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An improved upper bound for ultraspherical coefficients

Mehdi Hamzehnejad[†*], Mohammad Mehdi Hosseini[‡], Abbas Salemi[‡]

[†] *Department of Mathematics, Faculty of Science and Modern Technology, Graduate University of Advanced Technology, Kerman, Iran*
[‡] *Department of Applied Mathematics and Mahani Mathematical Research Center, Shahid Bahonar University of Kerman, Kerman, Iran*
Email(s): mhdhamzehnejad@gmail.com, mhosseini@uk.ac.ir, salemi@uk.ac.ir

Abstract. In this paper, new upper bounds for the ultraspherical coefficients of differentiable functions are presented. Using partial sums of ultraspherical polynomials, error approximations are presented to estimate differentiable functions. Also, an error estimate of the Gauss-Jacobi quadrature is obtained and we state an upper bound for Legendre coefficients which is sharper than upper bounds proposed so far. Numerical examples are given to assess the efficiency of the presented theoretical results.

Keywords: Ultraspherical coefficients, approximation error, upper bound, Gauss-Jacobi quadrature.
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1 Introduction

The estimation of the ultraspherical coefficients has fascinated a great attention in recent years because of their importance in spectral methods and the approximation theory [1, 2]. Suppose that a suitably smooth function f has the following expansion

$$f(x)=\sum_{n=0}^{\infty}u_nP_n^{(\alpha)}(x), \quad \alpha>-1, \tag{1}$$

where $P_n^{(\alpha)}(x)$, $n=0,1,\ldots$, denotes the ultraspherical polynomials and u_n is the ultraspherical coefficient defined by

$$u_n=\frac{1}{\gamma_n}\int_{-1}^{1}(1-x^2)^{\alpha}f(x)P_n^{(\alpha)}(x)dx, \tag{2}$$

in which,

$$\gamma_n=\frac{2^{2\alpha+1}\Gamma^2(\alpha+1)\Gamma(n+2\alpha+1)}{n!(2n+2\alpha+1)\Gamma^2(2\alpha+1)}. \tag{3}$$

*Corresponding author.
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Ultraspherical polynomials are considered as general forms of some famous orthogonal polynomials such as the Chebyshev, Legendre, and Gegenbauer polynomials [1]. It is demonstrated that the convergence rate of the Jacobi expansion and Gauss-type quadrature depends on the convergence rate of the Jacobi coefficients [1,3,5–7]. For example, in [1], the authors utilized the following inequality to present an approximation error bound for a function $f(x)$

$$\|f(x)-f_N(x)\|_{L^2_w[-1,1]} \leq \sum_{n=N+1}^{\infty} |u_n|\sqrt{\gamma_n}, \tag{4}$$

where $f_N(x)=\sum_{n=0}^{N} a_n P_n^{(\alpha)}(x)$ is the partial sums of ultraspherical polynomials.

According to the fundamental importance of the ultraspherical polynomials, a number of researchers have recently deepened studies on the estimation of the ultraspherical coefficients. As an instance, Xiang [6] proposed an upper bound for the Jacobi coefficients and then obtained some approximation error bounds for differentiable functions and the Jacobi-Gauss quadrature.

In this paper, an upper bound is established for the ultraspherical coefficients of differentiable functions, which is sharper compared to that given in [6]. For this purpose, we benefit from the telescoping series to obtain our suggested upper bound. Moreover, we show a new approximation error bound for a differentiable function $f(x)$, which exhibits a sharper trend in comparison with the trend observed for that mentioned in (4). Some numerical examples are also provided to verify the effectiveness of our upper bounds. Furthemore, it is discussed that in some cases, our proposed upper bound for the ultraspherical coefficients is much smaller than that presented by Xiang [6].

The rest of this paper is organized as follows. In Section 2, we review some basic definitions and results of ultraspherical polynomials. In Section 3, we derive a new upper bound for the ultraspherical coefficients of differentiable functions. Also, we provide a new error estimate on the approximation of a function $f(x)$ by partial sums of ultraspherical polynomials. In Section 4, numerical results are presented and we compare the proposed upper bound by the upper bounds were presented in [6]. Furthermore, we provide a new and sharper upper bound for Legendre coefficients which is sharper than the upper bounds proposed so far. In Section 5, by using the asymptotic of the ultraspherical coefficients, a new error estimate on the Jacobi-Gauss quadrature is derived.

2 Preliminary results on ultraspherical polynomials

In this section, we conduct a brief review on some results for the ultraspherical polynomials. The ultraspherical polynomials $P_n^{(\alpha)}(x)$ are a subclass of well-known Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ with $\alpha=\beta$. The relation between ultraspherical polynomials and Jacobi polynomials is as follow (see [1])

$$P_n^{(\alpha)}(x)=\frac{\Gamma(\alpha+1)\Gamma(n+2\alpha+1)}{\Gamma(2\alpha+1)\Gamma(n+\alpha+1)}P_n^{(\alpha,\alpha)}(x), \quad \alpha>-1. \tag{5}$$

The Jacobi polynomials satisfy the following inequality for all $x\in[-1,1]$ [2, Theorem 3.24].

$$|P_n^{(\alpha,\alpha)}(x)| \leq \begin{cases} \frac{1}{\sqrt{n}}, & -1<\alpha<\frac{-1}{2}, \\ |P_n^{(\alpha,\alpha)}(1)|=\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)}, & \frac{-1}{2}\leq\alpha. \end{cases} \tag{6}$$

Also, the following relation holds [2, eq. 3.109]

$$\int_{-1}^{1}\left(P_n^{(\alpha,\alpha)}(x)\right)^2(1-x^2)^{\alpha}dx=\gamma_n. \tag{7}$$

The following theorems will be useful.

Theorem 1. *[1, Theorem 5.1] Let*

$$f^{(r)}(x)=\sum_{n=0}^{\infty}u_n^{(r)}P_n^{(\alpha)}(x), \tag{8}$$

in which,

$$u_n^{(r)}=\frac{1}{\gamma_n}\int_{-1}^{1}(1-x^2)^{\alpha}f^{(r)}(x)P_n^{(\alpha)}(x)dx. \tag{9}$$

The following relation holds among the ultraspherical coefficients

$$u_n^{(r-1)}=\frac{1}{2n+2\alpha+1}\left(u_{n+1}^{(r)}-u_{n-1}^{(r)}\right),\quad n\geq 1. \tag{10}$$

Suppose $U=\sqrt{\int_{-1}^{1}\left|f^{(r+1)}(x)\right|^2dx}$. The following lemma provides an upper bound for $u_n^{(r)}$.

Theorem 2. *[6, Lemma 2.2] Assume that $f,f',\ldots,f^{(r)}$ are continuous on $[-1,1]$. If $U<\infty$ for some $r\geq 0$, then*

$$\left|u_n^{(r)}\right|\leq\delta_{\alpha}\lambda_{n,\alpha}U, \tag{11}$$

where

$$\delta_{\alpha}=\frac{\sqrt{2}|2\alpha+1|\Gamma(\alpha+2)\Gamma^2(2\alpha+1)}{2^{\alpha}\Gamma(2\alpha+3)\Gamma^2(\alpha+1)},\quad\lambda_{n,\alpha}=\sqrt{\frac{(n-1)!(2n+2\alpha+1)}{\Gamma(n+2\alpha+2)}}.$$

In [6], Xiang proposed the following upper bound for the ultraspherical coefficients of differentiable functions.

Theorem 3. *[6] Assume that $f,f',\ldots,f^{(r)}$ are continuous on $[-1,1]$. If $U<\infty$ for some $r\geq 1$, then*

$$|u_n|\leq\begin{cases}2^r\delta_{\alpha}\lambda_{n+r,\alpha}U\beta_{n,r,\alpha}, & -1<\alpha\leq\frac{-1}{2},\\ 2^r\delta_{\alpha}\lambda_{n-r,\alpha}U\beta_{n,r,\alpha}, & \frac{-1}{2}<\alpha,\end{cases} \tag{12}$$

where

$$\beta_{n,r,\alpha}=\frac{1}{(2n+2\alpha+1)\cdots(2n+2\alpha-2r+3)}.$$

3 New upper bound for the ultraspherical coefficients

Now, we provide the following upper bounds for ultraspherical coefficients.

Theorem 4. *Assume that $r \geq 1$ and $f, f', \dots, f^{(r)}$ are continuous on $[-1,1]$. If $U < \infty$ and $n \geq r+1$, then*

$$|u_n| \leq \begin{cases} 2^r \delta_\alpha \lambda_{n+r,\alpha} U \theta_{n,r,\alpha}, & -1 < \alpha \leq \frac{-1}{2}, \\ 2^r \delta_\alpha \lambda_{n-r,\alpha} U \theta_{n,r,\alpha}, & \frac{-1}{2} < \alpha, \end{cases} \tag{13}$$

where

$$\theta_{n,r,\alpha} = \frac{1}{\prod_{j=1}^{r}(2n+2\alpha-2r+4j-1)}.$$

Proof. From (10) and Lemma 2, we have

$$|u_n^{(r-1)}| \leq \frac{\left(|u_{n+1}^{(r)}| + |u_{n-1}^{(r)}|\right)}{2n+2\alpha+1} \leq \frac{(\delta_\alpha U \lambda_{n+1,\alpha} + \delta_\alpha U \lambda_{n-1,\alpha})}{2n+2\alpha+1}. \tag{14}$$

The coefficients $\lambda_{n,\alpha}$ are increasing for $-1 < \alpha \leq \frac{-1}{2}$ and are decreasing for $\frac{-1}{2} < \alpha$. So, we have

$$|u_n^{(r-1)}| \leq \begin{cases} \frac{2\delta_\alpha U \lambda_{n+1,\alpha}}{2n+2\alpha+1}, & -1 < \alpha \leq \frac{-1}{2}, \\ \frac{2\delta_\alpha U \lambda_{n-1,\alpha}}{2n+2\alpha+1}, & \frac{-1}{2} < \alpha. \end{cases} \tag{15}$$

If we apply (15) in (10), we obtain

$$\begin{aligned} |u_n^{(r-2)}| &\leq \frac{\left(|u_{n+1}^{(r-1)}| + |u_{n-1}^{(r-1)}|\right)}{2n+2\alpha+1} \\ &\leq \begin{cases} \frac{2\delta_\alpha U \lambda_{n+2,\alpha}}{(2n+2\alpha+1)(2n+2\alpha+3)} + \frac{2\delta_\alpha U \lambda_{n,\alpha}}{(2n+2\alpha-1)(2n+2\alpha+1)} & -1 < \alpha \leq \frac{-1}{2} \\ \frac{2\delta_\alpha U \lambda_{n,\alpha}}{(2n+2\alpha+1)(2n+2\alpha+3)} + \frac{2\delta_\alpha U \lambda_{n-2,\alpha}}{(2n+2\alpha-1)(2n+2\alpha+1)} & \frac{-1}{2} < \alpha \end{cases} \\ &\leq \begin{cases} \frac{2\delta_\alpha U \lambda_{n+2,\alpha}}{(2n+2\alpha+1)(2n+2\alpha+3)} + \frac{2\delta_\alpha U \lambda_{n+2,\alpha}}{(2n+2\alpha-1)(2n+2\alpha+1)}, & -1 < \alpha \leq \frac{-1}{2}, \\ \frac{2\delta_\alpha U \lambda_{n-2,\alpha}}{(2n+2\alpha+1)(2n+2\alpha+3)} + \frac{2\delta_\alpha U \lambda_{n-2,\alpha}}{(2n+2\alpha-1)(2n+2\alpha+1)}, & \frac{-1}{2} < \alpha, \end{cases} \\ &= \begin{cases} \frac{4\delta_\alpha U \lambda_{n+2,\alpha}}{(2n+2\alpha+3)(2n+2\alpha-1)}, & -1 < \alpha \leq \frac{-1}{2}, \\ \frac{4\delta_\alpha U \lambda_{n-2,\alpha}}{(2n+2\alpha+3)(2n+2\alpha-1)}, & \frac{-1}{2} < \alpha. \end{cases} \end{aligned}$$

If we continue the above process, by easy computation for integers $k \geq 3$, we obtain the following upper bound for $u_n^{(r-k)}$

$$|u_n^{(r-k)}| \leq \begin{cases} \frac{2^k \delta_\alpha \lambda_{n+k,\alpha} U}{(2n+2\alpha-2k+3)\cdots(2n+2\alpha-3)(2n+2\alpha+1)\cdots(2n+2\alpha+2k-1)}, & -1 < \alpha \leq \frac{-1}{2}, \\ \frac{2^k \delta_\alpha \lambda_{n-k,\alpha} U}{(2n+2\alpha-2k+3)\cdots(2n+2\alpha-3)(2n+2\alpha+1)\cdots(2n+2\alpha+2k-1)}, & \frac{-1}{2} < \alpha. \end{cases}$$

Then (12) holds when $k = r$. So the result holds. □

Now, using Theorem 4, we provide a sharper bound on the approximation error of $f(x)$.

Theorem 5. *Assume that $r \geq 2$ and $f, f', \ldots, f^{(r)}$ are continuous on $[-1,1]$. If $U < \infty$ and $N \geq r+1$, then*

$$\|f(x) - f_N(x)\|_{L^2_w[-1,1]} \leq \begin{cases} \frac{2^{r-\frac{1}{2}}U}{(r-1)(N+2\alpha+2)\prod_{j=1}^{r-1}(2N+2\alpha-2r+4j-1)}, & -1 < \alpha \leq \frac{-1}{2}, \\ \frac{2^{r-1}U}{(r-1)(N-r)\prod_{j=1}^{r-1}(2N+2\alpha-2r+4j-1)}, & \frac{-1}{2} < \alpha < 0, \\ \frac{2^{r-1}U\sqrt{\frac{(N+2\alpha+1)\cdots(N+2\alpha-r+2)}{(N+1)N\cdots(N-r+2)}}}{(r-1)(N-r)\prod_{j=1}^{r-1}(2N+2\alpha-2r+4j-1)}, & \alpha \geq 0. \end{cases} \tag{16}$$

Proof. Applying the inequality (4), we have

$$\begin{aligned} \|f(x) - f_N(x)\|_{L^2_w[-1,1]} &\leq \sum_{n=N+1}^{\infty} |u_n|\sqrt{\gamma_n} \\ &\leq \begin{cases} \sum_{n=N+1}^{\infty} \frac{2^r \delta_\alpha \lambda_{n+r,\alpha} U \sqrt{\gamma_n}}{\prod_{j=1}^{r}(2n+2\alpha-2r+4j-1)} & -1 < \alpha \leq \frac{-1}{2}, \\ \sum_{n=N+1}^{\infty} \frac{2^r \delta_\alpha \lambda_{n-r,\alpha} U \sqrt{\gamma_n}}{\prod_{j=1}^{r}(2n+2\alpha-2r+4j-1)} & \frac{-1}{2} < \alpha. \end{cases} \end{aligned} \tag{17}$$

Using the properties of the gamma function, for $n \geq r+1$ we have

$$\delta_\alpha \lambda_{n+r,\alpha}\sqrt{\gamma_n} \leq \sqrt{\frac{(2n+2\alpha+2r+1)}{(n+2\alpha+2)(n+2\alpha+1)(2n+2\alpha+1)}} \leq \frac{\sqrt{2}}{(n+2\alpha+1)}, \tag{18}$$

and with the results obtained in [6]

$$\begin{aligned} \delta_\alpha \lambda_{n-r,\alpha}\sqrt{\gamma_n} &= \sqrt{\frac{(n+2\alpha)\cdots(n+2\alpha-r+1)(2n+2\alpha-2r+1)}{n\cdots(n-r)(2n+2\alpha+1)(n+2\alpha-r+1)}} \\ &\leq \begin{cases} \frac{1}{n-r-1} & \frac{-1}{2} < \alpha < 0, \\ \frac{1}{n-r-1}\sqrt{\frac{(n+2\alpha)\cdots(n+2\alpha-r+1)}{n(n-1)\cdots(n-r+1)}} & 0 \leq \alpha. \end{cases} \end{aligned} \tag{19}$$

For $n \geq N+1$

$$\sqrt{\frac{(n+2\alpha)\cdots(n+2\alpha-r+1)}{n(n-1)\cdots(n-r+1)}} \leq \sqrt{\frac{(N+2\alpha+1)\cdots(N+2\alpha-r+2)}{(N+1)N\cdots(N-r+2)}}. \tag{20}$$

Therefore, for $-1 < \alpha \leq \frac{-1}{2}$, $n \geq r+1$ and $r \geq 2$, (17) becomes

$$\begin{aligned}
\|f(x) - f_N(x)\|_{L^2_w[-1,1]} &\leq 2^r U \sum_{n=N+1}^{\infty} \frac{\sqrt{2}}{(n+2\alpha+1)\prod_{j=1}^{r}(2n+2\alpha-2r+4j-1)} \\
&\leq \frac{2^r U}{(N+2\alpha+2)} \sum_{n=N+1}^{\infty} \frac{\sqrt{2}}{\prod_{j=1}^{r}(2n+2\alpha-2r+4j-1)} \\
&= \frac{2^r U}{(N+2\alpha+2)} \sum_{n=N+1}^{\infty} \frac{\sqrt{2}}{(2n+2\alpha+2r-1)^r \prod_{j=1}^{r}\left(1-\frac{4r-4j}{(2n+2\alpha+2r-1)}\right)} \\
&\leq \frac{2^r U}{(N+2\alpha+2)\prod_{j=1}^{r}\left(1-\frac{4r-4j}{(2N+2\alpha+2r-1)}\right)} \sum_{n=N+1}^{\infty} \frac{\sqrt{2}}{(2n+2\alpha+2r-1)^r} \\
&\leq \frac{2^r U}{(N+2\alpha+2)\prod_{j=1}^{r}\left(1-\frac{4r-4j}{(2N+2\alpha+2r-1)}\right)} \int_N^{\infty} \frac{\sqrt{2}}{(2x+2\alpha+2r-1)^r} dx \\
&= \frac{2^{r-\frac{1}{2}} U}{(r-1)(N+2\alpha+2)\prod_{j=1}^{r-1}(2N+2\alpha-2r+4j-1)}. \qquad (21)
\end{aligned}$$

Then (16) holds for $-1 < \alpha \leq \frac{-1}{2}$. By the similar method as above, the result holds for $\alpha > \frac{-1}{2}$. □

4 Comparison results

In the following, comparison results are provided to investigate the quality of the upper bounds discussed in this paper.

Remark 1. *The aim of this remark is to draw a comparison between our proposed upper bound* (13) *and the upper bound* (12)*. Note that the only difference between the upper bounds* (12) *and* (13) *are the denominators of the fractions which the denominators of the fractions corresponding to the upper bounds* (12) *and* (13) *are denoted by $\beta_{n,r,\alpha}$ and $\theta_{n,r,\alpha}$, respectively.*

Moreover, the numerical results for the values of $\beta_{n,r,\alpha}$ and $\theta_{n,r,\alpha}$ are listed in Table 1. The numerical results obtained from this table indicate that in all cases, the value of $\theta_{n,r,\alpha}$ is smaller than that of $\beta_{n,r,\alpha}$. In particular, for the case that r is close to n, the value of $\theta_{n,r,\alpha}$ is much smaller than $\beta_{n,r,\alpha}$. Therefore, it reveals from this remark that the proposed upper bound (13) *is sharper than the upper bound* (12)*.*

Table 1: A comparison between the values of $\beta_{n,r,\alpha}$ and $\theta_{n,r,\alpha}$.

n	r	α	$\beta_{n,r,\alpha}$	$\theta_{n,r,\alpha}$	n	r	α	$\beta_{n,r,\alpha}$	$\theta_{n,r,\alpha}$
5	2	0	0.01	0.008	20	10	-0.5	1.46×10^{-15}	1.49×10^{-16}
10	6	0	6.87×10^{-8}	1.66×10^{-8}	25	20	-0.5	7.38×10^{-30}	1.62×10^{-33}
20	15	0	7.93×10^{-22}	3.26×10^{-24}	30	27	-0.5	1.69×10^{-40}	1.82×10^{-46}
5	2	0.5	0.0083	0.0071	20	10	1	5.6×10^{-16}	6.78×10^{-17}
10	6	0.5	2.2×10^{-5}	1.5×10^{-6}	25	20	1	8.56×10^{-31}	3.42×10^{-34}
20	15	0.5	4.3×10^{-22}	2.06×10^{-24}	30	27	1	8.42×10^{-42}	2.35×10^{-47}

Example 1. In this example, we compare the approximation error of $f_N(x)$ for two functions $f(x) = x^{20}$ and $f(x) = 1 + x - \frac{1}{2}e^{-5(x-\frac{1}{2})^2}$. Figure 1 shows the numerical results of comparing the approximation error of $f_N(x)$ obtained from Theorem 5 and [6, Theorem 2.1] for these two functions.

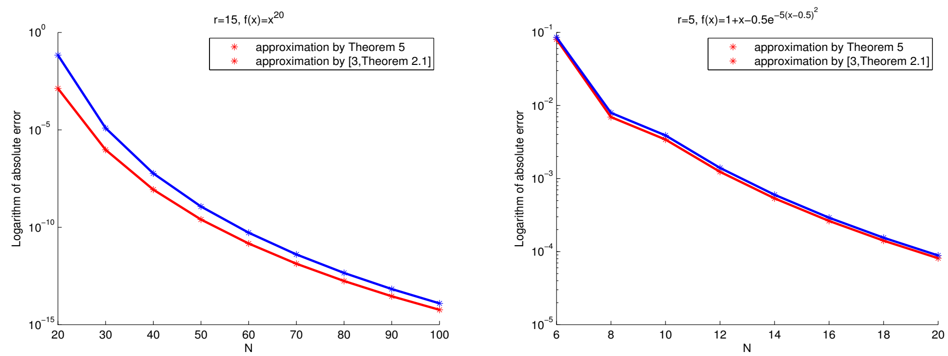

Figure 1: Approximation error of $f_N(x)$ evaluated by Theorem 5 and [6, Theorem 2.1].

4.1 A new and sharper bound for Legendre coefficients

Legendre polynomials $P_n(x)$ are the special cases of ultraspherical Jacobi polynomials with $\alpha = 0$ and satisfies the following inequality [4]:

$$(1-x^2)^{\frac{1}{4}}\,|P_n(x)| < \sqrt{\frac{2}{\pi(n+\frac{1}{2})}}, \quad x \in [-1,1],\ n \geq 0. \tag{22}$$

In [5, Theorem 2.2], Wang presented a new upper bound for Legendre coefficients of differentiable functions which is sharper than the upper bounds proposed so far.

Theorem 6. *[5] Suppose that $f, f', \ldots, f^{(r-1)}$ are absolutely continuous and the r^{th} derivative $f^{(r)}$ is of bounded variation and*

$$V_r = \|f^{(r)}\|_1 = \int_{-1}^{1} \frac{|f^{(r+1)}(x)|}{\sqrt[4]{1-x^2}} < \infty. \tag{23}$$

Then for $n \geq r+1$

$$|a_n| \leq \frac{2V_r}{\sqrt{\pi(2n-2r-1)}} \prod_{j=1}^{r} h_{n-j}, \tag{24}$$

where $h_{n-j} = (n-j+\frac{1}{2})^{-1}$.

By using (10), (22) and similar method as in the proof of Theorem 4, we can obtain a sharper upper bound as follows:

Theorem 7. *Suppose $f, f', \ldots, f^{(r-1)}$ are absolutely continuous and the r^{th} derivative $f^{(r)}$ is of bounded variation. Furthermore, let $V_r = \int_{-1}^{1} \frac{f^{(r+1)}(x)}{\sqrt[4]{1-x^2}} < \infty$. Then for $n \geq r+1$*

$$|u_n| \leq \begin{cases} \frac{\sqrt{2(n+r+\frac{3}{2})}V_r}{(n-r+\frac{1}{2})\cdots(n-\frac{1}{2})(n+\frac{3}{2})\cdots(n+r+\frac{1}{2})\sqrt{\pi}}, & r\ odd, \\ \frac{\sqrt{2(n+r+\frac{3}{2})}V_r}{(n-r+\frac{1}{2})\cdots(n-\frac{3}{2})(n+\frac{1}{2})\cdots(n+r+\frac{1}{2})\sqrt{\pi}}, & r\ even. \end{cases} \tag{25}$$

Proof. The Legendre coefficient of the function $f^{(r+1)}(x)$ is defined by

$$u_n^{(r+1)} = (n+\frac{1}{2})\int_{-1}^{1} f^{(r+1)}(x)P_n(x)dx. \tag{26}$$

By using (22), we have the following bound

$$\left|u_n^{(r+1)}\right| = (n+\frac{1}{2})\int_{-1}^{1}\left|f^{(r+1)}(x)\right||P_n(x)|\,dx \le \frac{\sqrt{2}(n+\frac{1}{2})}{\sqrt{\pi(n+\frac{1}{2})}}\int_{-1}^{1}\frac{\left|f^{(r+1)}(x)\right|}{\sqrt[4]{1-x^2}}dx = \frac{V_r\sqrt{2(n+\frac{1}{2})}}{\sqrt{\pi}}. \tag{27}$$

From (10) and (27), we obtain the following upper bound for $u_n^{(r)}$

$$\left|u_n^{(r)}\right| \le \frac{1}{2n+1}\left(\left|u_{n+1}^{(r+1)}\right| + \left|u_{n-1}^{(r+1)}\right|\right) \le \frac{V_r\sqrt{2(n+\frac{3}{2})}}{\sqrt{\pi}(n+\frac{1}{2})}. \tag{28}$$

By using the similar methods as in the proof of Theorem 4 and (10), the result holds. □

5 New error estimate for Gauss-Jacobi quadrature

The Gauss-type quadrature has been studied in several articles for analytic and differentiable functions on $[-1,1]$ [3, 6, 7]. In [3, 6] the authors obtained error estimates for the Gauss-type quadrature with the weight function $w(x) = 1$ using the Chebyshev series. Although the decay of the Chebyshev coefficients is faster than the other Jacobi coefficients, but the proposed error estimates are for a specific class of differentiable functions. The error estimates has been obtained for a differentiable function f such that $f, f', \ldots, f^{(r-1)}$ are absolutely continuous and the r^{th} derivative $f^{(r)}$ is of bounded variation and

$$V_r = \|f^{(r)}\|_1 = \int_{-1}^{1}\frac{|f^{(r+1)}(x)|}{\sqrt{1-x^2}} < \infty. \tag{29}$$

In this section, we estimate the Gauss-Jacobi quadrature to numerically compute integrals of the form $\int_{-1}^{1}(1-x^2)^{\alpha}f(x)dx$. We obtain an error estimate for Gauss-Jacobi quadrature of a differentiable function f such that $f, f', \ldots, f^{(r)}$ are continuous on $[-1,1]$ and $U < \infty$.

The Gauss-Jacobi quadrature formula in the ultraspherical case associated to the weight function $w(x) = (1-x^2)^{\alpha}$ on $[-1,1]$ satisfies

$$I[f] = \int_{-1}^{1}(1-x^2)^{\alpha}f(x)dx \approx \sum_{i=1}^{N} w_i f(x_i) = Q_N^{GJ}[f]. \tag{30}$$

Now we provide an error bound for Gauss-Jacobi quadrature of differentiable functions. Using the orthogonality of the Jacobi polynomials leads to

$$I[P_n^{\alpha}(x)] = \int_{-1}^{1}(1-x^2)^{\alpha}P_n^{\alpha}(x)dx = \int_{-1}^{1}(1-x^2)^{\alpha}P_n^{\alpha}(x)P_0^{\alpha}(x)dx = 0, \quad n \ge 1. \tag{31}$$

Since the Gauss quadrature is accurate for polynomials of degree $\le 2N-1$, therefore

$$I[P_n^{\alpha}(x)] = Q_N^{GJ}[P_n^{\alpha}(x)] = 0, \quad 0 \le n \le 2N-1, \tag{32}$$

So, we can see that

$$\left|I[f]-Q_N^{GJ}[f]\right| \leq \sum_{n=0}^{\infty} |u_n| \left|I[P_n^{\alpha}(x)]-Q_N^{GJ}[P_n^{\alpha}(x)]\right| = \sum_{n=2N}^{\infty} |u_n| \left|Q_N^{GJ}[P_n^{\alpha}(x)]\right|. \tag{33}$$

So, (33) implies that the error bound for Gauss-Jacobi quadrature depends on the Jacobi coefficients. In the following theorem, we provide an error estimate on the Gauss-Jacobi quadrature for $-1<\alpha\leq 0$.

Theorem 8. *Let $f, f', \cdots, f^{(r)}$ be continuous on $[-1,1]$ and $U<\infty$. Then for $N>\frac{2r+1}{4}$ and $r\geq 2$*

$$\left|I[f]-Q_N^{GJ}[f]\right| \leq \begin{cases} \frac{2^r\Gamma(\alpha+1)\delta_\alpha\gamma_0 U\sqrt{2}}{(r-1)\Gamma(2\alpha+1)\sqrt{N}\prod_{j=1}^{r-1}(4N+2\alpha-2r+4j-3)}, & -1<\alpha<\frac{-1}{2}, \\ \frac{2^r\delta_\alpha U\gamma_0\sqrt{2}}{(r-1)\Gamma(2\alpha+1)N\prod_{j=1}^{r-1}(4N+2\alpha-2r+4j-3)}, & \alpha=\frac{-1}{2}, \\ \frac{2^r\delta_\alpha U\gamma_0\sqrt{2}}{(r-1)\Gamma(2\alpha+1)\prod_{j=1}^{r-1}(4N+2\alpha-2r+4j-3)}, & \frac{-1}{2}<\alpha\leq 0. \end{cases} \tag{34}$$

Proof. By using (33) we have

$$\left|I[f]-Q_N^{GJ}[f]\right| = \left|\sum_{n=2N}^{\infty} u_n Q_N^{G}[P_n^{\alpha}(x)]\right| \leq \sum_{n=2N}^{\infty} |u_n||Q_N^{GJ}[P_n^{\alpha}(x)]|. \tag{35}$$

Now, we compute $\left|Q_N^{GJ}[P_n^{\alpha}(x)]\right|$ to obtain an upper bound for Gauss-Jacobi quadrature. Then

$$\left|Q_N^{GJ}[P_n^{\alpha}(x)]\right| = \left|\sum_{i=1}^{N} w_i P_n^{\alpha}(x_i)\right| \leq \sum_{i=1}^{N} w_i |P_n^{\alpha}(x_i)|. \tag{36}$$

Using (5) and (6) and by easy computations we have the following inequality for all $x\in[-1,1]$

$$|P_n^{\alpha}(x)| \leq \begin{cases} \frac{\Gamma(\alpha+1)\Gamma(n+2\alpha+1)}{\Gamma(2\alpha+1)\Gamma(n+\alpha+1)\sqrt{n}}, & -1<\alpha<\frac{-1}{2}, \\ \frac{\Gamma(n+2\alpha+1)}{\Gamma(2\alpha+1)\Gamma(n+1)}, & \frac{-1}{2}\leq\alpha. \end{cases} \tag{37}$$

So (36) becomes

$$\begin{aligned} \left|Q_N^{GJ}[P_n^{\alpha}(x)]\right| &\leq \begin{cases} \frac{\Gamma(\alpha+1)\Gamma(n+2\alpha+1)}{\Gamma(2\alpha+1)\Gamma(n+\alpha+1)\sqrt{n}} \sum_{i=1}^{N} w_i, & -1<\alpha<\frac{-1}{2}, \\ \frac{\Gamma(n+2\alpha+1)}{\Gamma(2\alpha+1)\Gamma(n+1)} \sum_{i=1}^{N} w_i, & \frac{-1}{2}\leq\alpha, \end{cases} \\ &= \begin{cases} \frac{\Gamma(\alpha+1)\Gamma(n+2\alpha+1)}{\Gamma(2\alpha+1)\Gamma(n+\alpha+1)\sqrt{n}} \int_{-1}^{1}(1-x^2)^{\alpha}dx, & -1<\alpha<\frac{-1}{2}, \\ \frac{\Gamma(n+2\alpha+1)}{\Gamma(2\alpha+1)\Gamma(n+1)} \int_{-1}^{1}(1-x^2)^{\alpha}dx, & \frac{-1}{2}\leq\alpha. \end{cases} \end{aligned} \tag{38}$$

Combining (5) with (7), we obtain

$$\int_{-1}^{1}(1-x^2)^{\alpha}dx = \int_{-1}^{1}(P_0^{\alpha}(x))^2(1-x^2)^{\alpha}dx = \gamma_0. \tag{39}$$

So by applying (38) and (39) we obtain

$$\left|I[f]-Q_N^{GJ}[f]\right| \leq \begin{cases} \sum_{n=2N}^{\infty} \frac{\Gamma(\alpha+1)\Gamma(n+2\alpha+1)\gamma_0}{\Gamma(2\alpha+1)\Gamma(n+\alpha+1)\sqrt{n}}|u_n|, & -1<\alpha<\frac{-1}{2}, \\ \sum_{n=2N}^{\infty} \frac{\Gamma(n+2\alpha+1)\gamma_0}{\Gamma(2\alpha+1)\Gamma(n+1)}|u_n|, & \frac{-1}{2}\leq\alpha. \end{cases} \tag{40}$$

If $-1<\alpha<\frac{-1}{2}$, we can see that $\frac{\Gamma(n+2\alpha+1)}{\Gamma(n+\alpha+1)}<1$ and $\lambda_{n+r,\alpha}\leq\sqrt{2}$. So (40) becomes

$$
\begin{aligned}
\left|I[f]-Q_N^{GJ}[f]\right| &\leq \sum_{n=2N}^{\infty}\frac{2^r\Gamma(\alpha+1)\delta_\alpha\lambda_{n+r,\alpha}\gamma_0 U}{\Gamma(2\alpha+1)\sqrt{n}\prod_{j=1}^{r-1}(2n+2\alpha-2r+4j-1)}\\
&\leq \sum_{n=2N}^{\infty}\frac{2^r\Gamma(\alpha+1)\delta_\alpha\gamma_0 U\sqrt{2}}{\Gamma(2\alpha+1)\sqrt{n}\prod_{j=1}^{r-1}(2n+2\alpha-2r+4j-1)}\\
&\leq \frac{2^r\Gamma(\alpha+1)\delta_\alpha\gamma_0 U\sqrt{2}}{\Gamma(2\alpha+1)\sqrt{N}}\sum_{n=2N}^{\infty}\frac{1}{\prod_{j=1}^{r-1}(2n+2\alpha-2r+4j-1)}.
\end{aligned}
$$

If $\frac{-1}{2}<\alpha\leq 0$, we can see that $\frac{\Gamma(n+2\alpha+1)}{\Gamma(n+1)}\leq 1$ and $\lambda_{n-r,\alpha}\leq\sqrt{2}$. So we have

$$
\left|I[f]-Q_N^{GJ}[f]\right|\leq\frac{2^r\delta_\alpha U\gamma_0\sqrt{2}}{\Gamma(2\alpha+1)}\sum_{n=2N}^{\infty}\frac{1}{\prod_{j=1}^{r-1}(2n+2\alpha-2r+4j-1)}.
$$

Also, for the case that $\alpha=\frac{-1}{2}$, we have $\frac{\Gamma(n+2\alpha+1)}{\Gamma(n+1)}=\frac{1}{n}$. So we get

$$
\left|I[f]-Q_N^{GJ}[f]\right|\leq\frac{2^r\delta_\alpha U\gamma_0\sqrt{2}}{\Gamma(2\alpha+1)N}\sum_{n=2N}^{\infty}\frac{1}{\prod_{j=1}^{r-1}(2n+2\alpha-2r+4j-1)}.
$$

By the similar method as in the proof of Theorem 5, the result holds. □

Remark 2. *The authors in [6] obtained error estimates for the Gauss-Jacobi quadrature for $w(x)=1$. Moreover, they showed that for functions with the conditions stated in the relation (29), the error estimates for the Gauss-Jacobi quadrature is equal to $O(N^{-r})$. Taking this discussion into consideration, in Theorem 8, we have presented the approximation errors for the Gauss-Jacobi quadrature for the weight function $w(x)=(1-x^2)^\alpha$, $-1<\alpha\leq 0$. In particular, we show that in the special case $\alpha=\frac{-1}{2}$, the error estimates for the Gauss-Jacobi quadrature is equal to $O(N^{-r})$.*

6 Conclusion

In this paper a new upper bounds for the ultraspherical Jacobi coefficients of differentiable functions were obtained (see Theorem 4). We compared these upper bounds by previous upper bounds which are presented in [6]. Moreover, we provided upper bounds on the approximation error of a function $f(x)$ by truncated ultraspherical Jacobi polynomial series. In specially, we provided a sharper upper bounds for Legendre coefficients. Finally, we derived a new error estimate on the Gauss-Jacobi quadrature.

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