni}$, $i = 1, ..., k$, such that

$$\widetilde{x}^k + \widetilde{a}_1 \widetilde{x}^{k-1} + \cdots + \widetilde{a}_{k-1}\widetilde{x} + \widetilde{a}_k = 0.$$

Now since M has natural structure as $R/(0 :_R M)-$module,

$$x^k M \subseteq \sum_{i=1}^{k} x^{k-i} I_{ni} M.$$

($\Rightarrow$) Since $x \in J_n$, there exists a positive integer k, such that

$$x^k M \subseteq \sum_{i=1}^{k} x^{k-i} I_{ni} M.$$

Let $L = \sum_{i=1}^{k} x^{k-i} I_{ni}$. Since M is a Noetherian R module, by [1, 2.1], we can see that there exist an integer $t \in \mathbf{N}$ and elements $c_1, ..., c_t \in R$ with $c_j \in L^j \subseteq \sum_{i=j}^{kj} x^{kj-i} I_{ni}$ such that

$$x^{kt} + c_1 x^{k(t-1)} + \cdots + c_{t-1} x^k + c_t \in (0 :_R M).$$

But for every $1 \leq j \leq t$, we have

$$c_j x^{k(t-j)} \in x^{k(t-j)} L^j \subseteq x^{k(t-j)} \sum_{i=j}^{kj} x^{kj-i} I_{ni} = \sum_{i=j}^{kj} x^{kt-i} I_{ni}$$

$$= x^{kt-j} I_{nj} + \cdots + x^{kt-kj} I_{nkj}.$$

This implies that

$$(\widetilde{x})^{kt} \in \sum_{i=1}^{kt} (\widetilde{x})^{kt-i} \widetilde{I}_{ni}$$

and so $\widetilde{x} \in \widetilde{K}_n$. □

In the above theorem, let R be a Noetherian ring and let $R-$module M be R. Then $Ann_R(R) = 0$ and in this case the above theorem is clear.

Corollary 3.3. *Let $\mathcal{F} = \{I_n\}_{n \geq 0}$ be a filtration on R and let M be a Noetherian $R-$module. For every $n \geq 0$, we assume that J_n contains all $x \in R$ such that*

$$x^k M \subseteq \sum_{i=1}^{k} x^{k-i} I_{ni} M$$

for a positive integer k. Then $\mathcal{G} = \{J_n\}_{n \geq 0}$ is a filtration on R.

Proof. It is clear that $J_0 = R$. By 3.2, we can see that J_n is an ideal of R for every $n \in \mathbf{N}$. Now let $x \in J_n$ and $y \in J_m$. Also let the integral closure of filtration $\widetilde{\mathcal{F}}$ on $\widetilde{R}$ be $(\widetilde{\mathcal{F}})^{-} = \{\widetilde{K}_n\}_{n \geq 0}$. We know from 3.2, $\widetilde{x} \in \widetilde{K}_n$ and $\widetilde{y} \in \widetilde{K}_m$. Since $(\widetilde{\mathcal{F}})^{-} = \{\widetilde{K}_n\}_{n \geq 0}$ is a filtration, we see that $\widetilde{xy} = \widetilde{x}\, \widetilde{y} \in \widetilde{K}_n \widetilde{K}_m \subseteq \widetilde{K}_{n+m}$. Then $xy \in J_{n+m}$ by 3.2. □

Definition 3.4. Let $\mathcal{F} = \{I_n\}_{n\geq 0}$ be a filtration on R and let M be a Noetherian $R-module$. For every n, let J_n be the set of all $x \in R$ such that

$$x^k M \subseteq \sum_{i=1}^{k} x^{k-i} I_{ni} M$$

for a positive integer k. By 3.3, $\{J_n\}_{n\geq 0}$ is a filtration on R. This filtration is denoted by $\mathcal{F}^{-(M)}$ and is called the integral closure of filtration $\mathcal{F}$ relative to M. We follows from 3.2, $\widetilde{(\mathcal{F}^{-(M)})} = (\widetilde{\mathcal{F}})^{-}$.

Remark 3.5. Let $\mathcal{F} = \{I_n\}_{n\geq 0}$ be a filtration on R and let M be a Noetherian $R-module$. Let for $x \in R$ and a positive integer k, we have $x^k M \subseteq \sum_{i=1}^{k} x^{k-i} I_{ni} M$. Then for every $r \geq 0$ we have

$$x^{k+r} M \subseteq \sum_{i=1}^{k+r} x^{(k+r)-i} I_{ni} M.$$

Theorem 3.6. *Let $\mathcal{F}$ and $\mathcal{G}$ be filtrations on R. Then for every Noetherian $R-module$ M, we have*

(a) $\mathcal{F} \leq \mathcal{F}^{-(M)}$;
(b) *if* $\mathcal{F} \leq \mathcal{G}$ *then* $\mathcal{F}^{-(M)} \leq \mathcal{G}^{-(M)}$;
(c) $(\mathcal{F}^{-(M)})^{-(M)} = \mathcal{F}^{-(M)}$;
(d) $\mathcal{F}^{-(M)}\mathcal{G}^{-(M)} \leq (\mathcal{F}\mathcal{G})^{-(M)}$.

Proof. (a) and (b) are clear.

(c) Let $\mathcal{F} = \{I_n\}_{n\geq 0}$, $\mathcal{F}^{-(M)} = \{J_n\}_{n\geq 0}$, and $(\mathcal{F}^{-(M)})^{-(M)} = \{K_n\}_{n\geq 0}$. Let the integral closure of filtration $\{\widetilde{J}_n\}_{n\geq 0}$ of ideals $\widetilde{R}$ be the filtration $(\{\widetilde{J}_n\}_{n\geq 0})^{-} = \{\widetilde{U}_n\}_{n\geq 0}$. We know from 3.2, $\{\widetilde{K}_n\}_{n\geq 0} = \{\widetilde{U}_n\}_{n\geq 0}$. This shows

$$\{\widetilde{K}_n\}_{n\geq 0} = (\{\widetilde{J}_n\}_{n\geq 0})^{-} = ((\{\widetilde{I}_n\}_{n\geq 0})^{-})^{-}.$$

By [3, 2.4(3)], we have $((\{\widetilde{I}_n\}_{n\geq 0})^{-})^{-} = (\{\widetilde{I}_n\}_{n\geq 0})^{-}$ and so $\{\widetilde{K}_n\}_{n\geq 0} = \{\widetilde{J}_n\}_{n\geq 0}$. Now since $Ann_R(M) \subseteq J_n$ for every $n \geq 0$, we can see that $(\mathcal{F}^{-(M)})^{-(M)} \leq \mathcal{F}^{-(M)}$. But by (a), we have $\mathcal{F}^{-(M)} \leq (\mathcal{F}^{-(M)})^{-(M)}$ and so $(\mathcal{F}^{-(M)})^{-(M)} = \mathcal{F}^{-(M)}$.

(d) Let $\mathcal{F} = \{I_n\}_{n\geq 0}$ and $\mathcal{G} = \{J_n\}_{n\geq 0}$. Further let $\mathcal{F}^{-(M)} = \{K_n\}_{n\geq 0}$, $\mathcal{G}^{-(M)} = \{L_n\}_{n\geq 0}$, and $(\mathcal{F}\mathcal{G})^{-(M)} = \{H_n\}_{n\geq 0}$. By 3.2, we know the integral closure of filtrations $\{\widetilde{I}_n\}_{n\geq 0}$, $\{\widetilde{J}_n\}_{n\geq 0}$, and $\{\widetilde{I}_n\widetilde{J}_n\}_{n\geq 0}$ of ideals $\widetilde{R}$ are $(\{\widetilde{I}_n\}_{n\geq 0})^{-} = \{\widetilde{K}_n\}_{n\geq 0}$, $(\{\widetilde{J}_n\}_{n\geq 0})^{-} = \{\widetilde{L}_n\}_{n\geq 0}$, and $(\{\widetilde{I}_n\widetilde{J}_n\}_{n\geq 0})^{-} = \{\widetilde{H}_n\}_{n\geq 0}$ respectively. Since $\widetilde{R}$ is a Noetherian ring by [3, 2.4(4)], we have $(\{\widetilde{I}_n\}_{n\geq 0})^{-}(\{\widetilde{J}_n\}_{n\geq 0})^{-} \leq (\{\widetilde{I}_n\widetilde{J}_n\}_{n\geq 0})^{-}$. This implies that $\{\widetilde{K}_n\}_{n\geq 0}\{\widetilde{L}_n\}_{n\geq 0} \leq \{\widetilde{H}_n\}_{n\geq 0}$. Now since $Ann_R(M) \subseteq H_n$ for every $n \geq 0$, we can conclude that $\mathcal{F}^{-(M)}\mathcal{G}^{-(M)} \leq (\mathcal{F}\mathcal{G})^{-(M)}$. □

Proposition 3.7. *Let $\mathcal{F} = \{I_n\}_{n\geq 0}$ be a Noetherian filtration of ideals R and let M be a Noetherian $R-$module. Let $\mathcal{F}^{-(M)} = \{J_n\}_{n\geq 0}$ be the integral closure of filtration $\mathcal{F}$ relative to M. If the filtration $\{\widetilde{J}_n\}_{n\geq 0}$ is a Noetherian filtration on $\widetilde{R}$ then the filtration $\mathcal{F}$ is a reduction of the filtration $\mathcal{F}^{-(M)}$ relative to M.*

Proof. We know from [4, 2.8], that the filtration $\widetilde{\mathcal{F}} = \{\widetilde{I}_n\}_{n\geq 0}$ is a reduction of the filtration $(\widetilde{\mathcal{F}})^- = \{\widetilde{J}_n\}_{n\geq 0}$. Then there exists a positive integer d such that

$$\widetilde{J}_n = \sum_{i=0}^{d} \widetilde{I}_{n-i}\widetilde{J}_i \quad for\ every\ n \geq 1.$$

Now we can see that

$$J_nM = \sum_{i=0}^{d} I_{n-i}J_iM \quad for\ every\ n \geq 1$$

and this completes the proof. □

Theorem 3.8. *Let $\mathcal{F} = \{I_n\}_{n\geq 0}$ be a filtration on R and let M be a Noetherian $R-module$. Let $\mathcal{F}^{-(M)} = \{J_n\}_{n\geq 0}$. Further for a non negative integer n and $x \in R$, let $L_k = Rx^k + x^{k-1}I_{n1} + x^{k-2}I_{n2} + \cdots + xI_{n(k-1)} + I_{nk}$ and $H_k = I_{nk}$. Then $x \in J_n$ if and only if the filtration $\{H_k\}_{k\geq 0}$ is a reduction of filtration $\{L_k\}_{k\geq 0}$ relative to M.*

Proof. ($\Rightarrow$) Let $x \in J_n$. Then there exists a positive integer k such that

$$x^kM \subseteq \sum_{i=1}^{k} x^{k-i}I_{ni}M.$$

Since $x^{k-i}I_{ni} \subseteq H_{k-(k-i)}L_{k-i}$ for every $1 \leq i \leq k$,

$$\sum_{i=1}^{k} x^{k-i}I_{ni}M \subseteq \sum_{i=0}^{k} H_{k-(k-i)}L_{k-i}M = \sum_{i=0}^{k} H_{k-i}L_iM.$$

But $x^kM \subseteq \sum_{i=1}^{k} x^{k-i}I_{ni}M$ and so

$$(Rx^k+x^{k-1}I_{n1}+x^{k-2}I_{n2}+\cdots+xI_{n(k-1)}+I_{nk})M = L_kM \subseteq \sum_{i=0}^{k} H_{k-i}L_iM.$$

It is easy to see that $\sum_{i=0}^{k} H_{k-i}L_iM \subseteq L_kM$. Then $L_kM = \sum_{i=0}^{k} H_{k-i}L_iM$. Now, we will show that

$$L_tM = \sum_{i=0}^{k} H_{t-i}L_iM \quad \textit{for every } t \geq 1.$$

First let $t < k$. Since $t < k$,

$$L_tM = H_0L_tM \subseteq \sum_{i=0}^{k} H_{t-i}L_iM.$$

Also we know $\sum_{i=0}^{k} H_{t-i}L_iM \subseteq L_tM$. Thus we have

$$L_tM = \sum_{i=0}^{k} H_{t-i}L_iM \quad \textit{for every } t \leq k-1.$$

Now let $t > k$. This is clear that

$$\sum_{i=0}^{k} H_{t-i}L_iM = \sum_{i=0}^{k} x^i I_{n(t-i)}M.$$

Since $x^kM \subseteq \sum_{i=1}^{k} x^{k-i}I_{ni}M$, we can see that

$$x^{k+r}M \subseteq \sum_{i=1}^{k} x^{k-i}I_{n(r+i)}M.$$

But by

$$x^{k+r}I_{n(t-(k+r))}M \subseteq \sum_{i=1}^{k} x^{k-i}I_{n(r+i)}I_{n(t-(k+r))}M$$

$$\subseteq \sum_{i=1}^{k} x^{k-i}I_{n(t-k+i)}M \subseteq \sum_{i=0}^{k} x^i I_{n(t-i)}M$$

we have

$$L_tM = (x^t + x^{t-1}I_{n1} + \cdots + x^{k+1}I_{n(t-(k+1))} + x^kI_{n(t-k)} + \cdots + xI_{n(t-1)} + I_{nt})M$$

$$\subseteq \sum_{i=0}^{k} H_{t-i}L_iM$$

and this implies that

$$L_t M = \sum_{i=0}^{k} H_{t-i} L_i M \quad for\ every\ t \geq 1.$$

($\Leftarrow$) Let $\{H_k\}_{k\geq 0}$ be a reduction of filtration $\{L_k\}_{k\geq 0}$ relative to M. Then there exists a positive integer d such that

$$L_k M = \sum_{i=0}^{d} H_{k-i} L_i M \quad for\ every\ k \geq 1.$$

So we can assume that $L_{d+1}M = \sum_{i=0}^{d} H_{((d+1)-i)} L_i M$. Now since

$$L_{d+1}M = \sum_{i=0}^{d} H_{((d+1)-i)} L_i M \subseteq \sum_{i=0}^{d} x^{(d-i)} I_{n(i+1)} M = \sum_{i=1}^{d+1} x^{((d+1)-i)} I_{ni} M,$$

we have $x^{d+1}M \subseteq \sum_{i=1}^{d+1} x^{((d+1)-i)} I_{ni} M$. Hence $x \in J_n$. □

Definition 3.9. (See [3, 3.1(2)].) Let $\mathcal{F} = \{I_n\}_{n\geq 0}$ be a filtration on a Noetherian ring R and $\mathcal{F}^- = \{J_n\}_{n\geq 0}$. Members of

$$A^-(\mathcal{F}) = \{P : P \in Ass(R/J_n)\ for\ some\ n \geq 1\}$$

are called the asymptotic prime divisors of $\mathcal{F}$.

Definition 3.10. Let $\mathcal{F} = \{I_n\}_{n\geq 0}$ be a filtration on a Noetherian ring R and M be a Noetherian $R-$module. Let $\mathcal{F}^{-(M)} = \{J_n\}_{n\geq 0}$. Members of

$$A^-(\mathcal{F}, M) = \{P : P \in Ass(R/J_n)\ for\ some\ n \geq 1\}$$

are called the asymptotic prime divisors of $\mathcal{F}$ relative to M.

Remark 3.11. Let $\mathcal{F} = \{I_n\}_{n\geq 0}$ be a Noetherian filtration of ideals of R and let M be a Noetherian $R-$module. Then $A^-(\mathcal{F}, M)$ is a finite set.

Proof. It is easy to see that $P \in A^-(\mathcal{F}, M)$ if and only if $\widetilde{P} \in A^-(\widetilde{\mathcal{F}})$. Since $\widetilde{\mathcal{F}}$ is a Noetherian filtration on Noetherian ring $\widetilde{R}$, we know from [3, 3.3(2)], that $A^-(\widetilde{\mathcal{F}})$ is a finite set and this completes the proof. □

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