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MAPPINGS BETWEEN THE LATTICES OF VARIETIES OF SUBMODULES

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ABSTRACT. Let R be a commutative ring with identity and M be an R-module. It is shown that the usual lattice $\mathcal{V}({}_RM)$ of varieties of submodules of M is a distributive lattice. If M is a semisimple R-module and the unary operation $'$ on $\mathcal{V}({}_RM)$ is defined by $(V(N))' = V(\tilde{N})$, where $M = N \oplus \tilde{N}$, then the lattice $\mathcal{V}({}_RM)$ with $'$ forms a Boolean algebra. In this paper, we examine the properties of certain mappings between $\mathcal{V}({}_RR)$ and $\mathcal{V}({}_RM)$, in particular considering when these mappings are lattice homomorphisms. It is shown that if M is a faithful primeful R-module, then $\mathcal{V}({}_RR)$ and $\mathcal{V}({}_RM)$ are isomorphic lattices, and therefore $\mathcal{V}({}_RM)$ and the lattice $\mathcal{R}(R)$ of radical ideals of R are anti-isomorphic lattices. Moreover, if R is a semisimple ring, then $\mathcal{V}({}_RR)$ and $\mathcal{V}({}_RM)$ are isomorphic Boolean algebras, and therefore $\mathcal{V}({}_RM)$ and $\mathcal{L}(R)$ are anti-isomorphic Boolean algebras.

1. Introduction

Throughout this paper all rings are commutative with identity and all modules are unitary. Let R be a ring, M be an R-module and $\mathcal{L}({}_RM)$ be the lattice of submodules of M with the following operations:

$$L \vee N = L + N \text{ and } L \wedge N = L \cap N,$$

for all submodules L and N of M. In case $M = R$, we write $\mathcal{L}(R)$ instead of $\mathcal{L}({}_RR)$ for convenience. Recently, the relationship between

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lattices $\mathcal{L}({}_RR)$ and $\mathcal{L}({}_RM)$ has been considered by examining the properties of a number of mappings between them (see [16, 17, 18]). For example, there are two mappings $\lambda\colon \mathcal{L}(R) \to \mathcal{L}({}_RM)$ defined by $\lambda(I) = IM$ and $\mu\colon \mathcal{L}({}_RM) \to \mathcal{L}(R)$ defined by $\mu(N) = (N : M)$, where $(N : M) = \{r \in R \mid rM \subseteq N\}$. In [16], it has been investigated conditions under which these mappings are homomorphisms or isomorphisms. An R-module M is called a λ-*module* (resp. μ-*module*) if λ (resp. μ) is a lattice homomorphism. A number of properties of these modules can also be found in [14].

A proper submodule P of an R-module M is called a prime submodule if for $r \in R$, $x \in M$, $rx \in P$ implies that $r \in (P : M)$ or $x \in P$ [8]. The set of all prime submodules of M is called the *spectrum* of M and is denoted by $\mathrm{Spec}(M)$. An R-module M is called *primeful* if $M = (0)$ or $M \neq (0)$ and for each prime ideal p of R containing $(0 : M)$ there exists a prime submodule P of M such that $(P : M) = p$ [9].

For a proper submodule N of an R-module M, the intersection of all prime submodules of M containing N is called the *radical* of N and denoted by $\mathrm{rad}\, N$; if there are no such prime submodules, $\mathrm{rad}\, N$ is M (see, for example, [13]). A submodule N of M is called a *radical submodule* if $\mathrm{rad}\, N = N$. For an ideal I of R, the radical of I is denoted by $\sqrt{I}$ and has the characterization $\sqrt{I} = \{x \in R \mid x^n \in I$ for some $n > 0\}$.

The collection of all radical submodules of M, denoted $\mathcal{R}({}_RM)$, is a lattice with the following operations:

$$L \vee N = \mathrm{rad}(L + N) \text{ and } L \wedge N = L \cap N,$$

for all radical submodules L and N of M [12]. For convenience, we write $\mathcal{R}(R)$ for $\mathcal{R}({}_RR)$. It is seen that the mapping $\rho\colon \mathcal{R}(R) \to \mathcal{R}({}_RM)$ defined by $\rho(I) = \mathrm{rad}(\lambda(I)) = \mathrm{rad}(IM)$ is a lattice homomorphism, but $\sigma\colon \mathcal{R}({}_RM) \to \mathcal{R}(R)$ defined by $\sigma(N) = \mu(N) = (N : M)$ is not. In [12], conditions under which M is a σ-module have been studied, i.e., σ is a lattice homomorphism.

For a submodule N of an R-module M, the *variety* of N is

$$V(N) = \{P \in \mathrm{Spec}(M) \mid (P : M) \supseteq (N : M)\}.$$

It is evident that $V((0)) = \mathrm{Spec}(M)$ and $V(M) = \emptyset$. The collection of all varieties of submodules of M forms a lattice which we shall denote by $\mathcal{V}({}_RM)$ with respect to the following operations:

$$V(L) \vee V(N) = V(L \cap N) \text{ and } V(L) \wedge V(N) = V(L + N),$$

for all submodules L and N of M. In particular, we shall denote the lattice $\mathcal{V}({}_RR)$ by $\mathcal{V}(R)$.

In this paper, we consider the mappings $\nu\colon \mathcal{V}(R) \to \mathcal{V}({}_RM)$ defined by

$$\nu(V(I)) = V(IM),$$

for every ideal I of R, and $\omega\colon \mathcal{V}({}_RM) \to \mathcal{V}(R)$ defined by

$$\omega(V(N)) = V((N : M)),$$

for every submodule N of M. It is shown that ν is always a lattice epimorphism (Theorem 2.2), but ω is not necessarily a lattice homomorphism (Example 2.3). We say that an R-module M is an *ω-module* if ω is a lattice homomorphism. It is shown that every cyclic module is an ω-module (Lemma 2.6). Also a vector space $\mathbb{V}$ over a field F is an ω-module if and only if $\dim_F \mathbb{V} \leq 1$ (Lemma 2.7). It is proved that a finitely generated R-module M is an ω-module if and only if M is a multiplication module (Theorem 2.8). It is shown that if M is a faithful primeful R-module, then $\mathcal{V}(R)$ and $\mathcal{V}({}_RM)$ are isomorphic lattices (Theorem 2.16), and $\mathcal{V}({}_RM)$ and $\mathcal{R}(R)$ are anti-isomorphic lattices (Theorem 2.18).

Note that $\mathcal{L}({}_RM)$ is not a distributive lattice in general, even if $M = R$. However, It is proved that $\mathcal{V}({}_RM)$ is a distributive lattice for any R-module M (Theorem 3.1). It is shown that if M is a semisimple R-module and $(V(N))'$ is defined to be the variety of direct sum complement of N, then $\mathcal{V}({}_RM)$ together with $'$ is a Boolean algebra (Theorem 3.4). Furthermore, if R is a semisimple ring, then $\mathcal{V}({}_RM)$ is a finite Boolean algebra, and therefore its cardinal number is 2^n for some positive integer n (Corollary 3.5). It is also proved that if R is a semisimple ring, then $\mathcal{L}(R)$ is a Boolean algebra. Moreover, if M is a faithful primeful R-module, then $\mathcal{L}(R)$ and $\mathcal{V}({}_RM)$ are anti-isomorphic Boolean algebras (Proposition 3.8).

2. Mappings between $\mathcal{V}(R)$, $\mathcal{V}({}_RM)$, and $\mathcal{R}({}_RM)$

Let R be a ring and let M be an R-module. Recall that the collection of varieties $\mathcal{V}({}_RM)$ of submodules of M is a lattice via the following operations:

$$V(L) \vee V(N) = V(L \cap N) \text{ and } V(L) \wedge V(N) = V(L + N),$$

where $\vee$ and $\wedge$ are respectively supremum and infimum of $\{V(L), V(N)\}$ with respect to inclusion.

The following lemma collects some facts about varieties of submodules which will be used in the sequel.

Lemma 2.1. *Let R be a ring and let M be an R-module. Then*

(1) $V(IM) = V(\sqrt{I}M)$ *for every ideal I of R.*

(2) $V((I\cap J)M)=V(IM\cap JM)=V(IM)\cup V(JM)$ *for all ideals* I *and* J *of* R.
(3) $V(IM+JM)=V(IM)\cap V(JM)$ *for all ideals* I *and* J *of* R.
(4) $V(N)=V((N:M)M)$ *for every submodule* N *of* M.

Proof. (1) Let I be any ideal of R. Then $IM\subseteq\sqrt{I}M$ and hence $V(\sqrt{I}M)\subseteq V(IM)$. For the reverse inclusion, let $P\in V(IM)$. Then we have

$$I\subseteq(IM:M)\subseteq(P:M).$$

Thus $\sqrt{I}\subseteq(P:M)$ and hence

$$(\sqrt{I}M:M)\subseteq((P:M)M:M)\subseteq(P:M).$$

It follows that $P\in V(\sqrt{I}M)$ and hence $V(IM)\subseteq V(\sqrt{I}M)$.
(2) Let I and J be two ideals of R. It is easily seen that

$$V(IM)\cup V(JM)\subseteq V(IM\cap JM)\subseteq V((I\cap J)M).$$

For the reverse inclusion, let $P\in V((I\cap J)M)$. Then

$$I\cap J\subseteq((I\cap J)M:M)\subseteq(P:M).$$

Now since $(P:M)$ is a prime ideal of R, we have $I\subseteq(P:M)$ or $J\subseteq(P:M)$. Therefore $(IM:M)\subseteq(P:M)$ or $(JM:M)\subseteq(P:M)$. It follows that $P\in V(IM)$ or $P\in V(JM)$ and hence $P\in V(IM)\cup V(JM)$, as required.
(3) Clearly $V(IM+JM)\subseteq V(IM)\cap V(JM)$. For the reverse inclusion, let $P\in V(IM)\cap V(JM)$. Then $I\subseteq(IM:M)\subseteq(P:M)$ and $J\subseteq(JM:M)\subseteq(P:M)$. Thus $(I+J)M\subseteq(P:M)M\subseteq P$. Therefore $((I+J)M:M)\subseteq(P:M)$ and hence $P\in V(IM+JM)$.
(4) Let N be any submodule of M. It is clear that $V(N)\subseteq V((N:M)M)$. For the reverse inclusion, let $P\in V((N:M)M)$. Then we have $(N:M)\subseteq((N:M)M:M)\subseteq(P:M)$ and hence $P\in V(N)$. □

Theorem 2.2. *Let* R *be a ring and* M *be an* R*-module. Then* $\nu\colon\mathcal{V}(R)\to\mathcal{V}(_RM)$ *defined by* $\nu(V(I))=V(IM)$ *is a lattice epimorphism. In particular,* $\mathcal{V}(_RM)$ *is isomorphic to a quotient of* $\mathcal{V}(R)$.

Proof. First we show that ν is well-defined. For this, let $V(I)=V(J)$ for ideals I and J of R and assume that $P\in V(IM)$. It follows that $I\subseteq(IM:M)\subseteq(P:M)$. Then $(P:M)\in V(I)$, and so $(P:M)\in V(J)$. Hence we have $(JM:M)\subseteq((P:M)M:M)\subseteq(P:M)$ which shows that $P\in V(JM)$. Therefore $V(IM)\subseteq V(JM)$. Similarly $V(JM)\subseteq V(IM)$, so that $V(IM)=V(JM)$.

Now, let I and J be two ideals of R. Then by Lemma 2.1(2), we have

$$\begin{aligned}\nu(V(I) \vee V(J)) &= \nu(V(I \cap J)) = V((I \cap J)M) = V(IM \cap JM)\\ &= V(IM) \vee V(JM) = \nu(V(I)) \vee \nu(V(J)).\end{aligned}$$

Also,

$$\begin{aligned}\nu(V(I) \wedge V(J)) &= \nu(V(I + J)) = V((I + J)M) = V(IM + JM)\\ &= V(IM) \wedge V(JM) = \nu(V(I)) \wedge \nu(V(J)).\end{aligned}$$

Thus ν is a lattice homomorphism.
Moreover, by Lemma 2.1(4), $\nu(V((N : M))) = V((N : M)M) = V(N)$ for any submodule N of M. Thus ν is an epimorphism.

Let $\sim$ be the relation on $\mathcal{V}(R)$ which is defined by $V(I){\sim}V(J)$ if and only if $V(IM) = V(JM)$. It is easily seen that $\sim$ is an equivalence relation on $\mathcal{V}(R)$. Furthermore, it is a congruence relation on $\mathcal{V}(R)$. For this, let $V(I_1){\sim}V(J_1)$ and $V(I_2){\sim}V(J_2)$, i.e., $V(I_1M) = V(J_1M)$ and $V(I_2M) = V(J_2M)$. Thus by Lemma 2.1(2),

$$\begin{aligned}V((I_1 \cap I_2)M) &= V(I_1M) \cup V(I_2M)\\ &= V(J_1M) \cup V(J_2M)\\ &= V((J_1 \cap J_2)M)\end{aligned}$$

which shows that $V(I_1) \vee V(I_2) \sim V(J_1) \vee V(J_2)$. Also, by Lemma 2.1(3),

$$\begin{aligned}V((I_1 + I_2)M) &= V(I_1M) \cap V(I_2M)\\ &= V(J_1M) \cap V(J_2M)\\ &= V((J_1 + J_2)M)\end{aligned}$$

which shows that $V(I_1) \wedge V(I_2) \sim V(J_1) \wedge V(J_2)$. Hence $\mathcal{V}(R)/{\sim}$, the set of all equivalence classes with respect to $\sim$, is a lattice with the following operations $\tilde{\vee}$ and $\tilde{\wedge}$:

$$V(I)/{\sim}\ \tilde{\vee}\ V(J)/{\sim} = (V(I) \vee V(J))/{\sim}$$

and

$$V(I)/{\sim}\ \tilde{\wedge}\ V(J)/{\sim} = (V(I) \wedge V(J))/{\sim}$$

for all $V(I)/{\sim}, V(J)/{\sim} \in \mathcal{V}(R)/{\sim}$. Now, it is easy to check that the mapping $\overline{\nu}\colon \mathcal{V}(R)/{\sim} \to \mathcal{V}({}_RM)$ defined by $\overline{\nu}(V(I)/{\sim}) = V(IM)$ is a lattice isomorphism. □

The following example shows that ω is not well-defined in general.

Example 2.3. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}(p^\infty)$. As mentioned in [11], M has no prime submodule. Thus $V((0)) = V(M) = \emptyset$. Hence

$\omega\colon \mathcal{V}({}_RM) \to \mathcal{V}(R)$ defined by $\omega(V(N)) = V((N : M))$ is not well-defined, since

$$\omega(V(0)) = V((0 : M)) = V(0) = \mathrm{Spec}(\mathbb{Z})$$

but

$$\omega(V(M)) = V((M : M)) = V(R) = \emptyset.$$

Recall that a non-zero module M is a primeful module if for every $p \in V((0 : M))$, there exists $P \in \mathrm{Spec}(M)$ such that $(P : M) = p$. Now we show that ω is a well defined mapping for primeful modules as a class of modules which contains all finitely generated modules and all projective modules over integral domains (see [9, Proposition 3.8 and Corollary 4.3]).

Theorem 2.4. *Let R be a ring and let M be a primeful R-module. Then ω is well-defined. Furthermore $\nu\omega = 1$ and therefore ω is an injection.*

Proof. Assume that $V(L) = V(N)$ for two submodules L and N of M. Let $p \in V((L : M))$. Since M is a primeful R-module, there exists $P \in \mathrm{Spec}(M)$ such that $(P : M) = p$. Thus $P \in V(L)$ and hence $P \in V(N)$. Therefore $p = (P : M) \supseteq (N : M)$. It follows that $V((L : M)) \subseteq V((N : M))$. Similarly, $V((N : M)) \subseteq V((L : M))$. Hence we have

$$\omega(V(L)) = V((L : M)) = V((N : M)) = \omega(V(N))$$

and thus ω is well-defined. Also, by Lemma 2.1(4),

$$\nu\omega(V(N)) = \nu(V((N : M))) = V((N : M)M) = V(N).$$

Therefore $\nu\omega = 1$ and hence ω is an injection. $\square$

Corollary 2.5. *Let R be a ring and M be a primeful R-module. Then ω is a monomorphism if and only if*

$$V(((L + N) : M)) = V((L : M) + (N : M)),$$

for all submodules L and N of M.

Proof. $\Rightarrow$) Follows from $\omega(V(L) \wedge V(N)) = \omega(V(L)) \wedge \omega(V(N))$.
$\Leftarrow$) Let L and N be any submodules of M. Then

$$\begin{aligned}\omega(V(L) \vee V(N)) &= \omega(V(L \cap N)) = V(((L \cap N) : M)) \\ &= V((L : M) \cap (N : M)) \\ &= V((L : M)) \vee V((N : M)) \\ &= \omega(V(L)) \vee \omega(V(N)).\end{aligned}$$

Next note that

$$\omega(V(L) \wedge V(N)) = \omega(V(L+N)) = V(((L+N) : M)),$$

and

$$\begin{aligned}\omega(V(L)) \wedge \omega(V(N)) &= V((L : M)) \wedge V((N : M)) \\ &= V((L : M) + (N : M)).\end{aligned}$$

The result follows. □

Recall that an R-module M is called an *ω-module* if ω is a lattice homomorphism. For example, the zero module is clearly an ω-module but there are many non-trivial examples as we show next.

Lemma 2.6. *Every cyclic module is an ω-module.*

Proof. By [16, Corollary 3.7], every cyclic module is a μ-module. Thus the assertion follows from [16, Lemma 3.1] and Corollary 2.5. □

The following lemma presents vector spaces with dimension greater than one as examples of primeful modules which are not ω-modules. We note that every proper subspace of any vector space $\mathbb{V}$ is a prime submodule of $\mathbb{V}$.

Lemma 2.7. *Let $\mathbb{V}$ be a vector space over a field F. Then $\mathbb{V}$ is an ω-module if and only if $\dim_F \mathbb{V} \leq 1$.*

Proof. First assume that $\mathbb{V}$ is an ω-module with $\dim_F \mathbb{V} > 1$. Let $\mathbb{W}_1$ and $\mathbb{W}_2$ be two non-zero subspaces of $\mathbb{V}$ such that $\mathbb{V} = \mathbb{W}_1 \oplus \mathbb{W}_2$, then we have

$$V(((\mathbb{W}_1 + \mathbb{W}_2) : \mathbb{V})) = V((\mathbb{V} : \mathbb{V})) = V(F) = \emptyset$$

and

$$V((\mathbb{W}_1 : \mathbb{V}) + (\mathbb{W}_2 : \mathbb{V})) = V((0)) = \{(0)\}.$$

Thus by Corollary 2.5, $\mathbb{V}$ is not an ω-module.
Conversely if $\dim_F \mathbb{V} \leq 1$, then by Lemma 2.6, $\mathbb{V}$ is an ω-module. □

An R-module M is called a *multiplication* module if for every submodule N of M there exists an ideal I of R such that $N = IM$ (see, for example, [4]). In this case, we can take $I = (N : M)$. The following theorem shows that multiplication modules and ω-modules coincide for finitely generated modules.

Theorem 2.8. *Let R be a ring and M be a finitely generated R-module. Then the following statements are equivalent:*

(1) *M is an ω-module.*
(2) *M is a σ-module.*

(3) *M is a μ-module.*
(4) *M is a multiplication module.*
(5) *$M/\mathfrak{m}M$ is a cyclic R-module for every maximal ideal $\mathfrak{m}$ of R.*

Proof. (1)⇒(2) Let L and N be any submodules of M. Then

$$V(((L+N):M)) = V((L:M)+(N:M))$$

implies that $\sqrt{(L+N):M} = \sqrt{(L:M)+(N:M)}$. Since M is finitely generated, by [10, Theorem 4.4],

$$(\mathrm{rad}(L+N):M) = \sqrt{(L+N):M} = \sqrt{(L:M)+(N:M)}.$$

Now, by [12, Lemma 2.2], M is a σ-module.
(2)⇔(3)⇔(4) Follows from [12, Theorem 2.11].
(3)⇒(1) Follows from Theorem 2.4, Corollary 2.5 and [16, Lemma 3.1].
(4)⇔(5) By [4, Corollary 1.5]. □

Corollary 2.9. *Every homomorphic image of a finitely generated ω-module is an ω-module.*

Proof. Follows from [16, Proposition 3.6] and Theorem 2.8 □

The following result shows that being ω-module is a local property for finitely generated modules.

Corollary 2.10. *Let R be a ring and M be a finitely generated R-module. Then the following statements are equivalent:*

(1) *M is an ω-module.*
(2) *M_p is an ω-module for all prime ideals p of R.*
(3) *$M_\mathfrak{m}$ is an ω-module for all maximal ideals $\mathfrak{m}$ of R.*

Proof. By Theorem 2.8 and [12, Theorem 2.19]. □

An R-module M is called a *distributive module* if $\mathcal{L}({}_RM)$ is a distributive lattice. As shown in the next theorem, distributive modules coincide with modules whose 2-generated submodules are ω modules. An R-module M is called a *chain module* if the set of submodules of M is linearly ordered by inclusion. It is clear that every chain module is distributive and they coincide over local rings. In this case, these modules also coincide with *Bezout modules* which are modules that their finitely generated submodules are cyclic [1, Proposition 1.3].

Theorem 2.11. *Let R be a ring and M be an R-module. Then the following statements* (1)-(6) *are equivalent. In particular, if R is a local ring, then all of the following statements are equivalent:*

(1) *Every finitely generated submodule of M is an ω-module.*
(2) *Every 2-generated submodule of M is an ω-module.*

(3) *Every finitely generated submodule of M is a multiplication module.*
(4) $R = (Rx : Ry) + (Ry : Rx)$ *for all* $x, y \in M$.
(5) $R = (L : N) + (N : L)$ *for all finitely generated submodules* L, N *of* M.
(6) M *is a distributive module.*
(7) M *is a chain module.*
(8) *The set of cyclic submodules of M is linearly ordered by inclusion.*
(9) M *is a Bezout module.*
(10) *Every 2-generated submodule of M is cyclic.*

Proof. (1) $\Rightarrow$ (2) Clear.
(2) $\Rightarrow$ (4) Let $M = Rx + Ry$ be a 2-generated ω-module. Then by Corollary 2.5,

$$\begin{aligned}\emptyset = V(R) &= V((Rx + Ry : Rx + Ry)) \\ &= V((Rx : (Rx + Ry)) + (Ry : (Rx + Ry))) \\ &= V((Rx : Ry) + (Ry : Rx)).\end{aligned}$$

Thus $(Rx : Ry) + (Ry : Rx) = R$.
(4) $\Rightarrow$ (1) By [12, Corollary 2.14], every finitely generated submodule of M is a μ-module. Thus (1) follows from Theorem 2.8.
(3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) Follows from Theorem 2.8 and [16, Corollary 3.9].
(4) $\Leftrightarrow$ (6) Follows from [19, Theorem 1.6].
Now, if R is a local ring, then by [1, Proposition 1.3] and [12, Corollary 2.17], all of the statements (1)-(10) are equivalent. □

Now, we examine the conditions under which ν and ω are lattice isomorphisms.

Lemma 2.12. *Let ω be a well-defined mapping. Then*
(1) $\nu\omega\nu = \nu$.
(2) $\omega\nu\omega = \omega$.

Proof. (1) Let I be an ideal of R. Then by Lemma 2.1(4),

$$\begin{aligned}\nu\omega\nu(V(I)) &= \nu\omega(V(IM)) = \nu(V((IM : M))) \\ &= V((IM : M)M) = V(IM) = \nu(V(I)).\end{aligned}$$

Thus $\nu\omega\nu = \nu$.
(2) By Theorem 2.4, $\nu\omega = 1$ and hence $\omega\nu\omega = \omega$. □

The following result is a consequence of Lemma 2.12.

Theorem 2.13. *Let R be a ring and let ω be a well-defined mapping. Then the following statements are equivalent:*

(1) ν *is an injection.*
(2) $\omega\nu = 1$.
(3) $V(I) = V((IM : M))$ *for every ideal* I *of* R.
(4) ω *is a surjection.*

Proof. (1) $\Rightarrow$ (2) Since $\nu\omega\nu = \nu$, we have $\nu\omega\nu(V(I)) = \nu(V(I))$ for all $V(I) \in \mathcal{V}(R)$. Since ν is injective, we get $\omega\nu(V(I)) = V(I)$ for all $V(I) \in \mathcal{V}(R)$. Thus $\omega\nu = 1$.
(2) $\Rightarrow$ (1), (4) Clear.
(2) $\Leftrightarrow$ (3) Clear.
(4) $\Rightarrow$ (2) Let $V(I) \in \mathcal{V}(R)$. Since ω is a surjection, there exists $V(N) \in \mathcal{V}({}_RM)$ such that $\omega(V(N)) = V(I)$. Thus

$$\omega\nu(V(I)) = \omega\nu(\omega(V(N)) = \omega\nu\omega(V(N)) = \omega(V(N)) = V(I).$$

□

Corollary 2.14. *Let* R *be a ring and* M *be a primeful* R*-module. Then* ν *is a bijection if and only if* ω *is a bijection. In this case* ν *and* ω *are inverses of each other, and therefore* $\mathcal{V}(R)$ *and* $\mathcal{V}({}_RM)$ *are isomorphic lattices.*

Proof. By Theorem 2.2, ν is a surjection and by Theorem 2.4, ω is an injection. Thus by Theorem 2.13, ν is a bijection if and only if ω is a bijection. In this case, ν and ω are inverses of each other and by Theorem 2.2, ν and ω are lattice isomorphisms. □

Lemma 2.15. (cf. [9, Proposition 3. 1]) *Let* M *be a non-zero primeful* R*-module and* I *a radical ideal of* R*. Then* $(IM : M) = I$ *if and only if* $(0 : M) \subseteq I$.

Theorem 2.16. *Let* R *be a ring and let* M *be a faithful primeful* R*-module. Then the mappings* ν *and* ω *are lattice isomorphisms. In particular,* ν *and* ω *are inverses of each other.*

Proof. Let I be an ideal of R. Then by Lemma 2.15,

$$V(I) = V(\sqrt{I}) = V((\sqrt{I}M : M)) \subseteq V((IM : M)).$$

Now, let p be a prime ideal of R containing $(IM : M)$. Thus $I \subseteq p$ and hence $\sqrt{I} \subseteq p$. It follows that $(\sqrt{I}M : M) \subseteq (pM : M)$. Using again Lemma 2.15, $(\sqrt{I}M : M) \subseteq p$. Thus $V((IM : M)) \subseteq V((\sqrt{I}M : M))$ and hence $V(I) = V((IM : M))$. Therefore by Theorem 2.13, ν is an injection and hence by Theorem 2.2, ν is an isomorphism. Then by Corollary 2.14, ω is the inverse of ν and hence ω is an isomorphism. □

Theorem 2.17. *Let* R *be a ring and let* M *be a faithful multiplication* R*-module. Then the following statements are equivalent:*

(1) *ω is a lattice isomorphism.*
(2) *ν is a lattice isomorphism.*
(3) *M is a finitely generated R-module.*
(4) *M is a primeful R-module.*
Moreover if R is a domain, then the statements $(1)-(4)$ *are also equivalent to:*
(5) *M is a projective R-module.*

Proof. $(1) \Rightarrow (2)$ By Corollary 2.14.
$(2) \Rightarrow (3)$ By [9, Proposition 3.8], it is enough to show that $(pM : M) = p$ for all prime ideals p of R. Assume on the contrary that $(pM : M) \neq p$ for some prime ideal p of R. Hence $p \notin V((pM : M))$ and thus $V(p) \neq V((pM : M))$. Since ν is an injective, $V(pM) \neq V((pM : M)M)$ which contradicts Lemma 2.1(4).
$(3) \Rightarrow (4)$ By [9, Proposition 3.8].
$(4) \Rightarrow (1)$ By Theorem 2.16.
Now if R is a domain, then
$(3) \Rightarrow (5)$ By [15, Theorem 11].
$(5) \Rightarrow (4)$ By [9, Corollary 4.3] □

A mapping φ from a lattice L to a lattice L' is said to be an *anti-homomorphism*, provided

$$\varphi(a \vee b) = \varphi(a) \wedge \varphi(b) \text{ and } \varphi(a \wedge b) = \varphi(a) \vee \varphi(b)$$

for all $a, b \in L$.

As usual, a bijective (resp. injective, surjective) anti-homomorphism is called an *anti-isomorphism* (resp. *anti-monomorphism*, *anti-epimorphism*). Recall that $\mathcal{R}({}_RM)$ is the lattice of radical submodules of an R-module M. We end this section by providing conditions under which $\mathcal{V}({}_RM)$ and $\mathcal{R}({}_RM)$ are anti-isomorphic lattices.

Theorem 2.18. *Let R be a ring and M be a faithful primeful R-module, then $\mathcal{V}({}_RM)$ and $\mathcal{R}(R)$ are anti-isomorphic lattices.*

Proof. First, we show that the mapping $\varphi \colon \mathcal{R}(R) \to \mathcal{V}(R)$ defined by $\varphi(I) = V(I)$ is an anti-isomorphism. Note that φ is a surjection, since $\varphi(\sqrt{I}) = V(\sqrt{I}) = V(I)$. It is easily seen that φ is an injection. Moreover, $\varphi(I \wedge J) = V(I \wedge J) = V(I \cap J) = V(I) \vee V(J) = \varphi(I) \vee \varphi(J)$. Now, by [18, Lemma 1.1], φ is an anti-isomorphism. Hence by Theorem 2.16, $\nu\varphi \colon \mathcal{R}(R) \to \mathcal{V}({}_RM)$ given by $\nu\varphi(I) = V(IM)$ is a lattice anti-isomorphism. □

Corollary 2.19. *Let R be a ring and let M be a faithful primeful R-module. Then*

(1) *The assignment* $V(N) \mapsto \mathrm{rad}((N:M)M)$ *is an anti-monomorphism from* $\mathcal{V}({}_RM)$ *to* $\mathcal{R}({}_RM)$.
(2) *The assignment* $N \mapsto V(N)$ *is an anti-epimorphism from* $\mathcal{R}({}_RM)$ *to* $\mathcal{V}({}_RM)$.

Proof. (1) By [12, Corollary 3.6], $\rho\colon \mathcal{R}(R) \to \mathcal{R}({}_RM)$ defined by $\rho(I) = \mathrm{rad}(IM)$ is a lattice monomorphism and by Theorm 2.16, $\omega\colon \mathcal{V}({}_RM) \to \mathcal{V}(R)$ defined by $\omega(V(N)) = V((N:M))$ is a lattice isomorphism. Also, let φ be as in the proof of Theorem 2.18. Then by Lemma 2.1, the mapping $\rho\varphi^{-1}\omega\colon \mathcal{V}({}_RM) \to \mathcal{R}({}_RM)$ which assigns $V(N)$ to $\mathrm{rad}((N:M)M)$ is a lattice anti-monomorphism.
(2) By [12, Corollary 3.6], $\sigma\colon \mathcal{R}({}_RM) \to \mathcal{R}(R)$ defined by $\sigma(N) = (N:M)$ is a lattice epimorphism and by Theorm 2.16, $\nu\colon \mathcal{V}(R) \to \mathcal{V}({}_RM)$ defined by $\nu(I) = V(IM)$ is a lattice isomorphism. Now by Lemma 2.1(4), the mapping $\nu\varphi\sigma\colon \mathcal{R}({}_RM) \to \mathcal{V}({}_RM)$ which assigns N to $V(N)$ is a lattice anti-epimorphism where φ is the anti-isomorphism given in the proof of Theorem 2.18. □

Corollary 2.20. *Let R be a ring and let M be a faithful primeful multiplication R-module. Then $\mathcal{V}({}_RM)$ and $\mathcal{R}({}_RM)$ are anti-isomorphic lattices.*

Proof. By the proof of (6) $\Rightarrow$ (1) in [12, Theorem 3.8], the mapping $\rho\colon \mathcal{R}(R) \to \mathcal{R}({}_RM)$ defined by $\rho(I) = \mathrm{rad}(IM)$ is a lattice isomorphism. Thus by Theorem 2.18, $\mathcal{V}({}_RM)$ and $\mathcal{R}({}_RM)$ are anti-isomorphic lattices. □

Remark 2.21. Let $\mathbb{V}$ be a vector space over a field F. Then, since every proper subspace of $\mathbb{V}$ is prime, $\mathcal{R}({}_F\mathbb{V})$ is the set of all subspaces of $\mathbb{V}$. In particular, for any subspace $\mathbb{W}$ of $\mathbb{V}$, $V(\mathbb{W})$ is the set of all proper subspaces of $\mathbb{V}$, and hence $\mathcal{V}({}_F\mathbb{V}) = \{\emptyset, \mathcal{L}({}_F\mathbb{V}) \setminus \{\mathbb{V}\}\}$. Therefore the lattices $\mathcal{R}({}_F\mathbb{V})$ and $\mathcal{V}({}_F\mathbb{V})$ are isomorphic if and only if $\dim_F \mathbb{V} \le 1$.

3. $\mathcal{V}({}_RM)$ as a Boolean algebra

In this section, we find conditions for an R-module M for which $\mathcal{V}({}_RM)$ is a Boolean algebra, and in particular, ν and ω are Boolean algebra homomorphisms.

Although $\mathcal{L}({}_RM)$ is not necessarily distributive, the following theorem shows that $\mathcal{V}({}_RM)$ is a distributive lattice for any R-module M.

Theorem 3.1. *Let R be a ring and let M be an R-module. Then $\mathcal{V}({}_RM)$ is a distributive lattice.*

Proof. Let I, J and H be any ideals of R. Since $(I \cap J) + (I \cap H) \subseteq I \cap (J + H)$, we have $V(I \cap (J + H)) \subseteq V((I \cap J) + (I \cap H))$. For the reverse inclusion, let p be a prime ideal of R containing $(I\cap J)+(I\cap H)$. If $p \not\supseteq I\cap(J+H)$, then there exist $i \in I, j \in J$ and $h \in H$ such that $i = j+h$ and $i \notin p$. Therefore $i^2 \notin p$ and hence $ij+ih \notin p$, a contradiction. Thus $p \supseteq I\cap(J+H)$ and so $V(I\cap J+I\cap H) \subseteq V(I\cap(J+H))$. Hence $\mathcal{V}(R)$ is distributive. Now, since $\mathcal{V}(_RM)$ is isomorphic to a quotient of $\mathcal{V}(R)$ by Theorem 2.2, $\mathcal{V}(_RM)$ is a distributive lattice. □

The following example shows that $\mathcal{R}(_RM)$ is not in general distributive.

Example 3.2. Let $\mathbb{V} = F \oplus F$ be a vector space over a field F and let $\mathbb{W}_1 = F \oplus 0$, $\mathbb{W}_2 = 0 \oplus F$ and $\mathbb{W}_3 = \{(x, x) \mid x \in F\}$. Then

$$\begin{aligned}(\mathbb{W}_1 \vee \mathbb{W}_2) \wedge \mathbb{W}_3 &= (\mathrm{rad}(\mathbb{W}_1 + \mathbb{W}_2)) \cap \mathbb{W}_3 = (\mathrm{rad}\,\mathbb{V}) \cap \mathbb{W}_3 \\ &= \mathbb{V} \cap \mathbb{W}_3 = \mathbb{W}_3\end{aligned}$$

and

$$\begin{aligned}(\mathbb{W}_1 \wedge \mathbb{W}_3) \vee (\mathbb{W}_2 \wedge \mathbb{W}_3) &= \mathrm{rad}((\mathbb{W}_1 \cap \mathbb{W}_3) + (\mathbb{W}_2 \cap \mathbb{W}_3)) \\ &= \mathrm{rad}((0, 0)) = ((0, 0)).\end{aligned}$$

Hence $\mathcal{R}(_F\mathbb{V})$ is not a distributive lattice.

Corollary 3.3. *Let R be a ring and let M be a faithful primeful multiplication R-module. Then $\mathcal{R}(_RM)$ is a distributive lattice.*

Proof. By Corollary 2.20 and Theorem 3.1. □

We recall that a distributive lattice $(L, \vee, \wedge)$ is a Boolean algebra if there is a unary operation $'$ on L and two constants $\mathbf{0}$ and $\mathbf{1}$ such that $x \wedge x' = \mathbf{0}$ and $x \vee x' = \mathbf{1}$ for all $x \in L$.

Let M be a semisimple R-module and N be a submodule of M. Then, by definition, there is a submodule $\tilde{N}$ of M such that $M = N\oplus\tilde{N}$. We define the unary operation $'$ on $\mathcal{V}(_RM)$ by $(V(N))' = V(\tilde{N})$.

Theorem 3.4. *Let M be a semisimple R-module and $'$ be as above. Then $\mathcal{V}(_RM)$ is a Boolean algebra with $\mathbf{0} = V(M)$ and $\mathbf{1} = V((0))$. In particular, if R is a semisimple ring, then $\mathcal{V}(R)$ is a Boolean algebra with $\mathbf{0} = V(R)$ and $\mathbf{1} = V((0))$.*

Proof. By Theorem 3.1, $\mathcal{V}(_RM)$ is distributive. Let $V(N) \in \mathcal{V}(_RM)$. Since M is semisimple, there is a submodule $\tilde{N}$ of M such that $M = N \oplus \tilde{N}$. Hence we have $V(N)\wedge V(N)' = V(N)\wedge V(\tilde{N}) = V(N+\tilde{N}) = V(M) = \mathbf{0}$ and $V(N)\vee V(N)' = V(N)\vee V(\tilde{N}) = V(N\cap\tilde{N}) = V((0)) = \mathbf{1}$. □

Corollary 3.5. *Let R be a semisimple ring and M be an R-module. Then $\mathcal{V}({}_RM)$ is isomorphic to the Boolean algebra of all subsets of some finite set X, and therefore its cardinal number is 2^n for some positive integer n.*

Proof. Since R is a semisimple ring, R is Artinian and then it has only a finite number of prime ideals. Thus $\mathcal{R}(R)$ is finite and hence by Theorem 2.18, $\mathcal{V}(R)$ is finite. Hence by Theorem 2.2 and Theorem 3.4, $\mathcal{V}({}_RM)$ is a finite Boolean algebra. Now the result follows from [3, Corollary IV.1.10]. □

Corollary 3.6. *Let R be a ring and M be a semisimple R-module. Then $\mathcal{V}({}_RM)$ is a Boolean ring with the following operations:*

$V(L) + V(N) = V((L + \tilde{N}) \cap (\tilde{L} + N))$ *and* $V(L) \cdot V(N) = V(L + N)$ *where* $M = L \oplus \tilde{L} = N \oplus \tilde{N}$.

Proof. Follows from [3, Theorem IV.2.3]. □

Let A and B be Boolean algebras. A mapping $f\colon A \to B$ is called a *Boolean algebra homomorphism* if f is a lattice homomorphism, $f(\mathbf{0}) = \mathbf{0}$, $f(\mathbf{1}) = \mathbf{1}$ and $f(a') = f(a)'$ for all $a \in A$. One can easily see that a lattice homomorphism $f\colon A \to B$ between Boolean algebras A and B respects the complementary operation $'$ if and only if $f(\mathbf{0}) = \mathbf{0}$ and $f(\mathbf{1}) = \mathbf{1}$.

As usual, a Boolean algebra homomorphism $f\colon A \to B$ is called a *Boolean algebra isomorphism* if $f\colon A \to B$ is a bijection.

Theorem 3.7. *Let R be a semisimple ring. Then*

1. *ν is a Boolean algebra homomorphism.*
2. *If M is a faithful primeful R-module, then ω is a Boolean algebra isomorphism.*

Proof. (1) By Theorem 3.4, $\mathcal{V}({}_RM)$ and $\mathcal{V}(R)$ are Boolean algebras and by Theorem 2.2, ν is always a lattice homomorphism. Also, $\nu(\mathbf{0}) = \nu(V(R)) = V(RM) = V(M) = \mathbf{0}$ and $\nu(\mathbf{1}) = \nu(V(0)) = V(0M) = V(0) = \mathbf{1}$. Hence, as noted above, ν is a Boolean algebra homomorphism.
(2) By Theorem 2.16, ω is a lattice isomorphism. Also, we have $\omega(\mathbf{0}) = \omega(V(M)) = V((M : M)) = V(R) = \mathbf{0}$ and $\omega(\mathbf{1}) = \omega(V((0))) = V((0 : M)) = V((0)) = \mathbf{1}$, as required. □

Corollary 3.8. *Let R be a semisimple ring. Then*

(1) *$\mathcal{L}(R)$ is a Boolean algebra with the least element $\mathbf{0} = (0)$, greatest element $\mathbf{1} = R$ and the unary operation $I' = J$ where ${}_RR = I \oplus J$.*

(2) *If M is a faithful primeful R-module, then $\mathcal{V}({}_RM)$ and $\mathcal{L}(R)$ are anti-isomorphic Boolean algebras.*

Proof. (1) Since R is semisimple, $\mathcal{L}(R)$ is a distributive lattice by [7, Exercise 1.2.5]. It is easily seen that $\varphi\colon \mathcal{L}(R) \to \mathcal{V}(R)$ defined by $\varphi(I) = V(I)$ is an anti-isomorphism. Also, we have

$$\mathbf{0} = \varphi^{-1}(\mathbf{1}) = \varphi^{-1}(V((0))) = (0),$$
$$\mathbf{1} = \varphi^{-1}(\mathbf{0}) = \varphi^{-1}(V(R)) = R$$

and for every $I \in \mathcal{L}(R)$, $I' = \varphi^{-1}(V(I)') = \varphi^{-1}(V(J)) = J$ where ${}_RR = I \oplus J$. Hence $\mathcal{L}(R)$ is a Boolean algebra which is anti-isomorphic to the Boolean algebra $\mathcal{V}(R)$.
(2) By (1) and Theorem 3.7(2). □

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