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Knowledge Generation based on Free Energy Principle of Brain, Active Inference, and Abstract Concepts

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ABSTRACT

The brain's free energy principle (FEP), and active inference, proposed by Karl Friston is a model that, based on the Bayesian inference, shows uncertainly how the concepts are generated based on the stimuli perceived by the human. In this model, it is assumed that the concepts exist in a hidden and real form in the environment, and the agent should identify and encode the concepts in his brain through the indirect perception of the stimuli of these concepts. This process takes place based on the Bayesian inference in the declarative or procedural real concepts (concepts in the environment) generation requiring the agent's actions and perceptions by the active inference process. Declarative concepts are concepts that do not require any action on the environment to learn and are learned directly through the transfer of knowledge. But procedural concepts are concepts that require the selection of different actions on the environment to learn (such as driving). In the current study, objectification or construction of abstract concepts (concepts that do not exist in the environment but are formulated by the agent through the reception of environmental stimuli in his brain) is based on active inference. In the proposed model, which is an extension to the active inference model, the policies must be identified or generated by the agent because these policies do not already exist. The identification or construction of these policies to generate or objectify the abstract concepts would mean knowledge generation and learning how the concepts are generated.

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1. Introduction

The brain free energy model proposed by Karl Friston is among the models proposed for different brain functions [1, 2, 3]. This model has been proposed using a combination of physical views such as the energy and entropy [4, 5, 6], Bayesian inference and probabilities, information theory [7, 8], distribution functions (especially Dirichlet, categorical and normal distributions), Markov process [9, 10], Kullback-Leibler criterion, and considering the feature of brain self-organization in a complex but powerful model of the brain [11, 12]. The execution process based on this model is called active inference.

According to the active inference model, environmental concepts are indirectly inferred in a probabilistic process based on the Markov model [4, 1]. In active inference, agents infer the actions that will result in visiting concepts of low expected free energy. It is done by sampling actions from a prior belief about policies according to how much expected free energy that policy will induce.

The real concepts (concepts or states in the environment) are formulated and encoded in the agent's brain as a probabilistic process [13] through the perception of sensory stimuli, which are related to the characteristics of each concept in the brain. In this case, the concepts are updated with the perception of new sensory data in Bayesian inference [14]. In this state, based on the free energy principle, the increase in the sensory exchange entropy (i.e., increasing the agent's surprise) created in the brain must be minimized in the shortest possible time to prevent the brain's collapse [1, 11]. Entropy is the uncertainty or surprise caused by the inconsistency of the prior model of the environment (previously coded concepts or agent's prior beliefs [15]) with new sensory data (stimuli). The more different these new sensory perceptions with the characteristics expected by the concepts encoded in the brain are, the greater the entropy and surprise. Free energy of the brain is indicative of the difference between [16] the new sensory stimuli and the expected stimuli, and minimization of this energy in an inferential process [17]. According to Bayesian inference, it leads to concepts updating in the brain.

Thus, the free energy minimization corresponds to evasion of the surprise that maximizes evidence for perceptual concepts. In this model, we will have three categories of uncertain variables [15]: hidden concepts present in the environment that should be perceived by the agent through stimulus, the concepts' characteristics or stimuli, and the policies intended to minimize the free energy. The free energy minimization leads to learning of concepts or updating them. This learning takes place in two processes, namely perception, and action [18].

- Perception is changing the expectations to reduce entropy and prediction error.
- Action is changing the agent's composition by biological factor's impact on the environment to change the sensory stimuli to avoid surprise. Under active inference, policies (sequences of actions) correspond to sequences of "control states"- a type of hidden concept that agents can directly inference.

Crucially, under active inference, both action and perception are realizations of the single drive to minimize surprise.

In summary, active inference casts perceptions as optimizing beliefs about the causes of sensory (stimuli) that minimize surprise (i.e. free energy) and action in terms of policies that minimize uncertainly (i.e. expected free energy) [19].

In this case, if the learning process only takes place in the form of sensory perceptions from the environment or knowledge transfer from another agent without the need to implement a hierarchy of actions, the learning is of declarative type based on the Bayesian inference. Thus, we are faced with two sets of uncertain variables, i.e., the concepts and the stimuli. However, besides these cases, if we need to perform a set of actions for learning, then, in addition to the probabilistic Bayesian inference, in this state, the variational Bayesian is used instead of the exact Bayesian it is required to adopt suitable policies in each step for learning and the free energy minimization (e.g. learning how to drive) [14, 20, 21]. This type of learning is of procedural type and inference is referred to as 'active inference'. The free energy minimization and active inference process variables are presented in ***Table 1***.

. ***Table 1.*** Variables of brain free energy model [19]

Symbol	Explanation
o	Sensory stimuli variable
s	Hidden concepts variable
$a_\tau = \pi(\tau)$	Specific action on the time scale τ
T	All time scales
$\tau \in [1, 2, ..., T]$	Time scales from the first scale to T scale
π	The selected policy for a specific action in each time scale
$\pi_\pi = (a_{\pi,1}, a_{\pi,2}, ... a_{\pi,T})$	Set of policies related to each action in each time scale
$\boldsymbol{O} = [o_1, ..., o_T]$	Set of sensory stimuli in all time scales
$\boldsymbol{S} = [s_1, ..., s_T]$	Set of hidden concepts in all time scales
μ	Encoding concepts in the brain [18]
F	Variable free energy

In the active inference and utilization of variational Bayesian [11], instead of posterior probabilistic distribution of concepts, $p(s|o)$ whose direct computationalization is not possible, we use the estimated distribution function $q(s)$, and the free energy can be minimized by getting the behaviors of these two functions closer to each other. In this case, the agent needs presuppositions that are indicative of a generation of concepts based on the sensory stimuli [22, 23]. The active inference and free energy minimization lead to the evolution of the concepts generation model relative to time, in a way that maximize the evidence for performed observations (stimuli). Due to the generation of concepts from the perceived stimuli, practically, we are faced with a generative model of concepts from the stimuli called the generative model or probabilistic distribution p(s,o) [11, 1]. The generative model for Markov decision processes can be parameterized in probabilistic distribution $p(s. o. \pi)$.

μ is the internalized and encoded states of the model in the brain, and a is the action needed to impact the sensory stimuli to minimize the free energy. μ will change with the perception of new sensory data and in case of entropy change and identification of the difference between the perceived and predicted data.

The hidden environmental concepts become parametric and are encoded by the internal states of the brain, i.e. μ. To select suitable actions, it is required to adopt policies (π) that lead to free energy minimization with selective actions [13]. These actions can change in the discrete time scales based on the new sensory perceptions [24], i.e., the actions will be a function of the intended policies in each time scale, such as τ. The set of the active inference process variables [25] based on selective policies and actions in different phases are presented in ***Tables 1*** and ***2***.

Table 2. The generative model functions and variables [19]

A	Similarity matrix (mapping the concepts to stimuli) or likelihood matrix
B	Transfer matrix (mapping prior concepts to new ones)
C_τ	The prior distribution of stimuli
D	The prior distribution of concepts (prior beliefs)
γ	Precision parameter
a,b,d	Dirichlet parameters

$$p(A) = Dir(a), p(B) = Dir(b), p(D) = Dir(d)$$

$$p(o_\tau | s_\tau) = Cat(A), p(s_{\tau+1} | s_\tau, \pi) = Cat(B_{\pi\tau})$$

$$p(o_\tau) = Cat(C), p(s_1) = Cat(D), p(\pi) = \sigma(-\gamma G(\pi))$$

$$G(\pi) = \sum_\tau^T G(\pi, \tau)$$

In ***Table 2***, the G function is the expected free energy at future time step τ under policy π [11, 26].

The agent's perception and learning of concepts in the action-perception cycle is summarized in ***Figure 1***.

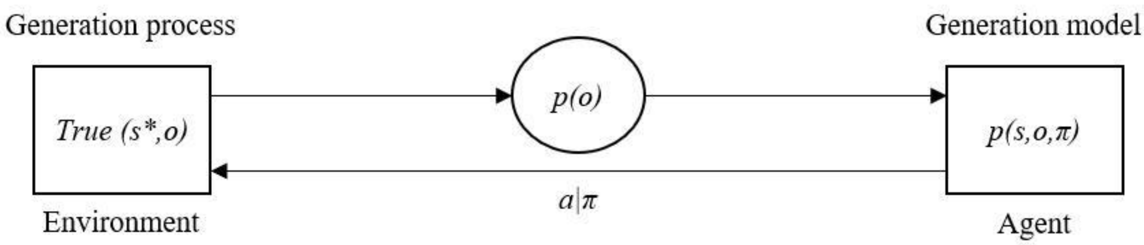

Figure 1. Generation process and model in the interaction between the environment and agent under the Bayesian inference and agent's actions on the environment [21]

The left block in ***Figure 1*** is indicative of the environment, and the right block is indicative of the agent (brain). There are real concepts or states in the environment (s^*) whose sensory stimuli enter the environment under a generative process. After the agent perceives these stimuli [27], it is possible for the agent to identify or generate them under the generative model [28]. Thus, the assumption is that the concepts are hidden in the environment and can be generated by observations perceived by the agent (generative model). Regarding the presence of different factors such as noise, interference, the agent's inexactitude, various physical factors affecting sensory stimuli, limitation of different senses in the agent such as visual or auditory problems, perception of the sensory stimuli generated in the environment is done under an uncertain and probabilistic process [13] with the probability *p(o)*.

There are two connecting links between the environment and the agent; one from the environment to the agent to transfer the uncertain sensory stimuli, and the other from the agent to the environment to select the policy and impact the environment so that he can fulfill procedural learning under an uncertain process and variational Bayesian inference known as the active inference.

Regarding the agent's initial prediction of the concepts that generate these perceptual stimuli, generating or updating the concept in the agent's brain becomes viable, and in case there is no prior prediction of these stimuli, it is required to generate a new concept by combining the stimuli perceived by the agent. This process is done under the complete Bayesian inference. If during the learning, the stimuli perceived by the agent are associated with impacting the environment or being influenced by it (such as step-by-step learning of driving), and the fact that this learning is allowed by different exercises and adoption of various control polices in several stages, , it is required that the agent adopt policies in each step that beside impacting the environment [29, 30], he learns the procedural concepts or actions (a) such as driving.

Since the generative model of concepts from the stimuli is based on the selected policies, it can be shown as a joint probability density function in *Eq.(1)* [31, 16].

$$p(o.s.\pi) = p(o|s)p(s|\pi)p(\pi) \tag{1}$$

In this model, the assumption is that the concepts and stimuli exist in the environment in a realistic and certain form, and they are inferred and generated in an uncertain and probabilistic manner. The generated concepts should be one of the concepts which are most likely to exist in the environment. The innovation in the current study is the attention paid to concepts that can be generated in agent's brain in an abstract form by the perception of the sensory stimuli from the environment, and they can be considered in the improvement of the model for knowledge generation by active inference process and generation of abstract concepts in a Bayesian inference process. In this context, knowledge generation means learning how to objectify and construct concepts produced in the agent's mind. To objectify an abstract concept that does not exist in the environment and is produced in the brain of an innovative agent, it is necessary for the agent to discover and clarify the relationships between the components of the concept, its applications, and how it works. It means the production of knowledge by the agent.

The free energy equations in the perception and action processes can be written as ***Eq.(2)*** [1]:

$$\textit{Perception to optimize the bound} \quad \begin{cases} F = Divergence + Surprise \\ \quad = D_{KL}\left(q(s|\mu)\|p(s|o)\right) - lnp(o) \\ \mu = \underset{\mu}{arg\,min}\, Divergence \end{cases}$$

$$\tag{2}$$

$$\textit{Action to minimize the surprise border} \quad \begin{cases} F = Complexity - Accuracy \\ \quad = D_{KL}\left(q\|p(s)\right) - \langle lnp\left(o(a)|s,m\right)\rangle_q \\ a = \underset{a}{arg\,max}\, Accuracy \end{cases}$$

In these equations, q is the estimated probability function by the agent from the concepts existing in the environment that is approximately estimated by the agent due to the impossibility of calculating the posterior function $p(s|o)$. The operator D_{KL} is the divergence between these functions (Kullback-Leibler divergence). The closer this divergence to zero, the closer the behavior of the functions of q and $p(s|o)$.

2. Generation of Abstract Concepts

In the active inference model, generation of real concepts is done by environmental stimuli, while we know that in the mind of an agent, by perceiving sensory stimuli, it is possible to produce concepts that do not exist in the environment (abstract concepts). Obviously, human agents such as scientists, inventors, innovators, artists, and creative people can generate more abstract and non-objective concepts by perceiving the sensory stimuli from the environment than others. Finally, they can present innovation by objectifying or generating their abstract concepts. For example, before making their inventions, the inventors should mentally produce abstract concepts and then, in a trial and error and learning process, objectify and make these inventions.

Accordingly, the differences between the normal and innovative people who generate knowledge can be mentioned as below [32, 33]:

- Innovative people, with the perception of the environmental stimuli are much abler to generate abstract concepts that do not exist in the environment compared to normal people.
- The innovative people can create new concepts and create knowledge related to their abstract concepts by objectify them.
- Normal people are usually satisfied with the same real concepts in the environment and have no special ability to expand the abstract concepts. In other words, the environment or world of such people is the same environment they live in, and the existence of concepts in their minds is formulated through adaptation with real concepts.
- Innovative people's world is not limited to the real world, but they can focus on a more extensive world by generating abstract concepts and then objectifying or making them. In other words, creative people act beyond the environment to develop their abstract concepts.

Both innovative and normal people act in a probabilistic method based on the Bayesian inference in generating mental concepts, whether they are real or not, and update these concepts with the perception of new sensory stimuli [15].

3. Generation of Knowledge and New Concepts

The agent can innovate or generate knowledge based on mental inferences and abstract concepts or objectify his abstract knowledge. This type of objectification is equivalent to the generation of concepts that have not previously existed in the environment and have been generated based on stimuli perceived from the environment. These concepts can include various innovative, explorative, or artistic achievements generated in the procedural process by an agent who has previously generated them in his mind in an abstract form [33]. This procedural process is different from the generation or learning of real concepts existing in the environment, which are explained in the following:

- In learning the procedural concepts, the knowledge of generation or method of learning these concepts already exists, and the agent is required to learn this knowledge by transferring the knowledge to implement the required procedures. Since we need different actions on the environment in these conditions, the generation of concepts is done in a hierarchy of selective actions (an outcome of the agent’s predicted policies) based on which the agent can predict his learning. Regarding the existence of previous knowledge, the policies the agent adopts are pre-determined, and the agent's duty is to properly select them in terms of prior and posterior and settings relevant to each policy (knowledge generation process). Whereas, when objectifying the abstract concepts, the agent is required to generate the knowledge. Thus, the policy selected by the agent must be discovered and implemented by himself. These policies are implemented through error and trial or based on the staged inference in implementing procedures leading to innovation.
- In normal learning, the learning takes place through minimization of free energy by the agent, while in the learning and innovation, first, the agent should discover and find different actions and policies by energy consumption. Then, by carrying these policies out in the form of environmental actions, he can fulfill his knowledge and innovation relevant learning by free energy minimization.
- In addition to the generation of new and beyond-environmental concepts, innovative people can create extensive changes in the environment.
- In the free energy model of brain and active inference, normal people can acquire skills by free energy minimization that leads to spontaneity or creation different skills in the environment. In contrast, in the proposed model, which is an extension of the active inference model, the innovative people, by the generation of schemata and knowledge and skills that did not previously exit, fulfill this learning process and knowledge generation through generation or creation of new actions.

4. Model of Knowledge Generation and Learning in Innovative People

Learning and knowledge generation by an agent based on the characteristics mentioned in previous section takes place in the two branches: generation or construction of real concepts (model of brain free energy), and construction or generation of abstract concepts, knowledge generation, and skill acquisition. In the free energy model and the active inference, we only have the first branch, while in the model proposed in the current study, both branches have been considered, and a model for knowledge generation in the innovative and creative agents has been presented.

The proposed model has been shown in ***Figure 2.*** The left side shows the environment and generation of real stimuli from real concepts. The stimuli transmitted from the environment to the agent are perceived with the probability $p(o)$. The agent can generate real and abstract concepts in two separate branches by these stimuli. In one branch, generation of real concepts and acquiring existing skills through knowledge acquisition and generative model $p(o.s.\pi)$ is fulfilled. This generative model can lead to the transfer and acquisition of declarative knowledge, or under a probabilistic process and selection of different policies (π) that are pre-determined, lead to the skill acquisition. In the other branch, during the generative and abstract process $p(o.s'.\pi')$, the agent can generate abstract concepts (s') in his mind which do not exist in the real world, and finally, these concepts can be encoded in his brain without the existence of a real form.

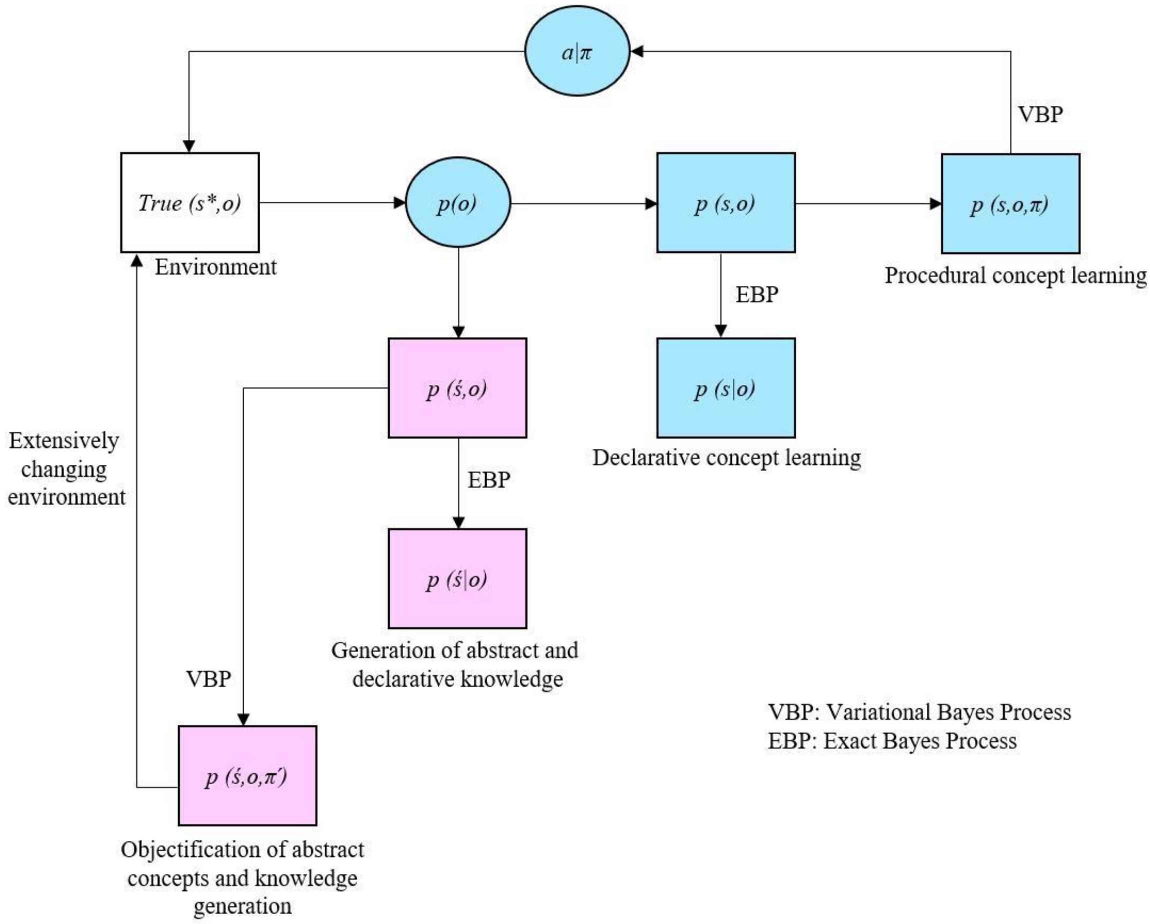

Figure 2. Real concepts generation and learning process in blue and abstract concepts in pink, under the framework of Bayesian inference for declarative concepts and active inference for procedural concepts and knowledge generation

If the agent is from innovative type, he can do this with the aim of objectification or physical construction of his abstract concepts during the learning and knowledge generation stage. In this method, similar to the basic skill acquisition process, it is required to realize the objectification and generation of concepts with a hierarchy of policies. However, unlike the basic state, these policies are not predetermined, and the agent should himself generate these policies by consuming energy.

This policy generation and its ultimate utilization for constructing and objectifying the concepts will mean knowledge generation or innovation. In the free energy model of brain, we only have concept generation from the stimuli while the policies are pre-determined; however, in the proposed model, there is both concept generation and policy generation. This state is shown in ***Figure 2***, which is presented as a more detailed part of ***Figure 3***.

Generation of new concepts by the agent can impact or change the environment either limitedly or extensively. ***Figure 3*** shows a general scheme of some policies considered by the agent to generate new environmental concepts. For example, for generating abstract concepts, the agent can predict the components and their relationships, how the environment impacts them, and how they are assembled in the framework of his policies and then construct or produce them. Likewise, if his selected policies are not in line with his predictions in the generation of the concept, and he does not receive an appropriate answer, he can change his policy under the active inference process. This collection of policies which finally allow for the objectification of abstract concepts can become a new knowledge for innovation or invention.

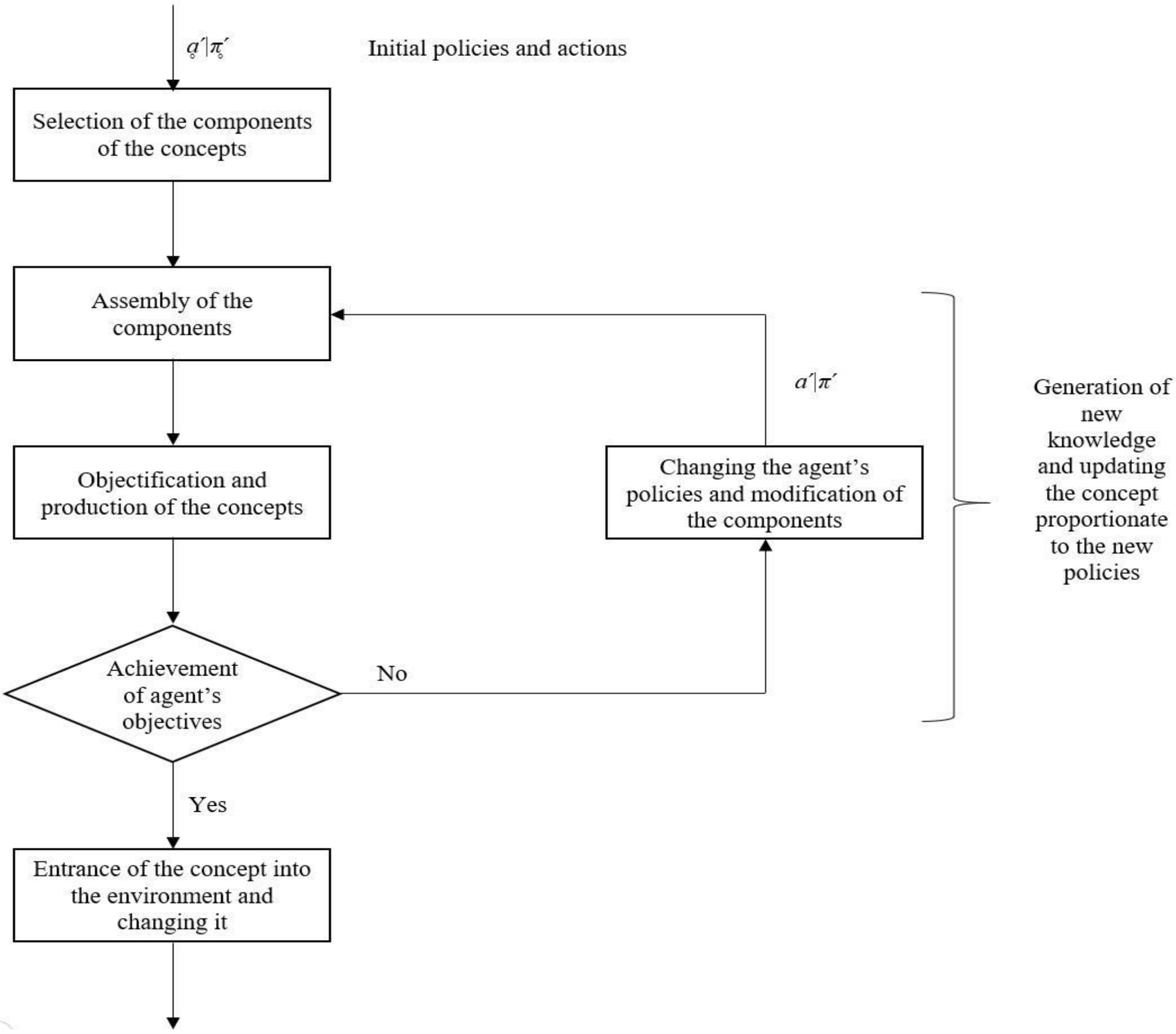

Figure 3*.* Active inference process in the generation of abstract concepts with the production of new policies and knowledge associated with learning

In the active inference model, the real concepts do not go through any changes, however, their transmitted stimuli may go through different changes over time, while in the proposed model, the abstract concepts are generated by the agent based on some specifically predicted characteristics and stimuli, and based on his efforts for the objectification of them, and these concepts are changed under different and variable policies to achieve the predicted objectives. In this model, there is the expectation that concepts are changed when objectifying them to achieve the goals intended by the agent, while the concepts' stimuli or characteristics can remain unchanged. In other words, the abstract concepts can change from the abstraction phase to objective production, based on the agent's knowledge which is an outcome of his selected policies.

Figure 3 shows the concept generation by the agent to their objectification in several stages. These stages include selecting the concept's components, connecting and assembling the components, and objectification and construction. If the characteristics of this type of generation are not consistent with the characteristics initially intended by the agent, he has to revise his policy and adopt a new one for the components and their assembly to fulfill the generation of his abstract concept. This process has been shown in the cycle of selecting components, assembling them, objectifying the concept and placing it in the environment, lack of appropriate answer, modification of the components, and return to the selection of components. This cycle indicates new learning processes and, finally, the generation of knowledge of objectification of concepts that does not already exist in the real environment.

Due to the fact that the active inference model is a type of generative model, the proposed model that is developed from this model is categorized under generative models. The difference is that in the proposed model, the generation of abstract concepts is done by free energy minimization (as in the model of active inference), while the generation of policies related to the objectification of abstract concepts is fulfilled by energy consumption in the brain. In the proposed model, we will have three stages: In the first stage, abstract concepts are generated by free energy minimization; in the second stage, through energy consumption, different policies are generated by the agent for objectification; and in the third stage, based on the formulated policies and by free energy minimization, abstract concepts are objectified.

This model is unsupervised learning that can update itself using a combination of different stimuli as a generative model can generate new concepts of unsupervised received stimuli. In this model, the active inference process is used in the generation of procedural and conditional knowledge and the perception process is used to generate declarative knowledge.

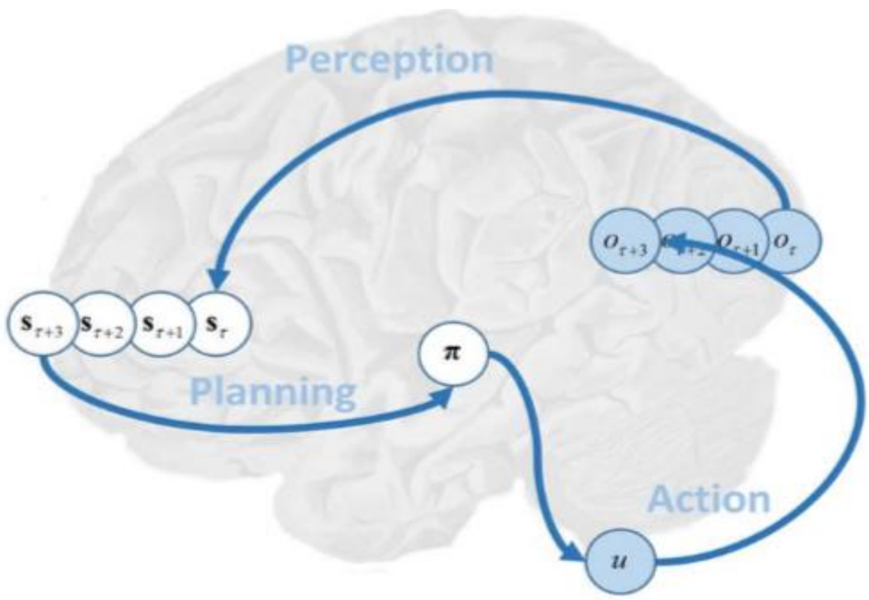

Figure 4. We have illustrated the active inference model graphically, and emphasized the propagation of beliefs through time. Starting from the back of the brain, sensory areas send messages to higher regions encoding beliefs about the causes of those sensations. These beliefs are propagated forwards in time, allowing for a plan of action into the future (presumably evaluated in cortico-striatal loops). Once a policy has been inferred, this is used to select an action (u) [34]

As shown in ***Figure 4***, the active inference model fully correlates between variables, how concepts are inferred, the presence of different policies, and neuroscience characteristics of the brain in terms of specialized areas in the brain such as perceptual region and prefrontal cortex and memory. In this figure, the areas associated with the active inference model variables are highlighted. For example, the cerebellum as the nucleus for producing and recording actions and the occipital region as the receptor for sensory stimuli (observations) are completely separable. For this reason, in the proposed model of Figure 2, there is an appropriate compromise between the classification of different types of memory and the neuroscience model of active inference of the brain. The actions are represented by the variable u in the cerebellum in this figure.

The evaluated and computational model of active inference is considered a logical benchmark for the proposed model. In practice, the foundation and validity of both models are the same.

5. Conclusion

The model of brain free energy is proposed with the assumption of real concepts and environment. And it properly presents skill acquisition or declarative learning, and knowledge transfer under the Bayesian inference, brain free energy characteristic, and entropic changes. However, this model needs to be developed in terms of knowledge generation and provision of abstract concepts by the agent, which is what presented in the proposed model in the current study. According to this model, it can be extended in two branches of real and abstract concepts with latter considering the skill acquisition, objectification of the abstract concepts, and knowledge generation. Both branches act under the active inference, while in the real concepts branch, we take the stimuli change and stabilization of the concepts into consideration, and in the abstract concepts branch, we consider the change in concepts and stabilization of the stimuli to generate knowledge and achieve the agent’s goals.

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References

[1] Friston, K., "The free energy principle: A unified brain theory?," *Nature Reviews Neuroscience,* vol. 11, no. 2, pp. 127-138, 2010.

[2] Friston, K., "A Free Energy Principle for Biological Systems," *MDP, Entropy,* vol. 14(11), pp. 2100-2121, 2012.

[3] Friston, K. J., Stephan, K. E., "Free-energy and the brain," *Synthese,* vol. 17, pp. 417-458, 2007.

[4] Friston, K., "A free energy principle for a particular physics," *arXiv: 1906.10184,* 2019.

[5] Szilard, L., "On the reduction of entropy in a thermodynamic system by the inference of an intelligent," *Z Physics,* vol. 53, pp. 840-858, 1929.

[6] Kirkadly, J. S., "Thermodynamics of human brain," *Biophysics,* vol. 5, pp. 981-986, 1965.

[7] Shannon, C. E., "A Mathematical Theory of Communication," *Bell System Technical Journal,* vol. 27, pp. 379-423 & 623-656, 1948.

[8] Sengupta, B.; Stemller, M. B.; Friston, K., "Information and Efficiency in the Nervous Systems," *PLos Comutational Biology,* 2013.

[9] Shen, Y.; Archambeau, C.; Cornford, D.; Opper, M.; Shawe-Taylor, J.; Barillec, R., "A comparision of variational and Markov chain monte carlo methods for inference in partially observed stochastic dynamic systems," *Journal of Signal Processing Systems,* vol. 61, no. 1, pp. 51-59, 2010.

[10] Todorov, E., "Linearly-solvable Markov decision problems," in *In Advances in Neural Information Processing Systems*, MIT Press, 2006, pp. 1369-1376.

[11] Friston, K.; FitzGerald, T.; Rigoli, F.; Schwartenbech, P; O'Doherty, J.; Pezzulo, G., "Active inference and learning," *Elsevier, Neuroscience and Behavioral Review*, vol. 68, pp. 862-879, 2016.

[12] Friston, K. J.; Kilner, J.; Harison, L., "A free energy principle for the brain," *Physiol. Paris,* Vols. 100(1-3), pp. 70-78, 2006.

[13] Parr, T.; Friston, K. J., "Uncertainty, epistemic and active inference," *Journal of Royal Society Interface,* vol. 14, no. 136, 2017.

[14] Greshman, S. J., "What does the free energy principle tell us about the brain?," *arXiv: 1901.07945v5 [q-bio.NC],* 2019.

[15] Knill, D., C.; Pouget, A., "The Bayesian brain: the role of uncertainty in neural coding and computation," *Trends Neuroscience,* pp. 301-308, 2004.

[16] Da Costa, L., Parr, T., Sajid, N., Veselic, S., Neacsu, V., Friston, K., "Active inference on discrete state-spaces: A synthesis," *Journal of Mathematical Psychology,* vol. 99, 2020.

[17] Mackay, D. J. C., "Free energy minmization algorithm for decoding and crypto analysis," *Elrctron. Lett.,* vol. 31, pp. 445-447, 1995.

[18] Sajid, N., Ball, P. J., Parr, T., Friston, K. J., "arXiv," 30 October 2020. [Online]. Available: arXiv: 1909.10863v3 [cs.AI]30 Oct 2020.

[19] Kaplan, R., Friston, K. J., "Planning and navigation as active inference," *Biological Cybernetic,* vol. 112, pp. 323-343, 2018.

[20] Ostwald, D.; Kirilina, E.; Strake, L.; Blankengurg, F., "A tutorial on variational Bayes for latent linear stochastic time-series models," *Elsevier, Journal of Mathematical Psychology,* vol. 60, pp. 1-19, 2014.

[21] Smith, R., Friston, K. J., Whyte, C., "A Step-by-Step Tutorial on Active Inference and its Application to Empirical Data," 2021. [Online]. Available: http://researchgate.net/publication/348153427.

[22] Friston, K.; Adams, R. A.; Perrit, L.; Breakspear, M., "Perception as hypotheses: saccades as experiments," *Front. Psychology,* vol. 3, p. 151, 2012.

[23] Bogacz, R., "A tutorial on free energy framework for modelling perception and learning," *Elsevier, Journal of Mathematical Psychology,* vol. 76, pp. 198-211, 2017.

[24] Friston, K. J., Parr, T., de Veris, B., "The graphical brain: Blief propagation and active inference," *Network Neuroscience,* vol. 1(4), pp. 381-414, 2017.

[25] Sajid, N.; Ball, P. J.; Friston, K., "Active inference: demystified and compred," *arXiv: 1909.10863v2 [cs.AI] 29 Jan 2020,* 2020.

[26] Catal, O.; Nauta, J.; Verbelen, T.; Simoens, P.; Dhoeht, B., "Bayesian policy selection using active inference," in *The proceedings of the workshop on"Struture & Priors in Reinforcement Learning"*, New Orleans, 2019.

[27] Estes, W. K., "On the organization and core concepts of learning theory and cognitive psychology," in *Handbook of learning and cognitive process*, Hillsdale, NJ, Erlbaum, 1978.

[28] Parr, T.; Friston, K., "The Anatomy of Inference: Generative Models and Brain Structure," *Frontiers in Computational Neuroscience,* vol. 12, no. 90, 2018.

[29] Friston, K., "The free-energy principle: a rough guid to the brain?," *Opinion, Trends in Cognitive Science,* vol. 13, no. 7, 2009.

[30] Smith, R.; Schwartenbeck, P.; Parr, T.; Friston, K., "An Active Inference Approach to modelling structure learning: concept learning as an example case," *Frontier in Computational Neuroscience,* vol. 14, no. 41, May, 2020.

[31] Catal, O., Wuthier, S., De Boom, C., Verbelen, T., Dhoedt, B., "Learning Generative State Space Models for Active Inference," *Frontiers in Computational Neuroscience,* vol. 14, 2020.

[32] Weisberg, R. W., "Problem solving and creativity," *The nature of creativity,* pp. 148-176, 1988.

[33] Weisberg, R. W., "Creativity and knowledge: A challenge to theories," in *Handbook of creativity,* New York, Cambridge University Press, 1999, pp. 226-250.

[34] Parr, T., Rikhye, R. V., Halassa, M. M., Friston, K. J., "Prefrontal Computation as Active Inference," *Cerebral Cortex,* vol. 00, no. 00, pp. 1-14, 2019.

[35] Kaplan, R., Friston, K. J., "Planning and navigation az active inference," *Biological Cybernetic,* vol. 112, pp. 323-343, 2018.