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A NEW RADICAL IN FREE MODULES

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Abstract. An R-module M is called torsion-free, if $rx = 0$ for $r \in R$ and $x \in M$ implies that $r = 0$ or $x = 0$. In this paper, we introduce the notions semi torsion-free modules and quasi torsion-free modules. We show that a submodule N of an R-module M is a P-primary submodule if and only if $\frac{R}{P}$-module $\frac{M}{N}$ is semi torsion-free. Also we define a new radical in free modules and find some characterizations of it. We prove that for P-submodule N of a free R-module F which $\sqrt{N} \subsetneqq F$, we have for any $r \in R$ and $m \in F$, $rm \in N$ implies $r \in \sqrt{P}$ or $m \in \sqrt{N}$ if and only if $\frac{R}{P}$-module $\frac{F}{N}$ is quasi torsion free.

1. Introduction

Throughout this paper, all rings are commutative with identity and all modules are unitary.

Let R be a ring, M be an R-module and N be a submodule of M. For any subset X of M, $(X :_R M) = \{r \in R : rM \subseteq X\}$. It is well-known that $(N :_R M)$ is an ideal of R. Also it is clear that for subsets X and Y of M which $X \subseteq Y$, we have $(X :_R M) \subseteq (Y :_R M)$. A submodule N of an R-module M is called I-submodule, if $I = (N :_R M)$. A proper submodule N of M is called a P-prime submodule, if $rm \in N$ for $r \in R$ and $m \in M$ implies that $m \in N$ or $r \in P = (N :_R M)$. It is well-known that a proper submodule N of M is P-prime if and only if P is a prime ideal of R and $\frac{M}{N}$ is torsion-free as an $\frac{R}{P}$-module [6,

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Theorem 1] . The radical of N is given by $rad_M(N) = \cap P$, where the intersection is over all prime submodules of M containing N. If there is no prime submodule containing N, then we put $rad_M(N) = M$(For more information about $rad_M(N)$, see [4],[5] and [7]). Also $Rad(M)$ is defined to be the intersection of all maximal submodules of M and if there is no maximal submodule, then we put $Rad(M) = M$.
A proper submodule Q of M is called a primary submodule provided that for any $r \in R$ and $m \in M$, $rm \in Q$ implies that $m \in Q$ or $r^n \in (Q :_R M)$ for some positive integer n. Let Q be a primary submodule of M, then the radical of the ideal $(Q :_R M)$ is a prime ideal of R [9, Propositions 15 and 18]. If $P = \sqrt{(Q :_R M)}$, then Q is called a P-primary submodule of M. Let F be a free R-module and P be a primary ideal of R. Then PF is a primary submodule of F and $(PF : F) = P$.

2. **Semi torsion-free modules**

In this section, we define semi torsion-free modules and find some characterizations of them. Let M ba an R-module and $Z_R(M) = \{r \in R : rm = 0, \exists m \in M\backslash\{0\}\}$.

Definition 2.1. An R-module M is called semi torsion-free, if for $r \in R$ and $x \in M$, $rx = 0$ implies that $x = 0$ or r is nilpotent.

If R is a domain, then any semi torsion-free module is torsion-free. It is clear that every torsion-free module is semi torsion-free; but in the following example we show that the converse is not true in general.

Example 2.2. i) $\mathbb{Z}_4$ as $\mathbb{Z}$-module is not torsion-free nor semi torsion-free.
ii) $F = \mathbb{Z}_4 \oplus \mathbb{Z}_4$ as $\mathbb{Z}_4$-module is semi torsion-free module; but it is not torsion-free module.

Theorem 2.3. *Let N be a P-submodule of an R-module M. Then the following conditions are equivalent:*
(a) N is a primary submodule of M;
(b) R/P-module M/N is semi torsion-free;
(c) For every $r \in R\backslash\sqrt{P}$, $N = \{m \in M : rm \in N\}$;
(d) For every ideal J of R such that $J \nsubseteq \sqrt{P}$, $N = \{m \in M : Jm \subseteq N\}$;
(e) For every $m \in M\backslash N$, $\sqrt{P} = \sqrt{(N :_R Rm)}$;
(f) For every submodule L of M such that $L \nsubseteq N$, $\sqrt{P} = \sqrt{(N :_R L)}$;
(g) For every $m \in M\backslash N$, $\sqrt{P} = \sqrt{Ann_R(m + N)}$;
(h) $\sqrt{P} = \sqrt{Z_R(M/N)}$.

Proof. $(a) \Rightarrow (b)$ Let for $r+P \in R/P$ and $m+N \in M/N$, $(r+P)(m+N) = 0_{M/N} = N$. So $rm \in N$. Now by part (a), $r \in \sqrt{P}$ or $m \in N$. If $r \in \sqrt{P}$, then

$$\exists t \in \mathbb{N};\ r^t \in P \Rightarrow (r+P)^t = r^t + P = P.$$

Therefore $r+P$ is nilpotent. If $m \in N$, then $m+N = N$. So R/P-module M/N is semi torsion-free.

$(b) \Rightarrow (c)$ Let $r \in R \backslash \sqrt{P}$ and $L = \{m \in M : rm \in N\}$. We show that $L = N$. It is clear that $N \subseteq L$. Now let $m \in L$. Hence $rm \in N$. Thus $(r+P)(m+N) = rm + N = N$. If $m \notin N$, then by part (b), there exists some $n \in \mathbb{N}$ such that $r^n \in P$, which results that $r \in \sqrt{P}$. This is a contradiction. So $m \in N$ and therefore $L \subseteq N$.

$(c) \Rightarrow (d)$ Let J be an ideal of R such that $J \nsubseteq \sqrt{P}$. So there exists some $r \in J \backslash \sqrt{P}$. By part (c), $N = \{m \in M : rm \in N\}$. Now put $L = \{m \in M : Jm \subseteq N\}$. We show that $L = N$. It is clear that $N \subseteq L$. Now let $m \in L$. Therefore $Jm \subseteq N$ and so $rm \in N$. Thus by part (c), $m \in N$.

$(d) \Rightarrow (e)$ Let $m \in M \backslash N$. So we have $P = (N :_R M) \subseteq (N :_R Rm)$. Hence $\sqrt{P} \subseteq \sqrt{(N :_R Rm)}$. Now let $r \in \sqrt{(N :_R Rm)}$. Therefore there exists some $n \in \mathbb{N}$ such that $r^n \in (N :_R Rm)$. So $r^n m \in N$. If $r \notin \sqrt{P}$, then $r^n \notin \sqrt{P}$. Now put $J = Rr^n$. Hence $J \nsubseteq \sqrt{P}$. By part (d), $N = \{m \in M : Jm \subseteq N\}$. On the other hand we have

$$r^n (Rm) = (Rr^n)\, m \subseteq N \Rightarrow Jm \subseteq N \Rightarrow m \in N.$$

This is a contradiction. So $r \in \sqrt{P}$. Therefore $\sqrt{(N :_R Rm)} \subseteq \sqrt{P}$. Hence $\sqrt{P} = \sqrt{(N :_R Rm)}$.

$(e) \Rightarrow (f)$ Let L be a submodule of M such that $L \nsubseteq N$. So there exists some $m \in L \backslash N$. By part (e), $\sqrt{P} = \sqrt{(N :_R Rm)}$. We have $P = (N :_R M) \subseteq (N :_R L)$ and so $\sqrt{P} \subseteq \sqrt{(N :_R L)}$. Now let $r \in \sqrt{(N :_R L)}$. So

$$\begin{aligned} \exists n \in \mathbb{N};\ r^n \in (N :_R L) \Rightarrow r^n L \subseteq N &\Rightarrow r^n m \in N \\ &\Rightarrow r^n (Rm) \subseteq N \\ &\Rightarrow r^n \in (N :_R Rm) \\ &\Rightarrow r \in \sqrt{(N :_R Rm)} = \sqrt{P}. \end{aligned}$$

Therefore $\sqrt{P} = \sqrt{(N :_R L)}$.

$(f) \Rightarrow (g)$ Let $m \in M \backslash N$. Put $L = N + Rm$. So $L \nsubseteq N$. Now by part (f),

$$\sqrt{P} = \sqrt{(N :_R L)} = \sqrt{(N :_R N + Rm)} = \sqrt{(N :_R Rm)}.$$

On the other hand we have $Ann_R(M/N) = (N :_R M)$ and so

$$\sqrt{Ann_R(M/N)} = \sqrt{(N :_R M)} = \sqrt{P}.$$

Thus

$$Ann_R(M/N) \subseteq Ann_R\,(m+N) \Rightarrow \sqrt{P} =$$
$$\sqrt{Ann_R(M/N)} \subseteq \sqrt{Ann_R\,(m+N)}.$$

Now let $r \in Ann_R\,(m+N)$. So

$$\begin{aligned} rm + N = r\,(m+N) = N &\Rightarrow rm \in N \\ &\Rightarrow r \in (N :_R Rm)\,. \end{aligned}$$

Thus $Ann_R\,(m+N) \subseteq (N :_R Rm)$ and then

$$\sqrt{Ann_R\,(m+N)} \subseteq \sqrt{(N :_R Rm)} = \sqrt{P}.$$

Therefore $\sqrt{P} = \sqrt{Ann_R\,(m+N)}$.

$(g) \Rightarrow (h)$ Let $m \in M\backslash N$. By part (g), $\sqrt{P} = \sqrt{Ann_R\,(m+N)}$. Let $t \in \sqrt{P}$. So

$$\begin{aligned} t \in \sqrt{Ann_R\,(m+N)} &\Rightarrow \exists n \in \mathbb{N};\; t^n\,(m+N) = N \\ &\Rightarrow t^n \in Z_R(M/N) \\ &\Rightarrow t \in \sqrt{Z_R(M/N)}. \end{aligned}$$

Hence $\sqrt{P} \subseteq \sqrt{Z_R(M/N)}$. Now let $r \in \sqrt{Z_R(M/N)}$. Therefore

$$\begin{aligned} \exists t \in \mathbb{N};\; r^t \in Z_R(M/N) &\Rightarrow \exists m \in M\backslash N;\; r^t\,(m+N) = N \\ &\Rightarrow r^t \in Ann_R\,(m+N) \\ &\Rightarrow r \in \sqrt{Ann_R\,(m+N)} = \sqrt{P}. \qquad (by\ part\ (g)) \end{aligned}$$

Hence $\sqrt{P} = \sqrt{Z_R(M/N)}$.

$(h) \Rightarrow (a)$ Let $r \in R$ and $m \in M$ such that $rm \in N$. So $rm + N = r(m+N) = N$. Now let $m \notin N$. Therefore $r \in Z_R(\frac{M}{N})$. By part (h), $\sqrt{P} = \sqrt{Z_R(\frac{M}{N})}$. Hence $r \in \sqrt{P}$. $\square$

3. **Quasi-radical subset of a free module**

In this section, we define a new radical for a non-empty subset of a free module and called it, quasi-radical and find some characterizations of it. Also we find the conditions that, quasi-radical of a subset of a free module, be a submodule. Let $N(R) = \{r \in R : \exists n \in \mathbb{N};\; r^n = 0\}$ be the nill-radical of R.

Remark 3.1. Let F be a free R-module with standard basis $\{e_\alpha\}_{\alpha\in\Gamma}$. So for any $y \in F$, there exist some $n \in \mathbb{N}$, $\alpha_i \in \Gamma$ and $r_{\alpha_i} \in R$ such that

$$y = r_{\alpha_1}e_{\alpha_1} + ... + r_{\alpha_n}e_{\alpha_n} = (r_\alpha)_{\alpha\in\Gamma},$$

where $r_\alpha = r_{\alpha_i}$, if $\alpha = \alpha_i$ and $r_\alpha = 0$, if $\alpha \neq \alpha_i$ for all $i(1 \leq i \leq n)$. Now for any $t \in \mathbb{N}$, y^t means

$$y^t = (r_\alpha)^t_{\alpha\in\Gamma} = (r^t_\alpha)_{\alpha\in\Gamma} = r^t_{\alpha_1}e_{\alpha_1} + ... + r^t_{\alpha_n}e_{\alpha_n}.$$

Definition 3.2. Let F be a free R-module and N be a subset of F. Define

$$\sqrt{N} = \left\{m \in F : \exists t \in \mathbb{N};\ m^t \in N\right\}$$

and called it quasi-recical of N.

Note that generally, $\sqrt{N}$ is not a submodule of F.

Example 3.3. Let $R = \mathbb{Z}$, $F = \mathbb{Z} \oplus \mathbb{Z}$ and $N = \langle (4,9) \rangle$. Then $(4,9), (2,3) \in \sqrt{N}$, but $(4,9) + (2,3) \notin \sqrt{N}$.

Lemma 3.4. *Let F be a free R-module and N be a submodule of F. Then $\sqrt{N}$ is closed under the modulus operation.*

Proof. The proof is obvious. □

Lemma 3.5. *Let I be an ideal of R and N and L be submodules of a free R-module F. Then we have the following statements:*
(a) $N \subseteq \sqrt{N}$ *and* $(N :_R F) \subseteq \left(\sqrt{N} :_R F\right)$.
(b) If $L \subseteq N$, *then* $\sqrt{L} \subseteq \sqrt{N}$.
(c) $\sqrt{\sqrt{N}} = \sqrt{N}$.
(d) $N = F$ *if and only if* $\sqrt{N} = F$.
(e) $\sqrt{N \cap L} \subseteq \sqrt{N} \cap \sqrt{L} = \sqrt{\sqrt{N} \cap \sqrt{L}}$.
(f) $\sqrt{N + L} \subseteq \sqrt{\sqrt{N} + \sqrt{L}}$.
(g) $\sqrt{IF} \subseteq \sqrt{\sqrt{I}F}$.
(h) $\sqrt{(N :_R F)} \subseteq (\sqrt{N} :_R F) \subseteq \sqrt{\left(\sqrt{N} :_R F\right)}$.
(i) If $N + L = F$, *then* $\sqrt{N} + \sqrt{L} = F$,
and if F is finitely generated, then
(j) $\sqrt{IF} = \sqrt{\sqrt{I}F}$.
(k) $(\sqrt{N} :_R F) \subseteq \sqrt{(radN :_R F)}$.
(l) $\sqrt{(N :_R F)} = (\sqrt{N} :_R F) = \sqrt{\left(\sqrt{N} :_R F\right)}$.

Proof. The proofs of (a) and (b) are clear.

(c) By part (a), $\sqrt{N} \subseteq \sqrt{\sqrt{N}}$. Now let $a \in \sqrt{\sqrt{N}}$. So

$$\begin{aligned} \exists t \in \mathbb{N};\ a^t \in \sqrt{N} &\Rightarrow \exists m \in \mathbb{N};\ a^{tm} \in N \\ &\Rightarrow a \in \sqrt{N} \\ &\Rightarrow \sqrt{\sqrt{N}} \subseteq \sqrt{N}. \end{aligned}$$

Hence $\sqrt{\sqrt{N}} = \sqrt{N}$.

(d) Let $N = F$, so $\sqrt{N} = \sqrt{F} = F$. Now let $\sqrt{N} = F$ and $\{e_i\}$ be a standard base for F. So for any i, $e_i \in \sqrt{N}$. Therefore there exist some $n_i \in \mathbb{N}$ such that $(e_i)^{n_i} \in N$. Thus for any i, $e_i = (e_i)^{n_i} \in N$ and hence $N = F$.

(e) Since $N \subseteq \sqrt{N}$ and $L \subseteq \sqrt{L}$, so $N \cap L \subseteq \sqrt{N} \cap \sqrt{L}$. Therefore by part (b), $\sqrt{N \cap L} \subseteq \sqrt{\sqrt{N} \cap \sqrt{L}}$. We show that $\sqrt{N} \cap \sqrt{L} = \sqrt{\sqrt{N} \cap \sqrt{L}}$. By part (a), $\sqrt{N} \cap \sqrt{L} \subseteq \sqrt{\sqrt{N} \cap \sqrt{L}}$. Let $x \in \sqrt{\sqrt{N} \cap \sqrt{L}}$. So

$$\begin{aligned} \exists n \in \mathbb{N};\ x^n \in \sqrt{N} \cap \sqrt{L} &\Rightarrow x^n \in \sqrt{N},\ x^n \in \sqrt{L} \\ &\Rightarrow \exists t, l \in \mathbb{N};\ x^{nt} \in N,\ x^{nl} \in L \\ &\Rightarrow x \in \sqrt{N},\ x \in \sqrt{L} \\ &\Rightarrow x \in \sqrt{N} \cap \sqrt{L} \\ &\Rightarrow \sqrt{\sqrt{N} \cap \sqrt{L}} \subseteq \sqrt{N} \cap \sqrt{L}. \end{aligned}$$

Hence $\sqrt{N \cap L} \subseteq \sqrt{N} \cap \sqrt{L} = \sqrt{\sqrt{N} \cap \sqrt{L}}$.

(f) The proof gets by parts (a) and (b).

(g) Since $\sqrt{I}$ is an ideal of R, $\sqrt{I}F$ is an R-submodule of F. Now by part (a), $\sqrt{I}F \subseteq \sqrt{\sqrt{I}F}$.

(h) Let $r \in \sqrt{(N :_R F)}$. So

$$\begin{aligned} \exists t \in \mathbb{N};\ r^t \in (N :_R F) &\Rightarrow r^t F \subseteq N \\ &\Rightarrow rF \subseteq \sqrt{N} \\ &\Rightarrow r \in \left(\sqrt{N} :_R F\right) \\ &\Rightarrow \sqrt{(N :_R F)} \subseteq \left(\sqrt{N} :_R F\right). \end{aligned}$$

On the other hand $\left(\sqrt{N} :_R F\right) \subseteq \sqrt{\left(\sqrt{N} :_R F\right)}$. So $\sqrt{(N :_R F)} \subseteq (\sqrt{N} :_R F) \subseteq \sqrt{\left(\sqrt{N} :_R F\right)}$.

(i) The proof is clear.

(j) Clearly $IF \subseteq \sqrt{I}F$. So by part (b), $\sqrt{IF} \subseteq \sqrt{\sqrt{I}F}$. Let $rank\,(F) = n$ and $\{e_i\}_{i=1}^n$ be a standard base for F. Now let $x \in$

$\sqrt{\sqrt{I}F}$. Therefore

$$\begin{aligned}
\exists t \in \mathbb{N};\ x^t \in \sqrt{I}F \Rightarrow \exists a_i \in \sqrt{I}\,(1 \leq i \leq n)\,;\ x^t = \sum_{i=1}^{n} a_i e_i = (a_1, \ldots, a_n) \\
&\Rightarrow \forall i\,(1 \leq i \leq n)\,,\ \exists t, t_i \in \mathbb{N};\ a_i^{t_i} \in I, x^t = (a_1, \ldots, a_n) \\
&\Rightarrow x^{tt_1 \ldots t_n} = \left(\left(a_1^{t_1}\right)^{t_2 \ldots t_n}, \ldots, \left(a_n^{t_n}\right)^{t_1 \ldots t_{n-1}}\right) \\
&\Rightarrow x^{tt_1 \ldots t_n} = \left(a_1^{t_1}\right)^{t_2 \ldots t_n} e_1 + \ldots + \left(a_n^{t_n}\right)^{t_1 \ldots t_{n-1}} e_n \in IF \\
&\Rightarrow x^{tt_1 \ldots t_n} \in IF \\
&\Rightarrow x \in \sqrt{IF} \\
&\Rightarrow \sqrt{\sqrt{I}F} \subseteq \sqrt{IF}.
\end{aligned}$$

Hence $\sqrt{IF} = \sqrt{\sqrt{I}F}$.

(k) Let $x \in \left(\sqrt{N} :_R F\right)$ and $\{e_i\}_{i=1}^n$ be a standard base for F. So

$$\begin{aligned}
xF \subseteq \sqrt{N} &\Rightarrow \forall i,\ xe_i \in \sqrt{N} \\
&\Rightarrow \forall i,\ \exists n_i \in \mathbb{N};\ x^{n_i} e_i \in N \subseteq rad_F(N) \\
&\Rightarrow \forall i,\ x^t e_i \in rad_F(N),\ where\ t = max\{n_i\}_{i=1}^n \\
&\Rightarrow x^t F \subseteq rad_F(N) \\
&\Rightarrow x^t \in (rad_F(N) :_R F) \\
&\Rightarrow x \in \sqrt{(rad_F(N) :_R F)} \\
&\Rightarrow \left(\sqrt{N} :_R F\right) \subseteq \sqrt{(rad_F(N) :_R F)}.
\end{aligned}$$

(l) Let $x \in \sqrt{\left(\sqrt{N} :_R F\right)}$ and $\{e_i\}_{i=1}^n$ be a standard base for F. So

$$\begin{aligned}
\exists n \in \mathbb{N};\ x^n \in \left(\sqrt{N} :_R F\right) &\Rightarrow x^n F \subseteq \sqrt{N} \\
&\Rightarrow \forall i,\ x^n e_i \in \sqrt{N} \\
&\Rightarrow \forall i,\ \exists t_i \in \mathbb{N};\ (x^n e_i)^{t_i} \in N \\
&\Rightarrow \forall i,\ x^{nt} e_i \in N,\ where\ t = max\{t_i\}_{i=1}^n \\
&\Rightarrow x^{nt} F \subseteq N \\
&\Rightarrow x^{nt} \in (N :_R F) \\
&\Rightarrow x \in \sqrt{(N :_R F)} \\
&\Rightarrow \sqrt{\left(\sqrt{N} :_R F\right)} \subseteq \sqrt{(N :_R F)}.
\end{aligned}$$

Now by part (h), we get

$$\sqrt{\left(\sqrt{N} :_R F\right)} \subseteq \sqrt{(N :_R F)} \subseteq \left(\sqrt{N} :_R F\right) \subseteq \sqrt{\left(\sqrt{N} :_R F\right)}.$$

Hence $\sqrt{(N :_R F)} = (\sqrt{N} :_R F) = \sqrt{\left(\sqrt{N} :_R F\right)}$. □

As the next example shows, generally in part (e) of the above lemma, the equality is not hold.

Example 3.6. Let $R = \mathbb{Z}$, $F = \mathbb{Z} \oplus \mathbb{Z}$, $N = \langle (2,3) \rangle$ and $L = \langle (4,9) \rangle$. Then $N \cap L = (0)$ and $(2,3) \in \sqrt{N} \cap \sqrt{L}$, but $(2,3) \notin \sqrt{N \cap L}$.

Proposition 3.7. *Let N and L be submodules of a free R-module F such that $(N :_R F) + (L :_R F) = R$ and $\sqrt{N \cap L}$ be submodule of F. Then $\sqrt{N \cap L} = \sqrt{N} \cap \sqrt{L}$.*

Proof. Since $(N :_R F) + (L :_R F) = R$, there exist $r_1 \in (N :_R F)$ and $r_2 \in (L :_R F)$ such that $r_1 + r_2 = 1$. By Lemma 3.5(e), $\sqrt{N \cap L} \subseteq \sqrt{N} \cap \sqrt{L}$. Now let $x \in \sqrt{N} \cap \sqrt{L}$. So

$$x \in \sqrt{N},\ x \in \sqrt{L} \Rightarrow \exists n, t \in \mathbb{N};\ x^t \in N,\ x^n \in L. \tag{3.1}$$

Since $r_1 \in (N :_R F)$, therefore we have

$$r_1^n \in (N :_R F) \Rightarrow r_1^n F \subseteq N \Rightarrow r_1^n x^n \in N$$

and on the other hand, $r_1^n x^n \in L$. So $r_1^n x^n \in N \cap L$. Similarly $r_2^t x^t \in N \cap L$. So

$$\begin{cases} (r_1 x)^n = r_1^n x^n \in N \cap L \Rightarrow r_1 x \in \sqrt{N \cap L} \\ \& \\ (r_2 x)^t = r_2^t x^t \in N \cap L \Rightarrow r_2 x \in \sqrt{N \cap L} \end{cases}$$

Now since $\sqrt{N \cap L}$ is a submodule of F, we have $r_1 x + r_2 x \in \sqrt{N \cap L}$. So

$$x = (r_1 + r_2)\, x \in \sqrt{N \cap L} \Rightarrow \sqrt{N} \cap \sqrt{L} \subseteq \sqrt{N \cap L}.$$

Hence $\sqrt{N \cap L} = \sqrt{N} \cap \sqrt{L}$. ☐

Corollary 3.8. *Let R be a ring such that every nonzero prime ideal of R is maximal. Let N and L be primary submodules of a free R-module F such that $(L :_R F) \neq 0$, $\sqrt{(N :_R F)} \nsubseteq \sqrt{(L :_R F)}$ and $\sqrt{N \cap L}$ be submodule of F. Then $\sqrt{N \cap L} = \sqrt{N} \cap \sqrt{L}$.*

Proof. Since $(L :_R F) \neq 0$, so $\sqrt{(N :_R F)} + \sqrt{(L :_R F)} = R$. Therefore $(N :_R F) + (L :_R F) = R$. Now by Proposition 3.7, $\sqrt{N \cap L} = \sqrt{N} \cap \sqrt{L}$. ☐

Proposition 3.9. *Let N and L be submodules of a free R-module F. Then $N \cap L = \sqrt{N} \cap \sqrt{L}$ if and only if $\sqrt{N \cap L} = N \cap L$ and $\sqrt{N \cap L} = \sqrt{N} \cap \sqrt{L}$.*

Proof. If $N \cap L = \sqrt{N} \cap \sqrt{L}$, then by Lemma 3.5(a) and (e),

$$\sqrt{N} \cap \sqrt{L} = N \cap L \subseteq \sqrt{N \cap L} \subseteq \sqrt{N} \cap \sqrt{L} \Rightarrow \sqrt{N \cap L} = \sqrt{N} \cap \sqrt{L} = N \cap L.$$

The other side of proposition is obvious. ☐

Proposition 3.10. *Let N and L be submodules of a finitely generated free R-module F. Then $(\sqrt{N\cap L} :_R F) = (\sqrt{N} :_R F)\cap(\sqrt{L} :_R F) = (\sqrt{N}\cap\sqrt{L} :_R F)$.*

Proof. By Lemma 3.5(e), $\sqrt{N\cap L}\subseteq\sqrt{N}\cap\sqrt{L}$, so $\left(\sqrt{N\cap L} :_R F\right)\subseteq\left(\sqrt{N}\cap\sqrt{L} :_R F\right)$. Now let $r\in\left(\sqrt{N}\cap\sqrt{L} :_R F\right)$ and $\{e_i\}_{i=1}^n$ be a standard base for F. Therefore

$$\begin{aligned} rF\subseteq\sqrt{N}\cap\sqrt{L} &\Rightarrow \forall i\,(1\le i\le n)\,,\ re_i\in\sqrt{N}\cap\sqrt{L}\\ &\Rightarrow \exists n,t\in\mathbb{N};\ \forall i\,(1\le i\le n)\,,\ (re_i)^n\in N, (re_i)^t\in L. \end{aligned}$$

Without loss of generality assume that $t\le n$. We have

$$\begin{aligned} \forall i\,(1\le i\le n)\,,\ (re_i)^n = r^n e_i\in N\cap L &\\ &\Rightarrow \forall i\,(1\le i\le n)\,,\ re_i\in\sqrt{N\cap L}\\ &\Rightarrow rF\subseteq\sqrt{N\cap L}\\ &\Rightarrow r\in\left(\sqrt{N\cap L} :_R F\right). \end{aligned}$$

Hence $\left(\sqrt{N\cap L} :_R F\right) = \left(\sqrt{N}\cap\sqrt{L} :_R F\right)$. It is clear that $(\sqrt{N} :_R F)\cap(\sqrt{L} :_R F) = (\sqrt{N}\cap\sqrt{L} :_R F)$. □

Proposition 3.11. *Let N and L be submodules of a finitely generated free R-module F such that $\sqrt{(N :_R F)} = (N :_R F)$ and $\sqrt{(L :_R F)} = (L :_R F)$. Then $(N\cap L :_R F) = (\sqrt{N\cap L} :_R F)$.*

Proof. Since $N\cap L\subseteq\sqrt{N\cap L}$. So $(N\cap L :_R F)\subseteq\left(\sqrt{N\cap L} :_R F\right)$. Let $r\in\left(\sqrt{N\cap L} :_R F\right)$ and $\{e_i\}_{i=1}^n$ be a standard base for F. Therefore

$$\begin{aligned} rF\subseteq\sqrt{N\cap L} &\Rightarrow \forall i\,(1\le i\le n)\,,\ re_i\in\sqrt{N\cap L}\\ &\Rightarrow \exists t\in\mathbb{N};\ \forall i\,(1\le i\le n)\,,\ r^t e_i\in N\cap L\\ &\Rightarrow \forall i\,(1\le i\le n)\,,\ r^t e_i\in N,\ r^t e_i\in L\\ &\Rightarrow r^t F\subseteq N,\ r^t F\subseteq L\\ &\Rightarrow r^t\in(N :_R F)\,,\ r^t\in(L :_R F)\\ &\Rightarrow r\in\sqrt{(N :_R F)} = (N :_R F)\,,\ r\in\sqrt{(L :_R F)} = (L :_R F)\\ &\Rightarrow r\in(N :_R F)\cap(L :_R F) = (N\cap L :_R F)\,. \end{aligned}$$

Hence $(N\cap L :_R F) = (\sqrt{N\cap L} :_R F)$. □

Proposition 3.12. *Let N be a submodule of finitely generated free R-module F such that $\sqrt{N} = N$. Then $\sqrt{(N :_R F)} = (N :_R F)$.*

Proof. Clearly $(N :_R F) \subseteq \sqrt{(N :_R F)}$. Let $r \in \sqrt{(N :_R F)}$ and $\{e_i\}_{i=1}^n$ be a standard base for F. So

$$\begin{aligned}\exists t \in \mathbb{N};\ r^t \in (N :_R F) &\Rightarrow r^t F \subseteq N \\ &\Rightarrow \forall i\,(1 \leq i \leq n)\,,\ (re_i)^t = r^t e_i \in N \\ &\Rightarrow \forall i\,(1 \leq i \leq n)\,,\ re_i \in \sqrt{N} \\ &\Rightarrow rF \subseteq \sqrt{N} = N \\ &\Rightarrow r \in (N :_R F)\,.\end{aligned}$$

Hence $\sqrt{(N :_R F)} = (N :_R F)$. □

In the following example, we show that the converse of the above proposition is not true in general.

Example 3.13. Let $R = \mathbb{Z}$, $F = \mathbb{Z} \oplus \mathbb{Z}$, $N =< (4, 9) >$. Then $\sqrt{(N :_R F)} = (N :_R F) = (0)$; but $(2, 3) \in \sqrt{N} \backslash N$.

Proposition 3.14. *Let F be a finitely generated free R-module. Then the following statements hold.*
(i) $N(R)F = \sqrt{0} \subseteq rad_F(0) \subseteq Rad(F)$.
(ii) If R is an integral domain, then $rad_F(0) = N(R)F = \sqrt{0}$.
(iii) If R is an Artinian ring, then $\sqrt{0} = Rad(F)$.

Proof. Let $\{e_i\}_{i=1}^n$ be a standard base for F.
(i) Since any maximal submodule is a prime submodule, so $rad_F(0) \subseteq Rad(F)$. Now we show that $N(R)F = \sqrt{0}$. Let $x \in \sqrt{0}$. So $x \in F$ and there exist some $r_i \in R$ such that

$$x = \sum_{i=1}^{n} r_i e_i = (r_1, \ldots, r_n)\,.$$

On the other hand

$$\begin{aligned}\exists t \in \mathbb{N};\ x^t = 0 &\Rightarrow \left(r_1^t, \ldots, r_n^t\right) = 0 \\ &\Rightarrow \forall j\,(1 \leq j \leq n)\,,\ r_j^t = 0 \\ &\Rightarrow \forall j\,(1 \leq j \leq n)\,,\ r_j \in N(R) \\ &\Rightarrow x = \sum_{i=1}^{n} r_i e_i \in N(R)F \\ &\Rightarrow \sqrt{0} \subseteq N(R)F.\end{aligned}$$

Now let $a \in N(R)$. So

$$\begin{aligned}\exists l \in \mathbb{N};\ a^n = 0 &\Rightarrow (ae_i)^n = 0,\ \forall i\,(1 \leq i \leq n) \\ &\Rightarrow ae_i \in \sqrt{0},\ \forall i\,(1 \leq i \leq n) \\ &\Rightarrow N(R)F \subseteq \sqrt{0}.\end{aligned}$$

So $N(R)F = \sqrt{0}$. Now we show that $N(R)F \subseteq rad_F(0)$. Let P be a prime submodule of F and $r \in N(R)$. So $r \in (P :_R F) \in Spec\,(R)$.

Therefore $rF \subseteq P$. So $N(R)F \subseteq rad_F(0)$. Hence $N(R)F = \sqrt{0} \subseteq rad_F(0) \subseteq RadF$.
(ii) Since R is a domain, so $\sqrt{0} = N(R)F = (0)$ and (0) is a prime submodule of F. Therefore $rad_F(0) = (0)$. Hence $rad_F(0) = N(R)F = \sqrt{0}$.
(iii) Since R is an Artinian ring, so by [3, Theorem 7.3], $\frac{R}{J(R)}$ is an Artinian and semi simple ring. Therefore by [1, Proposition 12] , $\frac{R}{J(R)}$-module $\frac{F}{J(R)F}$ is semi simple. So by [1, Proposition 13], $Rad\left(\frac{F}{J(R)F}\right) = 0$. Therefore $Rad(F) = J(R)F$. Now we get

$$Rad(F) = J(R)F = N(R)F = \sqrt{0} \Rightarrow Rad(F) = \sqrt{0}.$$

□

Theorem 3.15. *Let S be a multiplicative closed subset of R and N be a submodule of a free R-module F. Then $\sqrt{N_S} = (\sqrt{N})_S$.*

Proof. Let $\frac{a}{b} \in \sqrt{N_S}$. So

$$\begin{aligned} \exists t \in \mathbb{N};\ \left(\frac{a}{b}\right)^t \in N_S \Rightarrow \frac{a^t}{b^t} \in N_S & \\ & \Rightarrow \exists n \in N, r \in S;\ \frac{a^t}{b^t} = \frac{n}{r} \\ & \Rightarrow \exists u \in S;\ ua^t r = unb^t \in N \\ & \Rightarrow u^t a^t r^t \in N \\ & \Rightarrow uar \in \sqrt{N} \\ & \Rightarrow \frac{a}{b} = \frac{uar}{bur} \in \left(\sqrt{N}\right)_S. \end{aligned}$$

Now let $\frac{c}{d} \in \left(\sqrt{N}\right)_S$ for $c \in \sqrt{N}$ and $d \in S$. Then

$$\begin{aligned} \exists m \in \mathbb{N};\ c^m \in N \Rightarrow & \frac{c^m}{d^m} \in N_S \\ \Rightarrow & \frac{c}{d} \in \sqrt{N_S}. \end{aligned}$$

Hence $\sqrt{N_S} = (\sqrt{N})_S$. □

Theorem 3.16. *Let N be a primary submodule of a free R-module F. Then $\sqrt{N} = N$ if and only if $\sqrt{N_m} = N_m$, for all maximal ideal m.*

Proof. Let for all $m \in Max(R)$, $\sqrt{N_m} = N_m$. By Lemma 3.5 (a), $N \subseteq \sqrt{N}$. Let $a \in \sqrt{N}$ and $I = (N :_R F)$. Since $I \neq R$, so there exists some maximal ideal m of R such that $I \subseteq m$. Therefore by Theorem 3.15,

$$\frac{a}{1} \in \left(\sqrt{N}\right)_m = \sqrt{N_m} = N_m \Rightarrow \exists n \in N,\ t \in R\backslash m;\ \frac{a}{1} = \frac{n}{t}$$
$$\Rightarrow \exists u \in R\backslash m;\ uat = un \in N$$
$$\Rightarrow ut \in \sqrt{(N :_R F)} \text{ or } a \in N. \text{ (N } \textit{is primary)}$$

If $ut \in \sqrt{(N :_R F)}$, then $ut \in m$, which is a contradiction. Therefore $a \in N$. Hence $\sqrt{N} = N$. Now let $\sqrt{N} = N$ and m be a maximal ideal of R. Therefore by Theorem 3.15, $\sqrt{N_m} = \left(\sqrt{N}\right)_m = N_m$. □

Proposition 3.17. *Let N be a primary submodule of free R-module F such that $(\sqrt{N} :_R F) = (N :_R F)$. Then N is a prime submodule.*

Proof. Let $r \in R$ and $x \in F$ such that $rx \in N$. Since N is a primary submodule, so $x \in N$ or $r \in \sqrt{(N :_R F)}$. If $r \in \sqrt{(N :_R F)}$, then by Lemma 3.5(h), $r \in \left(\sqrt{N} :_R F\right)$. So $r \in (N :_R F)$. Hence N is a prime submodule. □

Corollary 3.18. *Let R be an integral domain and N be a primary submodule of free R-module F such that $(\sqrt{N} :_R F) = (0)$. Then N is a prime submodule.*

Proof. By Lemma 3.5(a), $N \subseteq \sqrt{N}$, so

$$\left(\sqrt{N}:_R F\right) = (0) \subseteq (N:_R F) \subseteq \left(\sqrt{N}:_R F\right) = (0) \Rightarrow (N:_R F) = \left(\sqrt{N}:_R F\right).$$

Now by Proposition 3.17, N is a prime submodule. □

Proposition 3.19. *Let R be a domain such that $dimR = 1$ and N be a primary submodule of a finitely generated free R-module F such that $\sqrt{N}$ is a submodule. Then N or $\sqrt{N}$ is a prime submodule.*

Proof. If $(N :_R F) = (0)$, then N is a prime submodule. If $(N :_R F) \neq (0)$, then $\sqrt{(N :_R F)}$ is a nonzero prime ideal and as $dimR = 1$, is a maximal ideal. By Lemma 3.5(l), $\sqrt{(N :_R F)} = (\sqrt{N} :_R F)$, so $(\sqrt{N} :_R F)$ is a maximal ideal. Thus $\sqrt{N}$ is a prime submodule. □

Lemma 3.20. *Let N_1 and N_2 be primary submodules of finitely generated free R-module F such that $\sqrt{N_1} = \sqrt{N_2}$. Then $N_1 \cap N_2$ is a primary submodule.*

Proof. Let $r \in R$ and $x \in F$ such that $rx \in N_1 \cap N_2$. So $rx \in N_1$ and $rx \in N_2$. Since N_1, N_2 are primary submodules, therefore

$$\begin{cases} r \in \sqrt{(N_1{:}_R F)} \text{ or } x \in N_1 \\ \& \\ r \in \sqrt{(N_2{:}_R F)} \text{ or } x \in N_2 \end{cases}$$

Let $\{e_i\}_{i=1}^n$ be a standard base for F. If $x \in N_1$ and $x \in N_2$, then $x \in N_1 \cap N_2$. If $r \in \sqrt{(N_1{:}_R F)}$, then

$$\begin{aligned} \exists t \in \mathbb{N};\ r^t \in (N_1{:}_R F) &\Rightarrow r^t F \subseteq N_1 \\ &\Rightarrow \forall i,\ r^t e_i \in N_1 \\ &\Rightarrow \forall i,\ (re_i)^t = r^t {e_i}^t = r^t e_i \in N_1 \\ &\Rightarrow \forall i,\ re_i \in \sqrt{N_1} = \sqrt{N_2} \\ &\Rightarrow rF \subseteq \sqrt{N_1},\ rF \subseteq \sqrt{N_2} \\ &\Rightarrow r \in \left(\sqrt{N_1}{:}_R F\right),\ r \in \left(\sqrt{N_2}{:}_R F\right) \\ &\Rightarrow r \in \left(\sqrt{N_1}{:}_R F\right) \cap \left(\sqrt{N_2}{:}_R F\right) = \left(\sqrt{N_1} \cap \sqrt{N_2}{:}_R F\right) \\ &\Rightarrow r \in \left(\sqrt{N_1} \cap \sqrt{N_2}{:}_R F\right) \\ &\Rightarrow r \in \left(\sqrt{N_1 \cap N_2}{:}_R F\right)\ (By\ Proposition\ 3.10) \\ &\Rightarrow rF \subseteq \sqrt{N_1 \cap N_2} \\ &\Rightarrow \exists n \in \mathbb{N};\ r^n F \subseteq N_1 \cap N_2\ (F\ is\ f.g.) \\ &\Rightarrow r^n \in (N_1 \cap N_2{:}_R F) \\ &\Rightarrow r \in \sqrt{(N_1 \cap N_2{:}_R F)} \end{aligned}$$

Similarly if $r \in \sqrt{(N_2{:}_R F)}$, then $r \in \sqrt{(N_1 \cap N_2{:}_R F)}$. Hence $N_1 \cap N_2$ is a primary submodule. □

Proposition 3.21. *Let P_i $(1 \leq i \leq n)$ be prime ideals of R and $N = P_1 \oplus ... \oplus P_n \oplus R \oplus ... \oplus R$. Then $\sqrt{N} = N$.*

Proof. Let $(s_1, \ldots, s_n, r_1, \ldots, r_m) \in \sqrt{N}$. So

$$\begin{aligned} \exists t \in \mathbb{N};\ \left(s_1^t, \ldots, s_n^t, r_1^t, \ldots, r_m^t\right) &= (s_1, \ldots, s_n, r_1, \ldots, r_m)^t \in N \\ \Rightarrow \forall i\,(1 \leq i \leq n),\ s_i^t \in P_i. \end{aligned}$$

Now since for all $i\,(1 \leq i \leq n)$, P_i is a prime ideal. Therefore for all $i\,(1 \leq i \leq n)$, $s_i \in P_i$. So $\sqrt{N} \subseteq N$. Hence by Lemma 3.5(a), $\sqrt{N} = N$. □

In the rest of this paper, we assume that N is a non-zero submodule of a free R-module F of rank n which is generated by the set

$$A = \left\{\left(a_i^1, a_i^2, \ldots, a_i^n\right) \in F : i \in \Lambda\right\}.$$

For $n = 2$, let $A = \{(a_i, b_i) \in F : i \in \Lambda\}$ and $\ell = \sum_{i,j \in \Lambda} R\Delta_{ij}$ where $\Delta_{ij} = a_i b_j - b_i a_j$ for $i, j \in \Lambda$.

Lemma 3.22. *Let $F = R \oplus R$ and N be as above. Then $\ell \subseteq (N :_R F) \subseteq \sqrt{\ell}$.*

Proof. [2, Lemma 2.1]. □

Theorem 3.23. *Let N be a submodule of free R-module F and $P = \sqrt{(N :_R F)}$.*
(a) If P is a prime ideal of R and $a_i^1, a_i^2, \ldots, a_i^n \in P$ for all $i \in \Lambda$, then $\sqrt{N} = P \oplus \ldots \oplus P$ and $\sqrt{N}$ is a prime submodule of F.
(b) If $(N :_R F)$ is a maximal ideal of R and for all j $(j = 1, \ldots, n)$ there exist some $i_j \in \Lambda$ such that $a_{i_j}^j \notin (N :_R F)$, then $N = \left(a_{i_1}^1 a_{i_2}^2 \ldots a_{i_n}^n\right) N + (N :_R F) F$ and N is a primary submodule of F.
(c) If $n = 2$, $\{a_i : i \in \Lambda\} \cup \{b_i : i \in \Lambda\} \not\subset P$ and N is a primary submodule of F such that $\sqrt{N}$ is closed under addition operation, then

$$\sqrt{N} = \left\{(m, n) \in F : \exists t \in \mathbb{N},\ m^t b_i - n^t a_i \in P,\ \forall i \in \Lambda\right\}.$$

In particular,
(i) If $a_i \in P$ for all $i \in \Lambda$ and $b_j \notin P$ for some $j \in \Lambda$ (resp. $b_i \in P$ for all $i \in \Lambda$ and $a_j \notin P$ for some $j \in \Lambda$), then $\sqrt{N} = P \oplus R$ (resp. $\sqrt{N} = R \oplus P$).
(ii) If $a_i \notin P$ and $b_i \in P$ for some $i \in \Lambda$ (resp. $b_i \notin P$ and $a_i \in P$ for some $i \in \Lambda$), then $\sqrt{N} = R \oplus P$ (resp. $\sqrt{N} = P \oplus R$).

Proof. (a) Since P is a prime ideal of R, it is clear that P is a primary ideal of R, PF is a primary submodule of F and $(PF :_R F) = P$ ([8, Result 1.4]).
As

$$a_i^1, a_i^2, \ldots, a_i^n \in P,\ \forall i \in \Lambda \Rightarrow \left(a_i^1, a_i^2, \ldots, a_i^n\right) \in P \oplus P \oplus \ldots \oplus P,\ \forall i \in \Lambda.$$

So $N \subseteq P \oplus \ldots \oplus P$ and hence $\sqrt{N} \subseteq \sqrt{P \oplus \ldots \oplus P}$. It is easily seen that $\sqrt{P \oplus \ldots \oplus P} = P \oplus \ldots \oplus P$. So we have $\sqrt{N} \subseteq P \oplus \ldots \oplus P$. By Lemma 3.5(h), $PF \subseteq \sqrt{N}$. So $\sqrt{N} = PF = P \oplus \ldots \oplus P$ and $\sqrt{N}$ is a primary submodule of F.
(b) Since $(N :_R F)$ is a maximal ideal of R, $(N :_R F)$ is a prime ideal of R and N is a primary submodule of F ([6] Proposition 2). Hence $a_{i_1}^1 a_{i_2}^2 \ldots a_{i_n}^n \notin (N :_R F)$. So

$$R\left(a_{i_1}^1 a_{i_2}^2 \ldots a_{i_n}^n\right) + (N :_R F) = R$$

Now let $(x_1, \ldots, x_n) \in N$, for some $x_i \in R$ $(1 \leq i \leq n)$. Hence there exist $r_1, \ldots, r_n \in R$ and $p_1, \ldots, p_n \in (N :_R F)$ such that for all j $(1 \leq j \leq n)$, $x_j = r_j\left(a_{i_1}^1 a_{i_2}^2 \ldots a_{i_n}^n\right) + p_j$. So

$$\begin{aligned}(x_1, \ldots, x_n) &= \left(r_1\left(a_{i_1}^1 a_{i_2}^2 \ldots a_{i_n}^n\right) + p_1, \ldots, r_n\left(a_{i_1}^1 a_{i_2}^2 \ldots a_{i_n}^n\right) + p_n\right)\\ &= \left(a_{i_1}^1 a_{i_2}^2 \ldots a_{i_n}^n\right)(r_1, \ldots, r_n) + (p_1, \ldots, p_n) \in N.\end{aligned}$$

Since $(p_1,\ldots,p_n)\in(N:_R F)F\subseteq N$, we have $\left(a^1_{i_1}a^2_{i_2}\ldots a^n_{i_n}\right)(r_1,\ldots,r_n)\in N$. Hence $a^1_{i_1}a^2_{i_2}\ldots a^n_{i_n}\in\sqrt{(N:_R F)}=(N:_R F)$ or $(r_1,\ldots,r_n)\in N$. As $a^1_{i_1}a^2_{i_2}\ldots a^n_{i_n}\notin(N:_R F)$ we have $(r_1,\ldots,r_n)\in N$. This implies that

$$\begin{aligned}(x_1,\ldots,x_n)&\in\left(a^1_{i_1}a^2_{i_2}\ldots a^n_{i_n}\right)N+(N:_R F)F\\ &\Rightarrow N\subseteq\left(a^1_{i_1}a^2_{i_2}\ldots a^n_{i_n}\right)N+(N:F)F\subseteq N+N=N\\ &\Rightarrow N=\left(a^1_{i_1}a^2_{i_2}\ldots a^n_{i_n}\right)N+(N:F)F.\end{aligned}$$

(c) We may assume that $a_1\notin P$. Put

$$T_P=\left\{(m,n)\in F:\exists t\in\mathbb{N},\ m^tb_i-n^ta_i\in P,\ \forall i\in\Lambda\right\}.$$

Now we show that $T_P=\sqrt{N}$. Assume that $(m,n)\in\sqrt{N}$. So there exists some $t\in\mathbb{N}$ such that $(m,n)^t=(m^t,n^t)\in N$. Thus $(m^t,n^t)=\sum_{i=1}^{s}r_i(a_i,b_i)$, for some $r_i\in R$ $(1\le i\le s)$. Hence

$$\left(m^t,n^t\right)=\left(\sum_{i=1}^{s}r_ia_i,\sum_{i=1}^{s}r_ib_i\right).$$

On the other hand, for all $j\in\Lambda$,

$$m^tb_j-n^ta_j=\sum_{i=1}^{s}r_ia_ib_j-\sum_{i=1}^{s}r_ib_ia_j=\sum_{i=1}^{s}r_i\left(a_ib_j-b_ia_j\right)=\sum_{i=1}^{s}r_i\Delta_{ij}.$$

By Lemma 3.22, $\sqrt{\ell}=\sqrt{(N:F)}=P$. As

$$\begin{aligned}\forall j\in\Lambda,\ \Delta_{ij}\in\ell\subseteq\sqrt{\ell}&\Rightarrow\forall j\in\Lambda,\ \Delta_{ij}\in P\\ &\Rightarrow\forall j\in\Lambda,\ \exists t\in\mathbb{N};\ m^tb_j-n^ta_j\in P\\ &\Rightarrow(m,n)\in T_P\\ &\Rightarrow\sqrt{N}\subseteq T_P.\end{aligned}$$

Now let $(r,s)\in T_P$. So

$$\begin{aligned}\exists t\in\mathbb{N};\ r^tb_i-s^ta_i\in P,\ \forall i\in\Lambda&\Rightarrow\exists p\in P;\ s^ta_1-r^tb_1=p\\ &\Rightarrow s^ta_1=r^tb_1+p.\end{aligned}$$

On the other hand we have

$$\begin{aligned}a_1\left(r^t,s^t\right)&=\left(a_1r^t,a_1s^t\right)\\ &=\left(a_1r^t,r^tb_1+p\right)\\ &=r^t\left(a_1,b_1\right)+(0,p)\in N+PF\subseteq\sqrt{N}+\sqrt{N}\subseteq\sqrt{N}\\ &\Rightarrow a_1\left(r^t,s^t\right)\in\sqrt{N}\\ &\Rightarrow\exists n\in\mathbb{N};\ a_1^n\left(r^{tn},s^{tn}\right)\in N\\ &\Rightarrow a_1^n\in\sqrt{(N:F)}=P\text{ or }(r,s)^{tn}\in N.\ (\text{N } \textit{is primary})\end{aligned}$$

If $a_1^n \in P$, then $a_1 \in \sqrt{P} = \sqrt{\sqrt{(N:F)}} = \sqrt{(N:F)} = P$ which is a contradiction. So $(r,s)^{tn} \in N$ and $(r,s) \in \sqrt{N}$. Therefore $T_P \subseteq \sqrt{N}$ and this completes the proof.
(i) Let $(x,y) \in \sqrt{N}$. So

$$\exists n \in \mathbb{N};\ (x,y)^n = (x^n, y^n) \in N$$

$$\Rightarrow (x^n, y^n) = \sum_{i=1}^{t} r_i (a_i, b_i);\ \exists r_i \in R\ (1 \leq i \leq t).$$

On the other hand

$$\begin{aligned} \forall i \in \Lambda\ a_i \in P \Rightarrow x^n = \sum_{i=1}^{t} r_i a_i \in P \\ \Rightarrow x \in \sqrt{P} = P \\ \Rightarrow (x,y) \in P \oplus R \\ \Rightarrow \sqrt{N} \subseteq P \oplus R. \end{aligned}$$

Now let $(x,y) \in P \oplus R$. So $x \in P, y \in R$ and therefore for all $i \in \Lambda$, $xb_i \in P$. Since for all $i \in \Lambda, a_i \in P$ we have for all $i \in \Lambda, ya_i \in P$. Hence for all $i \in \Lambda$ $xb_i - ya_i \in P$. So by part (c), $(x,y) \in \sqrt{N}$ and this completes the proof.
(ii) Let $(x,y) \in \sqrt{N}$. By part (c)

$$\exists n \in \mathbb{N};\ x^n b_j - y^n a_j \in P,\ \forall j \in \Lambda \Rightarrow x^n b_i - y^n a_i \in P$$

On the other hand $x^n b_i \in P$. So $y^n a_i \in P$. As P is a prime ideal of R, we have

$$\begin{aligned} & y^n \in P \text{ or } a_i \in P \Rightarrow y^n \in P \\ & \Rightarrow y \in \sqrt{P} = P \\ & \Rightarrow (x,y) \in R \oplus P \\ & \Rightarrow \sqrt{N} \subseteq R \oplus P. \end{aligned}$$

Now let $(m,n) \subseteq \sqrt{N} \subseteq R \oplus P$. Then $n \in P$ and so for all $r \in \mathbb{N}, n^r \in P$. If $m \in P$, then $\sqrt{N} \subseteq P \oplus P$. On the other hand we have $(a_i, b_i) \in N \subseteq \sqrt{N} \subseteq P \oplus P$, hence $a_i \in P$ which is a contradiction. Therefore $m \notin P$, since P is a prime ideal of R, for all $l \in \mathbb{N}$, $m^l \notin P$. By part (c),

$$\exists t \in \mathbb{N};\ m^t b_j - n^t a_j \in P\ \forall j \in \Lambda.$$

On the other hand $n^t a_j \in P$, for all $j \in \Lambda$. So $m^t b_j \in P$, for all $j \in \Lambda$. Since P is a prime ideal of R, hence $m^t \in P$ or $b_j \in P$, for all $j \in \Lambda$. If $m^t \in P$, this is a contradiction. So $b_j \in P$, for all $j \in \Lambda$ and since there exists $i \in \Lambda$ such that $a_i \notin P$. Therefore by part (i), $\sqrt{N} = R \oplus P$ and this completes the proof. $\square$

Corollary 3.24. *Let F be a free module of rank 2 and $N =< (a_i, b_i) \in F : i \in \Lambda >$ be a P-primary submodule of free R-module F such that $\sqrt{N}$ is a submodule. Then*
(a) If $(1, 0) \in N$, then $\sqrt{N} = R \oplus P$.
(b) If $(0, 1) \in N$, then $\sqrt{N} = P \oplus R$.

Proof. (a) It is clear that $N \neq P \oplus P$. So there exists $i \in \Lambda$ such that $a_i \notin P$. On the other hand $(1, 0) \in N \subseteq \sqrt{N}$. By Theorem 3.23(c), there exists some $n \in \mathbb{N}$ such that $1^n b_j - 0^n a_j \in P$, for all $j \in \Lambda$. So $b_j \in P$ for all $j \in \Lambda$ and since there exits some $i \in \Lambda$ such that $a_i \notin P$ by Theorem 3.23(c)(i), we heve $\sqrt{N} = R \oplus P$ and this completes the proof.
(b) The proof is similar to part (a). □

Lemma 3.25. *If $N = P \oplus P$ is a submodule of $F = R \oplus R$, then $\sqrt{N} = N$.*

Proof. The proof is obvious. □

Theorem 3.26. *Let N be a submodule of F with rank 2 which does not contain $(1, 0)$ and $(0, 1)$ and $\sqrt{N}$ is a submodule. Let $P = \sqrt{(N :_R F)}$ be a maximal ideal of R. Let there exist some $a, b \in R$ such that for all $n \in \mathbb{N}$, $(a^n, b^n) \in N$ and $Ra^n + Rb^n \nsubseteq P$. Then*

$$\sqrt{N} = \{(x, y) \in F : \exists n \in \mathbb{N};\ ay^n - bx^n \in P\}.$$

Proof. As P is maximal, N is a primary submodule of F. If $a \in P = \sqrt{(N : F)}$, then

$$\begin{aligned} \exists m \in \mathbb{N};\ a^m \in (N : F) &\Rightarrow a^m F \subseteq N \\ &\Rightarrow (a^m, 0) = a^m (1, 0) \in N. \end{aligned}$$

On the other hand $(a^m, b^m) \in N$ and therefore $(0, b^m) \in N$. This results that $b^m (0, 1) = (0, b^m) \in N$. Since N is a primary submodule of F and $(0, 1) \notin N$, so $b^m \in \sqrt{(N :_R F)}$. Hence $b \in \sqrt{(N :_R F)} = P$. So for all $n \in \mathbb{N}$, $a^n, b^n \in P$. Hence for all $n \in \mathbb{N}$, $Ra^n + Rb^n \subseteq P$, which is a contradiction. Therefore $a \notin P$ and similarly, $b \notin P$. Since P is a maximal ideal, we have $R = Ra + P$ and $R = Rb + P$. Therefore there exist some $r_1, r_2 \in R, p_1, p_2 \in P$ such that $1 = r_1 a + p_1$ and $1 = r_2 b + p_2$. Put

$$K = \{(x, y) \in F : \exists n \in \mathbb{N};\ ay^n - bx^n \in P\}.$$

We show that $\sqrt{N} = K$. Let $(c,d) \in \sqrt{N}$. So

$$\begin{aligned}\exists n \in \mathbb{N};\ (c^n, d^n) \in N \Rightarrow (ad^n - bc^n, 0) &= (ad^n - bc^n, bd^n - bd^n)\\ &= (ad^n, bd^n) - (bc^n, bd^n)\\ &= d^n\,(a,b) - b\,(c^n, d^n) \in N\\ \Rightarrow (ad^n - bc^n)\,(1,0) &= (ad^n - bc^n, 0) \in N\end{aligned}$$

and since N is a primary submodule and $(1,0) \notin N$, we have $ad^n - bc^n \in \sqrt{(N :_R F)} = P$. Therefore $(c,d) \in K$. This results $\sqrt{N} \subseteq K$. Now let $(c,d) \in K$. There exists some $m \in \mathbb{N}$ such that $ad^m - bc^m \in P$. So there exists $q \in P$ such that $bc^m = ad^m + q$. We have,

$$\begin{aligned}&(c^m, d^m) = (1.c^m, 1.d^m) = ((br_2 + p_2)\,c^m, (ar_1 + p_1)\,d^m))\\ &= (br_2c^m + p_2c^m, ar_1d^m + p_1d^m) = (br_2c^m, ar_1d^m) + (p_2c^m, p_1d^m)\,.\end{aligned}$$

Since $(p_2c^m, p_1d^m) \in P \oplus P \subseteq \sqrt{N}$, it is enough to show that $(br_2c^m, ar_1d^m) \in \sqrt{N}$. As $(a,b) \in N$, we have $r_1\,(a,b) \in N \subseteq \sqrt{N}$. So

$$\begin{aligned}(ar_1, br_1) + (p_1, 0) &= (ar_1 + p_1, br_1)\\ &= (1, br_1)\\ &= (br_2 + p_2, br_1)\\ &= (br_2, br_1) + (p_2, 0) \in \sqrt{N}.\end{aligned}$$

This results $b\,(r_2, r_1) = (br_2, br_1) \in \sqrt{N}$. So there exists $n \in \mathbb{N}$ such that $b^n\,(r_2^n, r_1^n) \in N$ and since N is a primary submodule of F, we have $(r_2^n, r_1^n) \in N$ or there exists $t \in \mathbb{N}$ such that $b^{nt} \in (N : F)$. If there exists $t \in \mathbb{N}$ such that $b^{nt} \in (N :_R F)$, then $b \in P$ which is a contradiction. Therefore $(r_2^n, r_1^n) \in N$ and hence $(r_2, r_1) \in \sqrt{N}$. Now we have

$$\begin{aligned}(r_2\,(bc^m)\,, ar_1d^m) &= (r_2\,(ad^m + q)\,, ar_1d^m)\\ &= (r_2ad^m, ar_1d^m) + (r_2q, 0)\\ &= ad^m\,(r_2, r_1) + (r_2q, 0)\,.\end{aligned}$$

Since $(r_2q, 0) \in P \oplus P \subseteq \sqrt{N}$, $ad^m\,(r_2, r_1) \in \sqrt{N}$, so $(r_2bc^m, ar_1d^m) \in \sqrt{N}$ and this results $(c^m, d^m) \in \sqrt{N}$. Therefore $(c,d) \in \sqrt{N}$. Hence $K \subseteq \sqrt{N}$ and this completes the proof. □

4. **Quasi torsion-free modules**

In this section, we define the notion "quasi torsion-free of a free module"and find some characterizations of it. Also by this notion, we find some other characterizations for quasi-radical of a subset of a free module. For an R-module M, $Z_R^*(M) = \{r \in R : \exists m \in M - \sqrt{0};\ rm = 0\}$.

Definition 4.1. A free R-module F is called quasi torsion-free, if for $r \in R, m \in M$, $rm = 0$ implies that r or m is nilpotent.

It is clear that all torsion-free modules are quasi torsion-free; but a quasi torsion-free module is not necessarily torsion-free.

Example 4.2. Consider $R = \mathbb{Z}_4$ and $M = \mathbb{Z}_4 \oplus \mathbb{Z}_4$. Then M is a quasi torsion-free; but is not a torsion-free R-module.

In the rest of this paper, we assume that N is a q-submodule of F such that $\dfrac{F}{N}$ is a free $\dfrac{R}{q}$-module.

Theorem 4.3. *Let N be a proper q-submodule of a free R-module $F = R \oplus R$. If $\dfrac{F}{N}$ as $\dfrac{R}{q}$-module is quasi torsion-free, then $P = \sqrt{(N :_R F)}$ is a prime ideal of R.*

Proof. Since $N \neq F$, so $(1, 0) \notin N$ or $(0, 1) \notin N$. Without loss of generality let $(1, 0) \notin N$. Let for $a, b \in R$, $ab \in P$. We show that $a \in P$ or $b \in P$. Since $ab \in P = \sqrt{(N : F)} = \sqrt{q}$, there exists $n \in \mathbb{N}$ such that $a^n b^n \in (N : F) = q$. Hence $a^n b^n F \subseteq N$. So $a^n (b^n, 0) = (a^n b^n, 0) = a^n b^n (1, 0) \in N$ and $(a^n + q)((b^n, 0) + N) = N$. As $\dfrac{F}{N}$ is quasi torsion-free $a^n + q$ is nilpotent or $(b^n, 0) + N$ is nilpotent. If $a^n + q$ is nilpotent, there exists some $m \in \mathbb{N}$ such that $a^{nm} + q = q$. Hence $a^{nm} \in q$ and so $a \in \sqrt{q} = P$. If $(b^n, 0) + N$ is nilpotent there exists some $t \in \mathbb{N}$ such that $(b^{nt}, 0) + N = N$ and so $(b^{nt} + q)((1, 0) + N) = N$. As $\dfrac{F}{N}$ is quasi torsion-free $b^{nt} + q$ is nilpotent or $(1, 0) \in N$. Since $(1, 0) \notin N$, so $b^{nt} + q$ is nilpotent. Hence there exists some $k \in \mathbb{N}$ such that $b^{nkt} + q = q$. Therefore $b^{nkt} \in q$, which results $b \in P$ and this completes the proof. □

Proposition 4.4. *Let N be a primary submodule of $F = R \oplus R$ such that $\sqrt{N}$ is a submodule and $(1, 0) \in N$ or $(0, 1) \in N$. Let $q = (N :_R F)$ and $P = \sqrt{q}$. Then $\dfrac{F}{N}$ as $\dfrac{R}{q}$-module is quasi torsion-free if and only if $\sqrt{N} = R \oplus P$ or $\sqrt{N} = P \oplus R$.*

Proof. Let $\dfrac{F}{N}$ be a quasi torsion-free $\dfrac{R}{q}$-module. By Corollary 3.24 we have: If $(1, 0) \in N$, then $\sqrt{N} = R \oplus P$ and if $(0, 1) \in N$, then $\sqrt{N} = P \oplus R$. Now let $\sqrt{N} = R \oplus P$ and for $(x, y) + N \in \dfrac{F}{N}$ and $r + q \in \dfrac{R}{q}$, $(r + q)((x, y) + N) = 0_{\frac{F}{N}} = N$. So $r(x, y) + N = N$

and $(rx, ry) \in N \subseteq \sqrt{N} = R \oplus P$. Therefore $rx \in R$ and $ry \in P = \sqrt{(N :_R F)}$. Since P is a prime ideal of R, we have $r \in P$ or $y \in P$. If $r \in P$, then there exists $n \in \mathbb{N}$ such that $r^n \in q$ which results $(r+q)^n = r^n + q = q = 0_{\frac{R}{q}}$. Therefore $r+q$ is nilpotent. If $y \in P$, then $(x, y) \in \sqrt{N}$ which results there exists some $t \in \mathbb{N}$ such that $(x^t, y^t) \in N$. Therefore $((x, y) + N)^t = (x^t, y^t) + N = N = 0_{\frac{F}{N}}$. Hence $(x, y) + N$ is nilpotent and this completes the proof. □

Theorem 4.5. *Let N be a P-submodule of a free R-module F and $\sqrt{N} \subsetneqq F$. Then the following conditions are equivalent.*

(a) *For any $r \in R$ and $m \in F$, $rm \in N$ implies $r \in \sqrt{P}$ or $m \in \sqrt{N}$;*

(b) *$\frac{R}{P}$-module $\frac{F}{N}$ is quasi torsion free;*

(c) *For any $r \in R$ such that $r \notin \sqrt{P}$, $\sqrt{N} = \{m \in F : \exists t \in \mathbb{N},\ rm^t \in N\}$;*

(d) *For any ideal J of R such that $J \not\subseteq \sqrt{P}$,*

$$\sqrt{N} = \{m \in F : \exists t \in \mathbb{N},\ Jm^t \subseteq N\};$$

(e) *For any $m \in F \backslash \sqrt{N}$, $\sqrt{P} = \sqrt{(N :_R Rm)}$;*

(f) *For any submodule L of F such that $L \not\subseteq \sqrt{N}$, $\sqrt{P} = \sqrt{(N :_R L)}$;*

(g) *For any $m \in F \backslash \sqrt{N}$, $\sqrt{P} = \sqrt{Ann_R(m + N)}$;*

(h) $\sqrt{P} = \sqrt{Z_R^*\left(\frac{F}{N}\right)}$.

Proof. $(a) \Rightarrow (b)$ Let for $r + P \in \frac{R}{P}$ and $m + N \in \frac{F}{N}$, $rm \in N$. By part (a), $r \in \sqrt{P}$ or $m \in \sqrt{N}$. If $r \in \sqrt{P}$, then

$$\exists t_1 \in \mathbb{N};\ r^{t_1} \in P \Rightarrow (r+P)^{t_1} = r^{t_1} + P = P.$$

Therefore $r + P$ is idempotent. If $m \in \sqrt{N}$, then

$$\exists t_2 \in \mathbb{N};\ m^{t_2} \in N \Rightarrow (m+N)^{t_2} = m^{t_2} + N = N.$$

Therefore $m + N$ is idempotent. This results $\frac{R}{P}$-module $\frac{F}{N}$ is quasi torsion free.

$(b) \Rightarrow (c)$ Let $r \in R$ such that $r \notin \sqrt{P}$ and put

$$L = \{m \in F : \exists t \in \mathbb{N},\ rm^t \in N\}$$

. Now we show that $L = \sqrt{N}$. It is clear that $\sqrt{N} \subseteq L$. Now let $m \in L$. Hence

$$\exists n \in \mathbb{N};\ rm^n \in N \Rightarrow (r+P)(m^n + N) = N.$$

By part (b), $r+P$ or m^n+N is nilpotent. If $r+P$ is nilpotent, then

$$\exists t \in \mathbb{N};\ r^t+P=(r+P)^t=P \Rightarrow r^t \in P \Rightarrow r \in \sqrt{P},$$

which is a contradiction. Therefore m^n+N is nilpotent. Hence

$$\exists t' \in N;\ (m^n+N)^{t'}=N \Rightarrow m^{nt'}+N=N \Rightarrow m^{nt'} \in N \Rightarrow m \in \sqrt{N}.$$

Therefore $L \subseteq \sqrt{N}$. This results $L=\sqrt{N}$.

$(c) \Rightarrow (d)$ Let J be an ideal of R such that $J \nsubseteq \sqrt{P}$. So there exists some $r \in J \backslash \sqrt{P}$. By part (c), $\sqrt{N}=\{m \in F : \exists t \in \mathbb{N},\ rm^t \in N\}$. Now put $L=\{m \in F : \exists t \in \mathbb{N},\ Jm^t \subseteq N\}$. We show that $L=\sqrt{N}$. Let $x \in \sqrt{N}$. So

$$\exists t \in \mathbb{N};\ x^t \in N \Rightarrow Jx^t \subseteq N \Rightarrow x \in L.$$

Now let $m \in L$. Therefore

$$\exists n \in \mathbb{N};\ Jm^n \subseteq N \Rightarrow rm^n \in N \Rightarrow m \in \sqrt{N}.$$

Hence $L=\sqrt{N}$.

$(d) \Rightarrow (e)$ Let $m \in F\backslash\sqrt{N}$. So we have $P=(N :_R F) \subseteq (N :_R Rm)$. Hence $\sqrt{P} \subseteq \sqrt{(N :_R Rm)}$. Now let $r \in \sqrt{(N :_R Rm)}$. Therefore

$$\exists n \in \mathbb{N};\ r^n \in (N :_R Rm) \Rightarrow r^n(Rm) \subseteq N \Rightarrow r^n m \in N.$$

If $r \notin \sqrt{P}$, then $r^n \notin \sqrt{P}$. Now put $J=Rr^n$. Hense $J \nsubseteq \sqrt{P}$. By part (d), $\sqrt{N}=\{m \in F : \exists t \in \mathbb{N},\ Jm^t \subseteq N\}$. On the other hand we have

$$r^n(Rm)=(r^nR)\,m \subseteq N \Rightarrow Jm \subseteq N \Rightarrow m \in \sqrt{N},$$

which is a contradiction. So $r \in \sqrt{P}$. Therefore $\sqrt{(N :_R Rm)} \subseteq \sqrt{P}$. Hence $\sqrt{P}=\sqrt{(N :_R Rm)}$.

$(e) \Rightarrow (f)$ Let L be a submodule of F such that $L \nsubseteq \sqrt{N}$. So there exists some $m \in L\backslash\sqrt{N}$. By part (e), $\sqrt{P}=\sqrt{(N :_R Rm)}$. We have

$$P=(N :_R F) \subseteq (N :_R L) \Rightarrow \sqrt{P} \subseteq \sqrt{(N :_R L)}.$$

Now let $r \in \sqrt{(N :_R L)}$. So

$$\begin{aligned}\exists n \in \mathbb{N};\ r^n \in (N :_R L) \Rightarrow r^nL \subseteq N &\Rightarrow r^n m \in N\\ &\Rightarrow r^n \in (N :_R Rm)\\ &\Rightarrow r \in \sqrt{(N :_R Rm)}=\sqrt{P}.\end{aligned}$$

Therefore $\sqrt{P}=\sqrt{(N :_R L)}$.

$(f) \Rightarrow (g)$ Let $m \in F\backslash\sqrt{N}$. Put $L=N+Rm$. So $L \nsubseteq \sqrt{N}$. Now by part (f),

$$\sqrt{P}=\sqrt{(N :_R L)}=\sqrt{(N :_R N+Rm)}=\sqrt{(N :_R Rm)}.$$

On the other hand

$$Ann_R\left(\frac{F}{N}\right) = (N :_R F) \Rightarrow \sqrt{Ann_R\left(\frac{F}{N}\right)} = \sqrt{(N :_R F)} = \sqrt{P}.$$

Thus

$$Ann_R\left(\frac{F}{N}\right) \subseteq Ann_R\left(m+N\right) \Rightarrow \sqrt{P} \subseteq \sqrt{Ann_R\left(m+N\right)}.$$

Now let $r \in Ann_R\left(m+N\right)$. So

$$\begin{aligned} rm + N = r\left(m+N\right) = N &\Rightarrow rm \in N \\ &\Rightarrow r \in \left(N :_R Rm\right). \end{aligned}$$

So $Ann_R\left(m+N\right) \subseteq (N :_R Rm)$. Thus $\sqrt{Ann_R\left(m+N\right)} \subseteq \sqrt{(N :_R Rm)}$ $= \sqrt{P}$. Therefore $\sqrt{P} = \sqrt{Ann_R\left(m+N\right)}$.

$(g) \Rightarrow (h)$ Let $m \in F\backslash\sqrt{N}$. By part (g), $\sqrt{P} = \sqrt{Ann_R\left(m+N\right)}$. Let $t \in \sqrt{P}$. So

$$\begin{aligned} t \in \sqrt{Ann_R\left(m+N\right)} &\Rightarrow \exists n \in \mathbb{N};\ t^n\left(m+N\right) = N \\ &\Rightarrow t^n \in Z_R^*\left(\frac{F}{N}\right) \\ &\Rightarrow t \in \sqrt{Z_R^*\left(\frac{F}{N}\right)}. \end{aligned}$$

Now let $r \in \sqrt{Z_R^*\left(\frac{F}{N}\right)}$. Therefore

$$\begin{aligned} \exists t \in \mathbb{N};\ r^t \in Z_R^*\left(\frac{F}{N}\right) &\Rightarrow \exists m \in F\backslash\sqrt{N};\ r^t\left(m+N\right) = N \\ &\Rightarrow r^t \in Ann_R\left(m+N\right) \\ &\Rightarrow r \in \sqrt{Ann_R\left(m+N\right)} = \sqrt{P}\ \textit{(by part (g))}. \end{aligned}$$

Hence $\sqrt{P} = \sqrt{Z_R^*\left(\frac{F}{N}\right)}$.

$(h) \Rightarrow (a)$ Let $r \in R$ and $m \in F$ such that $rm \in N$. So $rm + N = r\left(m+N\right) = N$. Now let $m \notin \sqrt{N}$. Therefore $r \in Z_R^*\left(\frac{F}{N}\right)$. By part (h), $\sqrt{P} = \sqrt{Z_R^*\left(\frac{F}{N}\right)}$. Hence $r \in \sqrt{P}$. □

Theorem 4.6. *Let F be a finitely generated R-module, S be a multiplicative closed subset of R and N be a prime submodule of F such that $S \cap (N :_R F) = \emptyset$. Then* $\sqrt{(N :_R F)_S} = \left(\sqrt{N} :_R F\right)_S$.

Proof. Since F is finitely generated, $(N :_R F)_S = (N_S :_{R_S} F_S)$. On the other hand by Theorem 3.5(l) and Theorem 3.15,

$$\sqrt{(N :_R F)_S} = \sqrt{(N_S :_{R_S} F_S)} = \left(\sqrt{N_S} :_{R_S} F_S\right) = \left(\left(\sqrt{N}\right)_S :_{R_S} F_S\right).$$

We show that $\left(\left(\sqrt{N}\right)_S :_{R_S} F_S\right) = \left(\sqrt{N} :_R F\right)_S$. Let $\frac{r}{t} \in \left(\left(\sqrt{N}\right)_S :_{R_S} F_S\right)$ and $\{e_i\}_{i=1}^n$ be a standard base for F. So

$$\begin{aligned}
\frac{r}{t}F_S \subseteq \left(\sqrt{N}\right)_S &\Rightarrow \forall i, \frac{r}{t}\cdot\frac{e_i}{1} \in \left(\sqrt{N}\right)_S \\
&\Rightarrow \forall i,\ \exists a \in \sqrt{N}, b \in S;\ \frac{re_i}{t} = \frac{a}{b} \\
&\Rightarrow \forall i,\ \exists u \in S;\ u\left(re_i b - ta\right) = 0 \\
&\Rightarrow \forall i,\ ure_i b = uta \in \sqrt{N} \\
&\Rightarrow \forall i,\ \exists n \in \mathbb{N};\ u^n r^n e_i^n b^n = (ure_i b)^n \in N \\
&\Rightarrow \forall i,\ r^n e_i^n \in N\ (N \text{ is prime and } S \cap (N :_R F) = \emptyset) \\
&\Rightarrow \forall i,\ re_i \in \sqrt{N} \\
&\Rightarrow rF \subseteq \sqrt{N}.
\end{aligned}$$

Therefore $\left(\left(\sqrt{N}\right)_S :_{R_S} F_S\right) \subseteq \left(\sqrt{N} :_R F\right)_S$. Now let $\frac{a}{b} \in \left(\sqrt{N} :_R F\right)_S$, for some $a \in R$ and $b \in S$. So

$$\begin{aligned}
\exists r \in \left(\sqrt{N} :_R F\right),\ t \in S;\ \frac{a}{b} = \frac{r}{t} &\Rightarrow \exists q \in S; q\left(at - br\right) = 0 \\
&\Rightarrow qat = qbr \in \left(\sqrt{N} :_R F\right) \\
&\Rightarrow (qat)\,F \subseteq \sqrt{N} \\
&\Rightarrow \forall i,\ qate_i \in \sqrt{N} \\
&\Rightarrow \forall i,\ \exists n \in \mathbb{N};\ q^n a^n e_i^n t^n \in N \\
&\Rightarrow \forall i,\ a^n e_i^n \in N \\
&\Rightarrow \forall i,\ ae_i \in \sqrt{N} \\
&\Rightarrow \forall i,\ \frac{a}{b}\cdot\frac{e_i}{1} = \frac{ae_i}{b\cdot 1} \in \left(\sqrt{N}\right)_S \\
&\Rightarrow \frac{a}{b}F_S \subseteq \left(\sqrt{N}\right)_S \\
&\Rightarrow \frac{a}{b} \in \left(\left(\sqrt{N}\right)_S :_{R_S} F_S\right).
\end{aligned}$$

Therefore $\left(\sqrt{N} :_R F\right)_S \subseteq \left(\left(\sqrt{N}\right)_S :_{R_S} F_S\right)$. Hence $\left(\left(\sqrt{N}\right)_S :_{R_S} F_S\right) = \left(\sqrt{N} :_R F\right)_S$. □

Theorem 4.7. *Let F be a free R-module and N be a primary submodule of F such that $\sqrt{N}$ is a submodule of F. Then $\sqrt{N}$ is a prime submodule.*

Proof. Let for $r \in R$ and $m \in F$, $rm \in \sqrt{N}$. So

$$\exists t \in N; r^t m^t = (rm)^t \in N \Rightarrow r^t \in \sqrt{(N :_R F)} \text{ or } m^t \in N.$$

If $m^t \in N$, then $m \in \sqrt{N}$, and if $r^t \in \sqrt{(N :_R F)}$, by Lemma 3.5(h),

$$r \in \sqrt{(N :_R F)} \subseteq \left(\sqrt{N} :_R F\right) \Rightarrow r \in \left(\sqrt{N} :_R F\right).$$

Therefore $\sqrt{N}$ is a prime submodule of F. □

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