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$*$-α-DERIVATION ON PRIME $*$-RINGS

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Abstract. Let $\Re$ be an associative ring with involution $*$. An additive map $\lambda \to \lambda^*$ of $\Re$ into itself is called an involution if the following conditions are satisfied $(i)(\lambda\mu)^* = \mu^*\lambda^*$, $(ii)(\lambda^*)^* = \lambda$ for all $\lambda, \mu \in \Re$. A ring equipped with an involution is called an $*$-ring or ring with involution. The aim of the present paper is to establish some results on $*$-α-derivations in $*$-rings and investigate the commutativity of prime $*$-rings admitting $*$-α-derivations on $\Re$ satisfying certain identities also prove that if $\Re$ admits a reverse $*$-α-derivation δ of $\Re$, then $\alpha \in Z(\Re)$ and some related results have also been discussed.

1. Preliminaries

Last few decades, several authors have investigated the relationship between the commutativity of the ring $\Re$ and certain specific types of derivations of $\Re$. The first result in this direction is due to Posner [3] who proved that if a ring $\Re$ admits a nonzero derivation δ such that $[\delta(\lambda), \lambda] \in Z(\Re)$ for all $\lambda \in \Re$, then $\Re$ is commutative. This result was subsequently, refined and extended by a number of authors. In [9], Bresar and Vukman showed that a prime ring must be commutative if $\Re$ admits a nonzero left derivation. Furthermore, Bresar and Vukman [9] studied the notions of a $*$-derivation and a Jordan $*$-derivations of $\Re$. In [2], Asma et al. generalized some identities on additive maps with $*$-rings. Recently, many authors have obtained commutativity theorems for prime and semiprime rings admitting derivation, generalized

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derivation. The aim of the present paper is to establish some results involving $*$-α-derivations in $*$-rings and investigate the commutativity of prime $*$-rings admitting $*$-α-derivations of $\Re$ satisfying certain identities and some related results have also been discussed.

2. Introduction

Throughout the paper $\Re$ will denote the associative ring and $Z(\Re)$ be the centre of $\Re$. For any $\lambda, \mu \in \Re$, a ring $\Re$ is said to be a prime if whenever $\lambda\Re\mu = \{0\}$ implies $\lambda = 0$ or $\mu = 0$ and is semiprime if for any $\lambda \in \Re$, $\lambda\Re\lambda = \{0\}$ implies $\lambda = 0$. For all $\lambda, \mu \in \Re$, as usual commutator, we shall write $[\lambda, \mu] = \lambda\mu - \mu\lambda$ and $\lambda \circ \mu = \lambda\mu + \mu\lambda$. Also, we shall frequently use the following identities and several well known fact about the prime ring without specific mention.

$$[\lambda\mu, \nu] = \lambda[\mu, \nu] + [\lambda, \nu]\mu$$

$$[\lambda, \mu\nu] = \mu[\lambda, \nu] + [\lambda, \mu]\nu$$

$$\lambda \circ \mu\nu = (\lambda \circ \mu)\nu - \mu[\lambda, \nu] = \mu(\lambda \circ \nu) + [\lambda, \mu]\nu$$

$$(\lambda\mu) \circ \nu = \lambda(\mu \circ \nu) - [\lambda, \nu]\mu = (\lambda \circ \nu)\mu + \lambda[\mu, \nu]$$

By derivation δ on $\Re$, we mean an additive mapping on $\Re$ satisfying $\delta(\lambda\mu) = \delta(\lambda)\mu + \lambda\delta(\mu)$ for all $\lambda, \mu \in \Re$. An additive map $\lambda \to \lambda^*$ of $\Re$ into itself is called an involution if the following conditions are satisfied $(i)(\lambda\mu)^* = \mu^*\lambda^*$, $(ii)(\lambda^*)^* = \lambda$ for all $\lambda, \mu \in \Re$. A ring equipped with an involution is called an $*$-ring or ring with involution. Let $\Re$ be a $*$-ring. An additive mapping δ on $\Re$ is said to be $*$-derivation if $\delta(\lambda\mu) = \delta(\lambda)\mu^* + \lambda\delta(\mu)$ for all $\lambda, \mu \in \Re$. An additive mapping δ on R is said to be reverse $*$-derivation if $\delta(\lambda\mu) = \delta(\mu)\lambda^* + \mu\delta(\lambda)$ for all $\lambda, \mu \in \Re$. An additive mapping δ on R is said to be $*$-α-derivation if $\delta(\lambda\mu) = \delta(\lambda)\mu^* + \alpha(\lambda)\delta(\mu)$ for all $\lambda, \mu \in \Re$. An additive mapping δ on $\Re$ is said to be reverse $*$-α-derivation if $\delta(\lambda\mu) = \delta(\mu)\lambda^* + \alpha(\mu)\delta(\lambda)$ for all $\lambda, \mu \in \Re$. An additive mapping $\digamma$ on $\Re$ is said to be a generalized derivation if there exists a derivation δ on $\Re$ such that $\digamma(\lambda\mu) = \digamma(\lambda)\mu + \lambda\delta(\mu)$ for all $\lambda, \mu \in \Re$. An additive mapping $\digamma$ on $\Re$ is said to be a generalized $*$-derivation if there exists a $*$-derivation δ on $\Re$ such that $\digamma(\lambda\mu) = \digamma(\lambda)\mu^* + \lambda\delta(\mu)$ for all $\lambda, \mu \in \Re$.

3. Main results

Theorem 3.1. *Let $\Re$ be a prime $*$-ring. If $\Re$ admits a nonzero $*$-α-derivation δ of $\Re$ such that $\delta(\lambda\mu) = \delta(\lambda)\delta(\mu)$ for all $\lambda, \mu \in \Re$, then $\alpha = \delta$.*

Proof. By hypothesis, we have

$$\delta(\lambda\mu) = \delta(\lambda)\mu^* + \lambda\delta(\mu) = \delta(\lambda)\delta(\mu) \text{ for all } \lambda, \mu \in \Re. \tag{3.1}$$

Replacing λ by $\lambda\nu$ in (3.1), we get

$$\delta(\lambda)\delta(\nu)\mu^* + \alpha(\lambda\nu)\delta(\mu) = \delta(\lambda)\delta(\nu)\delta(\mu) \tag{3.2}$$

$$= \delta(\lambda)\delta(\nu\mu) = \delta(\lambda)(\delta(\nu)\mu^* + \alpha(\nu)\delta(\mu)). \tag{3.3}$$

This implies that

$$\alpha(\lambda\nu)\delta(\mu) = \delta(\lambda)\alpha(\nu)\delta(\mu) \text{ for all } \lambda, \mu, \nu \in \Re. \tag{3.4}$$

Since α is an automorphism on $\Re$, we obtain

$$(\alpha(\lambda) - \delta(\lambda))\alpha(\nu)\delta(\mu) = 0 \text{ for all } \lambda, \mu, \nu \in \Re.$$

This gives that

$$(\lambda - \alpha^{-1}(\delta(\lambda))\Re\alpha^{-1}(\delta(\mu)) = \{0\} \text{ for all } \lambda, \mu \in \Re. \tag{3.5}$$

Since $\Re$ is prime, implies that either $(\lambda - \alpha^{-1}(\delta(\lambda)) = 0$ or $\alpha^{-1}(\delta(\mu)) = 0$. In the second case, $\delta = 0$ but in our hypothesis $\delta \neq 0$. In first case $\delta = \alpha$.

□

Theorem 3.2. *Let $\Re$ be a prime $*$-ring. If $\Re$ admits a nonzero $*$-α-derivation δ of $\Re$ such that $\delta(\lambda\mu) = \delta(\mu)\delta(\lambda)$ for all $\lambda, \mu \in \Re$, then $\delta(\lambda) = \lambda^*$.*

Proof. Suppose that

$$\delta(\lambda\mu) = \delta(\lambda)\mu^* + \alpha(\lambda)\delta(\mu) = \delta(\mu)\delta(\lambda) \text{ for all } \lambda, \mu \in \Re. \tag{3.6}$$

Replacing $\mu\lambda$ instead of μ in (3.6), we get

$$\delta(\lambda\mu\lambda) = \delta(\lambda)\mu^*\lambda^* + \alpha(\lambda)\delta(\lambda\mu) \text{ for all } \lambda, \mu \in \Re.$$

This implies that

$$\delta(\lambda\mu\lambda) = \delta(\lambda)\mu^*\lambda^* + \alpha(\lambda)\delta(\lambda)\delta(\mu) \text{ for all } \lambda, \mu \in \Re. \tag{3.7}$$

Again, we apply hypothesis

$$\delta(\lambda\mu\lambda) = \delta(\lambda\mu)\delta(\lambda) \text{ for all } \lambda, \mu \in \Re.$$

This implies that

$$\delta(\lambda\mu\lambda) = \delta(\lambda)\mu^*\delta(\lambda) + \alpha(\lambda)\delta(\mu)\delta(\lambda) \text{ for all } \lambda, \mu \in \Re. \tag{3.8}$$

Using equation (3.7) and (3.8), we get

$$\delta(\lambda)\mu^*\lambda^* = \delta(\lambda)\mu^*\delta(\lambda) \text{ for all } \lambda, \mu \in \Re.$$

This gives that

$$\delta(\lambda)\Re(\lambda^* - \delta(\lambda)) = \{0\} \text{ for all } \lambda, \mu \in \Re.$$

Using primeness of $\Re$, we find that either $\delta = 0$ or $\delta(\lambda) = \lambda^*$. In our hypothesis $\delta \neq 0$ this implies that $\delta(\lambda) = \lambda^*$.

□

Theorem 3.3. *Let $\Re$ be a prime $*$-ring and $a \in \Re$. If $\Re$ admits an $*$-α-derivation δ of $\Re$ and $[\delta(\lambda), a] = 0$, then either $\delta(a) = 0$ or $a \in Z(\Re)$.*

Proof. We have

$$[\delta(\lambda), a] = 0 \text{ for all } \lambda \in \Re. \tag{3.9}$$

Replacing $\lambda\mu$ instead of λ in (3.9), we find

$$\delta(\lambda)[\mu^*, a] + [\alpha(\lambda), a]\delta(\mu) = 0 \text{ for all } \lambda, \mu \in \Re. \tag{3.10}$$

Since α is an automorphism, we put $\alpha(\lambda) = \lambda$, we find that

$$\delta(\lambda)[\mu^*, a] + [\lambda, a]\delta(\mu) = 0 \text{ for all } \lambda, \mu \in \Re. \tag{3.11}$$

Substituting λ by a, we get

$$\delta(a)[\mu^*, a] = 0 \text{ for all } a, \mu \in \Re.$$

Using hypothesis, we find that $\delta(a)[\mu^*, a] = 0$ for all $\mu \in \Re$. Substituting μ^* in place of μ in that relation, we have $\delta(a)[\mu, a] = 0$ for all $\mu \in \Re$. Again, replacing $\nu\lambda$ instead of μ in the last relation, we obtain

$$\delta(a)\mu[\lambda, a] = 0 \text{ for all } \lambda, \mu \in \Re. \tag{3.12}$$

This implies that $\delta(a)\Re[\lambda, a] = \{0\}$ for all $\lambda \in R$. Since R is prime, we have $\delta(a) = 0$ or $a \in Z(R)$.

□

Theorem 3.4. *Let $\Re$ be a semiprime $*$-ring. If $\Re$ admits a reverse $*$-α-derivation δ of $\Re$, then $\alpha \in Z(\Re)$.*

Proof. By hypothesis, we have

$$\delta(\lambda\mu) = \delta(\mu)\lambda^* + \alpha(\mu)\delta(\lambda) \text{ for all } \lambda, \mu \in \Re. \tag{3.13}$$

Replacing $\lambda\mu$ instead of λ in (3.13), we find

$$\delta((\lambda\nu)\mu) = \delta(\mu)\nu^*\lambda^* + \alpha(\mu)\delta(\lambda\nu) \text{ for all } \lambda, \mu, \nu \in \Re.$$

Simplifying the above relation, we find that

$$\delta((\lambda\nu)\mu) = \delta(\mu)\nu^*\lambda^* + \alpha(\mu)\delta(\nu)\lambda^* + \alpha(\mu)\alpha(\nu)\delta(\lambda). \tag{3.14}$$

On the other hand, we have

$$\delta(\lambda(\nu\mu)) = \delta(\nu\mu)\lambda^* + \alpha(\nu\mu)\delta(\lambda) \text{ for all } \lambda, \mu, \nu \in \Re.$$

Simplifying the above relation, we find that

$$\delta(\lambda(\nu\mu) = \delta(\mu)\nu^*\lambda^* + \alpha(\mu)\delta(\nu)\lambda^* + \alpha(\nu)\alpha(\mu)\delta(\lambda). \tag{3.15}$$

Comparing (3.14) and (3.15), we get $[\alpha(\mu), \alpha(\nu)]\delta(\lambda) = 0$. Since α is an automorphism, so we can put $\alpha(\mu) = \mu$ in the last relation, we obtain

$$[\mu, \alpha(\nu)]\delta(\lambda) = 0 \text{ for all } \lambda, \mu, \nu \in \Re. \tag{3.16}$$

Now replacing μ by $\mu\iota$ in (3.16) and using (3.16), we get

$$[\mu, \alpha(\nu)]\iota\delta(\lambda) = 0 \text{ for all } \iota, \lambda, \mu, \nu \in \Re. \tag{3.17}$$

Multiplying the right side of (3.17) by $\alpha(\nu)\mu$, we have

$$[\mu, \alpha(\nu)]\iota\delta(\lambda)\alpha(\nu)\mu = 0 \text{ for all } \iota, \lambda, \mu, \nu \in \Re. \tag{3.18}$$

Multiplying the right side of (3.17) by $\mu\alpha(\nu)$, we have

$$[\mu, \alpha(\nu)]\iota\delta(\lambda)\mu\alpha(\nu) = 0 \text{ for all } \iota\lambda, \mu, \nu \in \Re. \tag{3.19}$$

Subtracting (3.19) from (3.18), we have $[\mu, \alpha(\nu)]\iota\delta(\lambda)[\mu, \alpha(\nu)] = 0$ for all $\iota, \lambda, \mu, \nu \in \Re$. This implies that $[\mu, \alpha(\nu)]\Re[\mu, \alpha(\nu)] = \{0\}$ for all $\mu, \nu \in \Re$. Since $\Re$ is semiprime, we have $[\mu, \alpha(\nu)] = 0$ for all $\mu, \nu \in \Re$ i.e., $\alpha \in Z(\Re)$.

□

Theorem 3.5. *Let $\Re$ be a prime $*$-ring. If $\Re$ admits a nonzero $*$-α-derivation δ of $\Re$ such that $\delta([\lambda, \mu]) = 0$ for all $\lambda, \mu \in \Re$, then $\Re$ is commutative.*

Proof. By hypothesis, we have

$$\delta([\lambda, \mu]) = 0 \text{ for all } \lambda, \mu \in \Re. \tag{3.20}$$

Replacing $\lambda\mu$ instead of λ in (3.20), we have

$$\delta([\lambda, \mu]\mu) = \delta([\lambda, \mu])\mu^* + \alpha([\lambda, \mu])\delta(\mu) \text{ for all } \lambda, \mu \in \Re.$$

By the above relation, we have

$$\alpha([\lambda, \mu])\delta(\mu) = 0 \text{ for all } \lambda, \mu \in \Re. \tag{3.21}$$

Replacing $\lambda\iota$ instead of λ in (3.21) and using (3.21), where $\iota \in \Re$, we get

$$\alpha([\lambda, \mu])\alpha(\iota)\delta(\mu) = 0 \text{ for all } \lambda, \mu, \iota \in \Re. \tag{3.22}$$

Since α is an automorphism on $\Re$, then we have

$$[\lambda, \mu]\iota\alpha^{-1}(\delta(\mu)) = 0 \text{ for all } \lambda, \mu, \iota \in \Re. \tag{3.23}$$

This implies that

$$[\lambda, \mu]\Re\alpha^{-1}(\delta(\mu)) = \{0\} \text{ for all } \lambda, \mu \in \Re. \tag{3.24}$$

Using primeness of $\Re$, we obtain either $[\lambda, \mu] = 0$ or $\alpha^{-1}(\delta(\mu)) = 0$ for all $\lambda, \mu \in \Re$. Let $K = \{\mu \in \Re | \alpha^{-1}(\delta(\mu)) = 0\}$ and $L = \{\mu \in \Re | [\lambda, \mu] = 0, \text{ for all } \lambda \in \Re\}$. Then K and L are both additive subgroups and $K \cup L = \Re$, but $(\Re, +)$ is not union of two its proper subgroups, which implies that either $K = \Re$ or $L = \Re$. In the former case, we have $\alpha^{-1}(\delta(\mu)) = 0$ i.e., $\delta(\mu) = 0$, which is a contradiction and in the second case, $\Re$ is commutative.

□

Theorem 3.6. *Let $\Re$ be a prime $*$-ring. If $\Re$ admits a nonzero $*$-α-derivation δ of $\Re$ such that $\delta(\lambda \circ \mu) = 0$ for all $\lambda, \mu \in \Re$, then $\Re$ is commutative.*

Proof. By hypothesis, we have

$$\delta(\lambda \circ \mu) = 0 \text{ for all } \lambda, \mu \in \Re. \tag{3.25}$$

Replacing $\lambda\mu$ instead of λ in (3.25), we have

$$\delta((\lambda \circ \mu)\mu) = \delta(\lambda \circ \mu)\mu^* + \alpha(\lambda \circ \mu)\delta(\mu) \text{ for all } \lambda, \mu \in \Re.$$

By the above relation, we have

$$\alpha(\lambda \circ \mu)\delta(\mu) = 0 \text{ for all } \lambda, \mu \in R. \tag{3.26}$$

Replacing $\lambda\iota$ instead of λ in (3.26) and using (3.26), where $\iota \in \Re$, we get

$$\alpha([\lambda, \mu])\alpha(\iota)\delta(\mu) = 0 \text{ for all } \lambda, \mu, \iota \in \Re. \tag{3.27}$$

Since α is an automorphism on $\Re$, then we have

$$[\lambda, \mu]\iota\alpha^{-1}(\delta(\mu)) = 0 \text{ for all } \lambda, \mu, \iota \in \Re. \tag{3.28}$$

This implies that

$$[\lambda, \mu]\Re\alpha^{-1}(\delta(\mu)) = \{0\} \text{ for all } \lambda, \mu \in \Re. \tag{3.29}$$

Using primeness of $\Re$, we obtain either $[\lambda, \mu] = 0$ or $\alpha^{-1}(\delta(\mu)) = 0$ for all $\lambda, \mu \in \Re$. Let $K = \{\mu \in \Re | \alpha^{-1}(\delta(\mu)) = 0\}$ and $L = \{\mu \in \Re | [\lambda, \mu] = 0 \text{ for all } \lambda \in \Re\}$. Then K and L are both additive subgroups and $K \cup L = \Re$, but $(\Re, +)$ is not union of two its proper subgroups, which implies that either $K = \Re$ or $L = \Re$. In the former case, we have $\alpha^{-1}(\delta(\mu)) = 0$ i.e., $\delta(\mu) = 0$, which is a contradiction and in the second case, $\Re$ is commutative.

□

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