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ON SOLUTIONS OF THE DIOPHANTINE EQUATION $F_{n_1} + F_{n_2} + F_{n_3} + F_{n_4} = 2^a$

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ABSTRACT. Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_0 = 0, F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$. In this paper, we solve all powers of two which are sums of four Fibonacci numbers with a few exceptions that we characterize.

1. INTRODUCTION

The equation $F_n - F_m = y^a$ has been well-studied. For instance Z. Siar and R. Keskin [1] have found all the solutions for $y = 2$, B. Demirtürk et al [2] and P. Tiebekabe et al [3] have independently determined all the solutions for $y = 3$ and finally F. Erduvan and R. Keskin [6] determined all solutions for $y = 5$ and conjectured that there are no solutions for $y > 7$.

There are comparatively fewer works on the equation $F_n + F_m = y^a$. J. J. Bravo and Luca [7] solved the case $y = 2$, and their result has been generalized by Pink and Ziegler in [8]. J. J. Bravo and E. Bravo [10] determined all solutions to the similar equation $F_n + F_m + F_l = 2^a$; and wrote that they expect that the equation

$$F_{n_1} + F_{n_2} + F_{n_3} + F_{n_4} = 2^a \tag{1.1}$$

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may be handled using the same method. Since the solution would involve more cases and harder computations, they leave this equation for other researchers. We have decided to tackle this difficult case.

Many problems similar to the one discussed in this paper have been investigated for the Fibonacci and Lucas sequences. For example, repdigits which are sums of at most three Fibonacci numbers were found in [12]; repdigits as sums of four Fibonacci or Lucas numbers were found in [15]; Fibonacci numbers which are sums of two repdigits were obtained in [14], while factorials which are sums of at most three Fibonacci numbers were found in [13]. In 2020, Das, A. and Saha [5], M. determined the number of Some families of cubic graphs.

Recall that the Zeckendorf representation [11] of a positive integer N is the representation

$$N = F_{m_1} + F_{m_2} + \ldots + F_{m_t}; \quad \text{with} \quad m_i - m_{i+1} \geqslant 2 \quad \text{for} \quad i = 1, \ldots, t-1.$$

Equation (1.1) is a particular case of Zeckendorf representation with $N = 2^a$ and $t = 4$.

This paper is subdivided as follows: In Section 2, we introduce auxiliary results used in Section 3 to prove the main theorem of this paper stated below.

Theorem 1.1. *All non-trivial solutions of the Diophantine equation (1.1) in positive integers n_1, n_2, n_3, n_4 and a with $n_1 \geqslant n_2 \geqslant n_3 \geqslant n_4$ are:*

$F_5 + 3F_2 = 2^3$	$F_7 + 3F_2 = 2^4$	$2F_4 + 2F_2 = 2^3$	$F_{13} + F_8 + 2F_2 = 2^8$
$F_4 + 2F_3 + F_2 = 2^3$	$F_6 + F_5 + F_3 + F_2 = 2^4$	$F_8 + F_6 + F_3 + F_2 = 2^5$	$F_{16} + F_9 + F_3 + F_2 = 2^{10}$
$F_{10} = F_5 + F_4 + F_2 = 2^6$	$3F_5 + F_2 = 2^4$	$F_8 + 2F_5 + F_2 = 2^5$	$2F_7 + F_5 + F_2 = 2^5$
$F_9 + F_8 + F_6 + F_2 = 2^6$	$3F_8 + F_2 = 2^6$	$F_{10} + F_5 + 2F_3 = 2^6$	$F_6 + 2F_4 + F_3 = 2^4$
$F_{11} + F_9 + F_4 + F_3 = 2^7$	$F_{13} + F_7 + F_6 + F_3 = 2^8$	$F_{12} + F_{11} + F_8 + F_3 = 2^8$	$F_{12} + 2F_{10} + F_3 = 2^8$
$F_{10} + 3F_4 = 2^6$	$2F_5 + 2F_4 = 2^4$	$F_8 + F_5 + 2F_4 = 2^5$	$2F_7 + 2F_4 = 2^5$
$F_7 + 2F_6 + F_4 = 2^5$	$F_{16} + F_8 + F_7 + F_4 = 2^{10}$	$F_{15} + F_{14} + F_9 + F_4 = 2^{10}$	$F_{13} + F_7 + 2F_5 = 2^8$
$F_{11} + F_8 + F_7 + F_5 = 2^7$	$2F_{10} + F_7 + F_5 = 2^7$	$F_{10} + 2F_9 + F_5 = 2^7$	$F_{16} + F_8 + 2F_6 = 2^{10}$
$F_{11} + 3F_7 = 2^7$	$4F_6 = 2^5$		

2. Auxiliary results

In this section, we give some important known definitions, proprieties, theorem and lemmas.

Definition 2.1. *For an algebraic number γ, we define its measure by the following identity :*

$$\mathrm{M}(\gamma) = |a_d| \prod_{i=1}^{d} \max\{1, |\gamma_i|\},$$

where γ_i are the roots of $f(x) = a_d \prod_{i=1}^{d}(x - \gamma_i)$, the minimal polynomial of γ.

Let us define now another height, deduced from the last one, called the absolute logarithmic height.

Definition 2.2. *(Absolute logarithmic height). For a non-zero algebraic number of degree d on $\mathbb{Q}$ where the minimal polynomial on $\mathbb{Z}$ is $f(x) = a_d \prod_{i=1}^{d}(x - \gamma_i)$, we denote by*

$$h(\gamma) = \frac{1}{d}\left(\log|a_d| + \sum_{i=1}^{d} \log\max\{1, |\gamma_i|\}\right) = \frac{1}{d}\log \mathrm{M}(\gamma).$$

the usual absolute logarithmic height of γ.

The following properties of the logarithmic height are well-known:

- $h(\gamma \pm \eta) \leq h(\gamma) + h(\eta) + \log 2$;
- $h(\gamma\eta^{\pm 1}) \leq h(\gamma) + h(\eta)$;
- $h(\gamma^k) = |k| h(\gamma) \quad k \in \mathbb{Z}$.

The nth Fibonacci number can be represented as

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \quad \text{for all} \quad n \geqslant 0.$$

where $(\alpha, \beta) := ((1+\sqrt{5})/2, (1-\sqrt{5})/2)$. The following inequalities

$$\alpha^{n-2} \leqslant F_n \leqslant \alpha^{n-1}$$

are well-known to hold for all $n \geqslant 1$ and can be proved by induction on n. The following theorem is deduced from Corollary 2.3 of Matveev [17].

Theorem 2.1 (Matveev [17])**.** *Let $n \geq 1$ an integer. Let $\mathbb{L}$ be a field of algebraic number of degree D. Let $\eta_1, \ldots, \eta_l$ non-zero elements of $\mathbb{L}$ and let $b_1, b_2, \ldots, b_l$ integers,*

$$B := \max\{|b_1|, ..., |b_l|\},$$

and

$$\Lambda := \eta_1^{b_1} \cdots \eta_l^{b_l} - 1 = \left(\prod_{i=1}^{l} \eta_i^{b_i}\right) - 1.$$

Let $A_1, \ldots, A_l$ reals numbers such that

$$A_j \geq \max\{Dh(\eta_j), |\log(\eta_j)|, 0.16\}, 1 \leq j \leq l.$$

Assume that $\Lambda \neq 0$, So we have

$$\log|\Lambda| > -3 \times 30^{l+4} \times (l+1)^{5.5} \times d^2 \times A_1 \ldots A_l (1+\log D)(1+\log nB).$$

Further, if $\mathbb{L}$ is real, then

$$\log|\Lambda| > -1.4 \times 30^{l+3} \times (l)^{4.5} \times d^2 \times A_1 \ldots A_l (1+\log D)(1+\log B).$$

The following two Lemmas are due Dujella and Pethő, and to Legendre respectively.

For a real number X, we write $\|X\| := \min\{| X - n | : n \in \mathbb{Z}\}$ for the distance of X to the nearest integer.

Lemma 2.1. *(Dujella and Pethő, [16]) Let M a positive integer, let p/q the convergent of the continued fraction expansion of κ such that $q > 6M$ and let A, B, μ real numbers such that $A > 0$ and $B > 1$. Let $\varepsilon := \|\mu q\| - M\,\|\kappa q\|$.*
If $\varepsilon > 0$ then there is no solution of the inequality

$$0 < m\kappa - n + \mu < AB^{-m}$$

in integers m and n with

$$\frac{\log(Aq/\varepsilon)}{\log B} \leqslant m \leqslant M.$$

Lemma 2.2. *(Legendre) Let τ real number such that x, y are integers such that*

$$\left|\tau - \frac{x}{y}\right| < \frac{1}{2y^2},$$

then $\dfrac{x}{y} = \dfrac{p_k}{q_k}$ is a convergent of τ.

Further,

$$\left|\tau - \frac{x}{y}\right| > \frac{1}{(q_{k+1}+2)y^2}.$$

3. Main result

Proof. Assume that

$$F_{n_1} + F_{n_2} + F_{n_3} + F_{n_4} = 2^a$$

holds.
Let us first find relation between n_1 and a.

Combining equation(1.1) with the well-known inequality $F_n \leqslant \alpha^{n-1}$ for all $n \geqslant 1$, one gets that

$$\begin{aligned} F_{n_1}+F_{n_2}+F_{n_3}+F_{n_4}=2^a \leqslant & \alpha^{n_1-1}+\alpha^{n_2-1}+\alpha^{n_3-1}+\alpha^{n_4-1} \\ < & 2^{n_1-1}+2^{n_2-1}+2^{n_3-1}+2^{n_4-1} \quad \because \alpha<2 \\ < & 2^{n_1-1}\left(1+2^{n_2-n_1}+2^{n_3-n_1}+2^{n_4-n_1}\right) \\ \leqslant & 2^{n_1-1}\left(1+1+2^{-1}+2^{-2}\right) \\ = & 2^{n_1-1}\left(2+2^{-1}+2^{-2}\right) \\ < & 2^{n_1+1}. \end{aligned}$$

Hence

$$2^a < 2^{n_1+1} \Longrightarrow a < n_1+1 \Longrightarrow a \leqslant n_1.$$

This inequality will help us to calculate some parameters.

Rewriting equation (1.1), we get

$$\frac{\alpha^{n_1}}{\sqrt{5}} - 2^a = \frac{\beta^{n_1}}{\sqrt{5}} - (F_{n_2}+F_{n_3}+F_{n_4}).$$

Taking absolute values on the above equation, we obtain

$$\left|\frac{\alpha^{n_1}}{\sqrt{5}} - 2^a\right| \leqslant \left|\frac{\beta^{n_1}}{\sqrt{5}}\right| + (F_{n_2}+F_{n_3}+F_{n_4}) < \frac{|\beta|^{n_1}}{\sqrt{5}} + (\alpha^{n_2}+\alpha^{n_3}+\alpha^{n_4}),$$

and

$$\left|\frac{\alpha^{n_1}}{\sqrt{5}} - 2^a\right| < \frac{1}{2} + \left(\alpha^{n_2}+\alpha^{n_3}+\alpha^{n_4}\right),$$

where we used $\quad F_n \leqslant \alpha^{n-1}$.

Dividing both sides of the above equation by $\alpha^{n_1}/\sqrt{5}$, we get

$$\begin{aligned} \left|1-2^a\cdot\alpha^{-n_1}\cdot\sqrt{5}\right| < & \frac{\sqrt{5}}{2\alpha^{n_1}} + \left(\alpha^{n_2-n_1}+\alpha^{n_3-n_1}+\alpha^{n_4-n_1}\right)\sqrt{5} \\ < & \frac{\sqrt{5}}{2\alpha^{n_1}} + \frac{\sqrt{5}}{\alpha^{n_1-n_2}} + \frac{\sqrt{5}}{\alpha^{n_1-n_3}} + \frac{\sqrt{5}}{\alpha^{n_1-n_4}}. \end{aligned}$$

Taking into account the assumption $n_4 \leqslant n_3 \leqslant n_3 \leqslant n_2 \leqslant n_1$, we get

$$|\Lambda_1| = \left|1-2^a\cdot\alpha^{-n_1}\cdot\sqrt{5}\right| < \frac{9}{\alpha^{n_1-n_2}} \tag{3.1}$$

Let us apply Matveev's theorem, with the following parameters $t := 3$ and

$$\gamma_1 := 2, \quad \gamma_2 := \alpha, \quad \gamma_3 := \sqrt{5}, \quad b_1 := a, \quad b_2 := -n, \quad \text{and} \quad b_3 := 1.$$

Since $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{K} := \mathbb{Q}(\sqrt{5})$, we can take $D := 2$. Before applying Matveev's theorem, we have to check the last condition: the left-hand side of (3.1) is not zero. Indeed, if it were zero, we would then get that $2^a\sqrt{5} = \alpha^n$. Squaring the previous relation, we get $\alpha^{2n} = 5\cdot 2^{2a} = 5\cdot 4^a$. This implies that $\alpha^{2n} \in \mathbb{Z}$, which is impossible, so we conclude that $\Lambda_1 \neq 0$.

The logarithmic height of γ_1, γ_2 and γ_3 are:

$h(\gamma_1) = \log 2 = 0.6931\ldots$, so we can choose $A_1 := 1.4$.

$h(\gamma_2) = \dfrac{1}{2}\log\alpha = 0.2406\ldots$, so we can choose $A_2 := 0.5$.

$h(\gamma_3) = \log\sqrt{5} = 0.8047\ldots$, it follows that we can choose $A_3 := 1.7$.

Since $a < n_1 + 1$, $B := \max\{|b_1|, |b_2|, |b_3|\} = n_1$. Matveev's result informs us that

$$\left|1 - 2^a \cdot \alpha^{n_1} \cdot \sqrt{5}\right| > \exp\left(-c_1 \cdot (1+\log n) \cdot 1.4 \cdot 0.5 \cdot 1.7\right), \tag{3.2}$$

where $c_1 := 1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 \cdot (1+\log 2) < 9.7 \times 10^{11}$.

Taking log in inequality (3.1), we get

$$\log|\Lambda_1| < \log 9 - (n_1 - n_2)\log\alpha.$$

Taking log in inequality (3.2), we get

$$\log|\Lambda_1| > 2.31 \times 10^{12} \log n_1.$$

Comparing the previous two inequalities, we get

$$(n_1 - n_2)\log\alpha - \log 9 < 2.31 \times 10^{12}\log n_1,$$

where we used $1 + \log n_1 < 2\log n_1$ which holds for all $n_1 \geqslant 3$. Then we have

$$(n_1 - n_2)\log\alpha < 2.32 \times 10^{12}\log n_1. \tag{3.3}$$

Let us now consider a second linear form in logarithms. We rewrite equation (1.1) as follows

$$\frac{\alpha^{n_1}}{\sqrt{5}} + \frac{\alpha^{n_2}}{\sqrt{5}} - 2^a = \frac{\beta^{n_1}}{\sqrt{5}} + \frac{\beta^{n_2}}{\sqrt{5}} - \left(F_{n_3} + F_{n_4}\right).$$

Taking absolute values on the above equation and the fact that $\beta = (1-\sqrt{5})/2$, we get

$$\begin{aligned}\left|\frac{\alpha^{n_1}}{\sqrt{5}}\left(1+\alpha^{n_2-n_1}\right) - 2^a\right| &\leqslant \frac{|\beta|^{n_1} + |\beta|^{n_2}}{\sqrt{5}} + F_{n_3} + F_{n_4} \\ &< \frac{1}{3} + \alpha^{n_3} + \alpha^{n_4} \quad \text{for all} \quad n_1 \geqslant 5 \quad \text{and} \quad n_2 \geqslant 5.\end{aligned}$$

Dividing both sides of the above inequality by $\dfrac{\alpha^{n_1}}{\sqrt{5}}\left(1+\alpha^{n_2-n_1}\right)$, we obtain

$$|\Lambda_2| = \left|1 - 2^a \cdot \alpha^{n_1} \cdot \sqrt{5}\left(1+\alpha^{n_2-n_1}\right)^{-1}\right| < \frac{6}{\alpha^{n_2-n_1}}. \tag{3.4}$$

Let us apply Matveev's theorem second time with the follow parameters

$$t := 3, \quad \gamma_1 := 2, \quad \gamma_2 := \alpha, \quad \gamma_3 := \sqrt{5}\left(1 + \alpha^{n_2 - n_1}\right)^{-1},$$

$$b_1 := a, \quad b_2 := -n_1, \quad \text{and} \quad b_3 := 1.$$

Since $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{K} := \mathbb{Q}(\sqrt{5})$, we can take $D := 2$. The left hand side of (3.4) is not zero, otherwise, we would get the relation

$$2^a\sqrt{5} = \alpha^{n_1} + \alpha^{n_2}. \tag{3.5}$$

Conjugating (3.5) in the field $\mathbb{Q}(\sqrt{5})$, we get

$$-2^a\sqrt{5} = \beta^{n_1} + \beta^{n_2}. \tag{3.6}$$

Combining (3.5) and (3.6), we get

$$\alpha^{n_1} < \alpha^{n_1} + \alpha^{n_2} = |\beta^{n_1} + \beta^{n_2}| \leqslant |\beta|^{n_1} + |\beta|^{n_2} < 1$$

which is impossible for $n_1 > 350$. Hence $\Lambda_2 \neq 0$. We know that, $h(\gamma_1) = \log 2$ and $h(\gamma_2) = \dfrac{1}{2}\log\alpha$. Let us now estimate $h(\gamma_3)$ by first observing that

$$\gamma_3 = \frac{\sqrt{5}}{1 + \alpha^{n_2 - n_1}} < \sqrt{5} \quad \text{and} \quad \gamma_3^{-1} = \frac{1 + \alpha^{n_2 - n_1}}{\sqrt{5}} < \frac{2}{\sqrt{5}},$$

so that $|\log\gamma_3| < 1$. Using proprieties of logarithmic height stated in Section 2, we have

$$h(\gamma_3) \leqslant \log\sqrt{5} + |n_2 - n_1|\left(\frac{\log\alpha}{2}\right) + \log 2 = \log(2\sqrt{5}) + (n_1 - n_2)\left(\frac{\log\alpha}{2}\right).$$

Hence, we can take $A_3 := 3 + (n_1 - n_2)\log\alpha > \max\{2h(\gamma_3), |\log\gamma_3|, 0.16\}$.

Matveev's theorem implies that

$$\exp\left(-c_2(1 + \log n_1) \cdot 1.4 \cdot 0.5 \cdot (3 + (n_1 - n_2)\log\alpha)\right)$$

where $c_2 := 1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 \cdot (1 + \log 2) < 9.7 \times 10^{11}$.

Since $(1 + \log n_1) < 2\log n_1$ hold for $n_1 \geqslant 3$, from (3.4), we have

$$(n_1 - n_3)\log\alpha - \log 6 < 1.4 \times 10^{12}\log n_1(3 + (n_1 - n_2)\log\alpha). \tag{3.7}$$

Putting relation (3.3) in the right-hand side of (3.7), we get

$$(n_1 - n_3)\log\alpha < 3.29 \times 10^{24}\log^2 n_1. \tag{3.8}$$

Let us consider a third linear form in logarithms. To this end, we again rewrite (1.1) as follows

$$\frac{\alpha^{n_1} + \alpha^{n_2} + \alpha^{n_3}}{\sqrt{5}} - 2^a = \frac{\beta^{n_1} + \beta^{n_2} + \beta^{n_3}}{\sqrt{5}} - F_{n_4}.$$

Taking absolute values on both sides, we obtain

$$\left|\frac{\alpha^{n_1}}{\sqrt{5}}\left(1+\alpha^{n_2-n_1}+\alpha^{n_3-n_1}\right)-2^a\right| \leqslant \frac{|\beta|^{n_1}+|\beta|^{n_2}+|\beta|^{n_3}}{\sqrt{5}}+F_{n_4}<\frac{3}{4}+\alpha^{n_4}$$

for all $\quad n_1>350, \quad$ and $\quad n_2, n_3, n_4 \geqslant 1.$

Thus we have

$$|\Lambda_3|=\left|1-2^a \cdot \alpha^{-n_1} \cdot \sqrt{5}\left(1+\alpha^{n_2-n_1}+\alpha^{n_3-n_1}\right)^{-1}\right|<\frac{3}{\alpha^{n_1-n_4}}. \quad (3.9)$$

In a third application of Matveev's theorem, we can take parameters

$$t:=3, \quad \gamma_1:=2, \quad \gamma_2:=\alpha, \quad \gamma_3:=\sqrt{5}\left(1+\alpha^{n_2-n_1}+\alpha^{n_3-n_1}\right)^{-1},$$

$$b_1:=a, \quad b_2:=-n, \quad \text{and,} \quad b_3:=1.$$

Since $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{K}:=\mathbb{Q}(\sqrt{5})$, we can take $D:=2$. Suppose, for a contradiction, that $|\Lambda_3|=0$. Then

$$2^a \sqrt{5}=\alpha^{n_1}+\alpha^{n_2}+\alpha^{n_3}.$$

Taking the conjugate in the field $\mathbb{Q}(\sqrt{5})$, we get

$$-2^a \sqrt{5}=\beta^{n_1}+\beta^{n_2}+\beta^{n_3},$$

which leads to

$$\alpha^{n_1}<\alpha^{n_1}+\alpha^{n_2}+\alpha^{n_3}=|\beta^{n_1}+\beta^{n_2}+\beta^{n_3}| \leqslant |\beta|^{n_1}+|\beta|^{n_2}+|\beta|^{n_3}<1$$

and leads to a contradiction since $n_1>350$. Hence $\Lambda_3 \neq 0$.

As we did before, we can take $A_1:=1.4, A_2:=0.5$ and $B:=n_1$. We can also see that

$$\gamma_3=\frac{\sqrt{5}}{1+\alpha^{n_2-n_1}+\alpha^{n_3-n_1}}<\sqrt{5} \quad \text{and} \quad \gamma_3^{-1}=\frac{1+\alpha^{n_2-n_1}+\alpha^{n_3-n_1}}{\sqrt{5}}<\frac{3}{\sqrt{5}},$$

so $|\log \gamma_3|<1$. Applying proprieties on logarithmic height, we estimate

$$\begin{aligned} h(\gamma_3) & \leqslant \log \sqrt{5}+|n_2-n_1|\left(\frac{\log \alpha}{2}\right)+|n_3-n_1|\left(\frac{\log \alpha}{2}\right)+\log 3 \\ & =\log(3 \sqrt{5})+(n_1-n_2)\left(\frac{\log \alpha}{2}\right)+(n_1-n_3)\left(\frac{\log \alpha}{2}\right); \end{aligned}$$

so we can take

$$A_3:=4+(n_1-n_2) \log \alpha+(n_1-n_3) \log \alpha>\max\{2 h(\gamma_3),|\log \gamma_3|, 0.16\}.$$

A lower bound on the left-hand side of (3.9) is

$$\exp(-c_3 \cdot(1+\log n_1) \cdot 1.4 \cdot 0.5 \cdot(4+(n_1-n_2) \log \alpha+(n_1-n_3) \log \alpha))$$

where $c_3=1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 \cdot(1+\log 2)<9.7 \times 10^{11}$.

From inequality (3.9), we have

$$(n_1 - n_4)\log\alpha < 1.4 \times 10^{12} \log n_1 \cdot (4 + (n_1 - n_2)\log\alpha + (n_1 - n_3)\log\alpha). \tag{3.10}$$

Combining equation (3.3) and (3.8) in the right-most terms of equation (3.10) and performing the respective calculations, we get

$$(n_1 - n_4)\log\alpha < 9.3 \times \log^3 n_1. \tag{3.11}$$

Let us now consider a fourth and last linear form in logarithms. Rerwriting (1.1) once again by separating large terms and small terms, we get

$$\frac{\alpha^{n_1} + \alpha^{n_2} + \alpha^{n_3} + \alpha^{n_4}}{\sqrt{5}} - 2^a = \frac{\beta^{n_1} + \beta^{n_2} + \beta^{n_3} + \beta^{n_4}}{\sqrt{5}}.$$

Taking absolute values on both sides, we get

$$\begin{aligned}\left|\frac{\alpha^{n_1}}{\sqrt{5}}\left(1 + \alpha^{n_2-n_1} + \alpha^{n_3-n_1} + \alpha^{n_4-n_1}\right) - 2^a\right| &\leqslant \frac{|\beta|^{n_1} + |\beta|^{n_2} + |\beta|^{n_3} + |\beta|^{n_4}}{\sqrt{5}} \\ &< \frac{4}{5}\end{aligned}$$

for all $\quad n_1 > 350, \quad$ and $\quad n_2, n_3, n_4 \geqslant 1$.

Dividing both sides of the above relation by the fist term of the RHS of the previous equation, we get

$$|\Lambda_4| = \left|1 - 2^a \cdot \alpha^{-n_1} \cdot \sqrt{5}\left(1 + \alpha^{n_2-n_1} + \alpha^{n_3-n_1} + \alpha^{n_4-n_1}\right)^{-1}\right| < \frac{2}{\alpha^{n_1}}. \tag{3.12}$$

In the last application of Matveev's theorem, we have the following parameters

$$\gamma_1 := 2, \quad \gamma_2 := \alpha, \quad \gamma_3 := \sqrt{5}\left(1 + \alpha^{n_2-n_1} + \alpha^{n_3-n_1} + \alpha^{n_4-n_1}\right)^{-1},$$

and we can also take $b_1 := a, \quad b_2 := -n$ and $b_3 := 1$. Since $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{K} := \mathbb{Q}(\sqrt{5})$, we can take $D := 2$. Suppose, for a contradiction, that $|\Lambda_4| = 0$. Then

$$2^a\sqrt{5} = \alpha^{n_1} + \alpha^{n_2} + \alpha^{n_3} + \alpha^{n_4}.$$

Conjugating the above relation in the field $\mathbb{Q}(\sqrt{5})$, we get

$$-2^a\sqrt{5} = \beta^{n_1} + \beta^{n_2} + \beta^{n_3} + \beta^{n_4}.$$

Combining the above two equations, we get

$$\begin{aligned}\alpha^{n_1} < \alpha^{n_1} + \alpha^{n_2} + \alpha^{n_3} + \alpha^{n_4} &= |\beta^{n_1} + \beta^{n_2} + \beta^{n_3} + \beta^{n_4}| \\ &\leqslant |\beta|^{n_1} + |\beta|^{n_2} + |\beta|^{n_3} + |\beta|^{n_4} < 1,\end{aligned}$$

which leads to a contradiction since $n_1 > 350$.

As before, here, we can take $A_1 := 1.4, A_2 := 0.5$ and $B := n_1$. Let us estimate $h(\gamma_3)$. We can see that,

$$\gamma_3 = \frac{\sqrt{5}}{1+\alpha^{n_2-n_1}+\alpha^{n_3-n_1}+\alpha^{n_4-n_1}} < \sqrt{5}$$

$$\text{and} \quad \gamma_3^{-1} = \frac{1+\alpha^{n_2-n_1}+\alpha^{n_3-n_1}+\alpha^{n_4-n_1}}{\sqrt{5}} < \frac{4}{\sqrt{5}}.$$

Hence $|\log\gamma_3| < 1$. Then

$$\begin{aligned} h(\gamma_3) &\leqslant \log(4\sqrt{5}) + |n_2-n_1|\left(\frac{\log\alpha}{2}\right) + |n_3-n_1|\left(\frac{\log\alpha}{2}\right) + |n_4-n_1|\left(\frac{\log\alpha}{2}\right) \\ &= \log(4\sqrt{5}) + (n_1-n_2)\left(\frac{\log\alpha}{2}\right) + (n_1-n_3)\left(\frac{\log\alpha}{2}\right) + (n_1-n_4)\left(\frac{\log\alpha}{2}\right); \end{aligned}$$

so we can take

$$A_3 := 5 + (n_1-n_2)\log\alpha + (n_1-n_3)\log\alpha + (n_1-n_4)\log\alpha.$$

Then a lower bound on the left-hand side of (3.12) is

$$\exp(-c_4\cdot(1+\log n_1)\cdot 1.4\cdot 0.5\cdot(5+(n_1-n_2)\log\alpha+(n_1-n_3)\log\alpha+(n_1-n_4)\log\alpha)),$$

where $c_4 = 1.4\cdot 30^6\cdot 3^{4.5}\cdot 2^2\cdot(1+\log 2) < 9.7\times 10^{11}$.

So, inequality (3.12) yields

$$n_1\log\alpha < 1.4\times 10^{12}\log n_1\cdot(5+(n_1-n_2)\log\alpha+(n_1-n_3)\log\alpha+(n_1-n_4)\log\alpha). \tag{3.13}$$

Using now (3.3), (3.8) and (3.11) in the right-most terms of the above inequality (3.13) and performing the respective calculation, we find that

$$n_1\log\alpha < 40.32\times 10^{48}\log^4 n_1.$$

With the help of *Mathematica*, we get from the previous inequality

$$n < 2.8\times 10^{58}.$$

We record what we have proved.

Lemma 3.1. *If (n_1, n_2, n_3, n_4, a) is a positive solution of (1.1) with $n_1 \geqslant n_2 \geqslant n_3 \geqslant n_4$, then*

$$a \leqslant n_1 < 2.8\times 10^{58}.$$

4. Reduction the bound on n

The goal of this section is to reduce the upper bound on n to a size that can be handled. To do this, we shall use Lemma 2.1 four times. Let us consider

$$z_1 := a\log 2 - n_1\log\alpha + \log\sqrt{5}. \tag{4.1}$$

From equation (4.1), (3.1) can be written as

$$|1-e^{z_1}| < \frac{9}{\alpha^{n_1-n_2}}. \tag{4.2}$$

Associating (1.1) and Binet's formula for the Fibonacci sequence, we have

$$\frac{\alpha^{n_1}}{\sqrt{5}} = F_{n_1} + \frac{\beta^{n_1}}{\sqrt{5}} < F_{n_1}+F_{n_2}+F_{n_3}+F_{n_4} = 2^a,$$

hence

$$\frac{\alpha^{n_1}}{\sqrt{5}} < 2^a,$$

which leads to $z_1 > 0$. This result, together with (4.2), give

$$0 < z_1 < e^{z_1} - 1 < \frac{9}{\alpha^{n_1-n_2}}.$$

Replacing (4.1) in the inequality and dividing both sides of the resulting inequality by $\log\alpha$, we get

$$0 < a\left(\frac{\log 2}{\log\alpha}\right) - n + \left(\frac{\log\sqrt{5}}{\log\alpha}\right) < \frac{9}{\log\alpha}\cdot\alpha^{n_1-n_2} < 19\cdot\alpha^{n_1-n_2}. \tag{4.3}$$

We put

$$\tau := \frac{\log 2}{\log\alpha}, \quad \mu := \frac{\log\sqrt{5}}{\log\alpha}, \quad A := 19, \quad \text{and} \quad B := \alpha.$$

τ is an irrational number. We also put $M := 2.8\times 10^{58}$, which is an upper bound on a by Lemma 2.1 applied to inequality (4.3), that

$$n_1 - n_2 < \frac{\log(Aq/\varepsilon)}{\log B},$$

where $q > 6M$ is a denominator of a convergent of the continued fraction of τ such that $\varepsilon := \|\mu q\| - M\,\|\tau q\| > 0$. A computation with *SageMath* revealed that if (n_1, n_2, n_3, n_4, a) is a possible solution of the equation 1.1, then

$$n_1 - n_2 \in [0, 314].$$

Let us now consider a second function, derived from (3.4) in order to find an improved upper bound on $n_1 - n_2$.

Put

$$z_2 := a \log 2 - n_1 \log \alpha + \log \Upsilon_1(n_1 - n_2)$$

where Υ is the function given by the formula $\Upsilon(t) := \sqrt{5}\,(1+\alpha^{-t})^{-1}$. From (3.4), we have

$$|1 - e^{z_2}| < \frac{6}{\alpha^{n_1 - n_3}}. \tag{4.4}$$

Using (1.1) and the Binet's formula for the Fibonacci sequence, we have

$$\frac{\alpha^{n_1}}{\sqrt{5}}+\frac{\alpha^{n_2}}{\sqrt{5}} = F_{n_1}+F_{n_2}+\frac{\beta^{n_1}}{\sqrt{5}}+\frac{\beta^{n_2}}{\sqrt{5}} < F_{n_1}+F_{n_2}+1 \leqslant F_{n_1}+F_{n_2}+F_{n_3}+F_{n_4} = 2^a.$$

Therefore $1 < 2^a\sqrt{5}\alpha^{-n_1}\left(1+\alpha^{n_2-n_1}\right)^{-1}$ and so $z_2 > 0$. This with (4.4) give

$$0 < z_2 \leqslant e^{z_2} - 1 < \frac{6}{\alpha^{n_1 - n_3}}.$$

Putting the expression of z_2 in the above inequality and arguing as in (4.3), we obtain

$$0 < a\left(\frac{\log 2}{\log \alpha}\right) - n_1 + \frac{\log \Upsilon_1(n_1 - n_2)}{\log \alpha} < 13 \cdot \alpha^{-(n_1 - n_3)}. \tag{4.5}$$

As before, we take $M := 2.8 \times 10^{58}$ as the upper bound on a, and, as explained before, we apply Lemma 2.1 to inequamity (4.5) for all choices $n_1 - n_2 \in [0, 314]$ except when $n_1 - n_2 = 2, 6$. With the help of *SageMath*, we find that if (n_1, n_2, n_3, n_4, a) is a possible solution of the equation (1.1) with $n_1 - n_2 \neq 2$ and $n_1 - n_2 \neq 6$, then $n_1 - n_3 \in [0, 314]$.

Next, we study the cases $n_1 - n_2 \in \{2, 6\}$. For these cases, when we apply Lemma 2.1 to the expression (4.5), the corresponding parameter μ appearing in Lemma 2.1 is

$$\frac{\log \Upsilon_1(t)}{\log \alpha} = \begin{cases} 1 & \text{if} \quad t = 2; \\ 3 - \dfrac{\log 2}{\log \alpha} & \text{if} \quad t = 6. \end{cases}$$

In both case, the parameters τ and μ are linearly dependent, which tell us that the corresponding value of ε from Lemma 2.1 is always negative and therefore the reduction method is not useful for reducing the bound on n in these instances. For this reason, we need to treat these cases differently.

However, we can see that if $t = 2$ and 6, then the resulting inequality from (4.5) has the shape $0 < |x\tau - y| < 13 \cdot \alpha^{-(n_1 - n_3)}$ with τ being an irrational number and $x, y \in \mathbb{Z}$. We will use the known properties of the convergents of continued fractions to obtain a nontrivial lower bound for $|x\tau - y|$.

For $n_1 - n_2 = 2$, from (4.5), we get that

$$0 < a\tau - (n_1 - 1) < 13 \cdot \alpha^{-(n_1 - n_3)}, \quad \text{where} \quad \tau = \frac{\log 2}{\log \alpha}. \tag{4.6}$$

Let $[a_1, a_2, a_3, a_4, \ldots] = [1, 2, 3, 1, \ldots]$ be the continued fraction of τ, and let p_k/q_k denote its kth convergent. By Lemma 2.2, we know that $a < 2.8 \times 10^{58}$. An inspection in *SageMath* reveals that

$$\begin{gathered} 12074711440474914515121100926577303328088091991053541856 85 \\ = q_{113} < 2.8 \times 10^{58} < q_{114} = \\ 28351096929195187169517686575841899309129196859170938821667. \end{gathered}$$

Furthermore, $a_M := \max\{a_i : i = 0, 1, \ldots, 114\} = 134$. So, from the proprieties of continued fractions, we obtain that

$$|a\tau - (n_1 - 1)| > \frac{1}{(a_M + 2)a}. \tag{4.7}$$

Comparing (4.6) and (4.7), we get

$$\alpha^{n_1 - n_3} < 13 \cdot (134 + 2)a.$$

Taking log on both sides of the above inequality and then dividing by $\log \alpha$, we get

$$n_1 - n_3 < 296.$$

In order to avoid repetition, we omits the details for the case $n_1 - n_2 = 6$. Here, we get $n_1 - n_3 < 314$.

This completes the analysis of the two special cases $n_1 - n_2 = 2$ and $n_1 - n_2 = 6$. Consequently $n_1 - n_3 \leqslant 314$ always holds.

Now let us use (3.9) in order to find improved upper bound on $n_1 - n_4$. Put

$$z_3 := a \log 2 - n_1 \log \alpha + \log \Upsilon_2(n_1 - n_2, n_1 - n_3),$$

where Υ_2 is the function given by the formula

$$\Upsilon_2(t, s) := \sqrt{5}\left(1 + \alpha^{-t} + \alpha^{-s}\right)^{-1}.$$

From (3.9), we have

$$|1 - e^{z_3}| < \frac{3}{\alpha^{n_1 - n_4}}. \tag{4.8}$$

Note that, $z_3 \neq 0$; thus, two cases arise: $z_3 > 0$ and $z_3 < 0$.

If $z_3 > 0$, then

$$0 < z_3 \leqslant e^{z_3} - 1 < \frac{3}{\alpha^{n_1 - n_4}}.$$

Suppose now $z_3 < 0$. It is easy to check that $3/\alpha^{n_1-n_4} < 1/2$ for all $n_1 > 350$ and $n_4 \geqslant 2$. From (4.8), we have that

$$|1 - e^{z_3}| < 1/2 \quad \text{and therefore} \quad e^{|z_3|} < 2.$$

Since $z_3 < 0$, we have:

$$0 < |z_3| \leqslant e^{|z_3|} - 1 = e^{|z_3|}\left|e^{|z_3|} - 1\right| < \frac{6}{\alpha^{n_1-n_4}},$$

which means

$$0 < |z_3| < \frac{6}{\alpha^{n_1-n_4}}$$

holds for $z_3 < 0$, $z_3 > 0$ and for all for all $n_1 > 350$, and $n_4 \geqslant 2$. Replacing the expression for z_3 in the above inequality and arguing again as before, we conclude that

$$0 < \left| a\left(\frac{\log 2}{\log \alpha}\right) - n_1 + \frac{\log \Upsilon_2(n_1-n_2, n_1-n_3)}{\log \alpha} \right| < 13 \cdot \alpha^{-(n_1-n_4)}. \tag{4.9}$$

Here, we also take, $M := 2.8 \times 10^{58}$ and we apply Lemma 2.1 in inequality (4.9) for all choices $n_1 - n_2 \in \{0, 314\}$ and $n_1 - n_3 \in \{0, 314\}$ except when

$$(n_1-n_2, n_1-n_3) \in \{(0,3),(1,1),(1,5),(3,0),(3,4),(4,3),(5,1),(7,8),(8,7)\}.$$

Indeed, with the help of *SageMath* we find that if (n_1, n_2, n_3, n_4, a) is a possible solution of equation (1.1), excluding the cases presented before, then $n_1 - n_4 \leqslant 314$.

SPECIAL CASES. We deal with the cases when

$$(n_1 - n_2, n_1 - n_3) \in \{(1,1),(3,0),(4,3),(5,1),(8,7)\}.$$

It is easy to check that

$$\frac{\log \Upsilon_2(t,s)}{\log \alpha} = \begin{cases} 0, & \text{if } (t,s) = (1,1); \\ 0, & \text{if } (t,s) = (3,0); \\ 1, & \text{if } (t,s) = (4,3); \\ 2 - \dfrac{\log 2}{\log \alpha}, & \text{if } (t,s) = (5,1); \\ 3 - \dfrac{\log 2}{\log \alpha}, & \text{if } (t,s) = (8,7). \end{cases}$$

As we explained before, when we apply Lemma 2.1 to the expression (4.9), the parameters τ and μ are linearly dependent, so the corresponding value of ε from Lemma 2.1 is negative in all cases. For this reason, we shall treat these cases differently.

Here, we have to solve the equations

$$F_{n_2+1} + 2F_{n_2} + F_{n_4} = 2^a,\ 2F_{n_2+3} + F_{n_2} + F_{n_4} = 2^a,\ F_{n_2+4} + F_{n_2} + F_{n_2+1} + F_{n_4} = 2^a,$$

$$F_{n_2+5} + F_{n_2} + F_{n_2+4} + F_{n_4} = 2^a, \quad \text{and} \quad F_{n_2+8} + F_{n_2} + F_{n_2+1} + F_{n_4} = 2^a \tag{4.10}$$

in positive integers n_2, n_4 and a. To do so, we recall the following well-known relation between the Fibonacci and the Lucas numbers:

$$L_k = F_{k-1} + F_{k+1} \quad \text{for all} \quad k \geqslant 1. \tag{4.11}$$

From (4.11) and (4.10), we have the following identities

$$F_{n_2+1} + 2F_{n_2} + F_{n_4} = F_{n_2+2} + F_{n_2} + F_{n_4} = F_{k+2} + F_k + F_m,$$

$$2F_{n_2+3} + F_{n_2} + F_{n_4} = F_{n_2+2} + F_{n_2+4} + F_{n_4} = F_{k+2} + F_{k+4} + F_m,$$

$$F_{n_2+4} + F_{n_2} + F_{n_2+1} + F_{n_4} = F_{n_2+2} + F_{n_2+4} + F_{n_4} = F_{k+2} + F_{k+4} + F_m, \tag{4.12}$$

$$F_{n_2+5}+F_{n_2}+F_{n_2+4}+F_{n_4} = 2F_{n_2+2}+2F_{n_2+4}+F_{n_4} = 2F_{k+2}+2F_{k+4}+F_m,$$

and $\quad F_{n_2+8}+F_{n_2}+F_{n_2+1}+F_{n_4} = 2F_{n_2+6}+2F_{n_2+4}+F_{n_4} = 2F_{k+6}+2F_{k+4}+F_m,$

which hold for all $k, m \geqslant 0$.

Equation (4.10) are transformed into the equations

$$L_{k+1}+F_m = 2^a, \quad L_{k+3}+F_m = 2^a, \quad 2L_{k+3}+F_m = 2^a, \quad 2L_{k+5}+F_m = 2^a, \tag{4.13}$$

to be resolved in positive integers k, m and a.

A quick search in SageMath and analytical resolution leads to :

$$(k, m, a) \in \{(4, 5, 4), (4, 8, 5)\} \quad \text{for} \quad L_{k+1} + F_m = 2^a,$$

$$(k, m, a) \in \{(2, 5, 4), (2, 8, 5), (4, 4, 5)\} \quad \text{for} \quad L_{k+3} + F_m = 2^a,$$

$$(k, m, a) = (5, 9, 7) \quad \text{for} \quad 2L_{k+3} + F_m = 2^a,$$

$$(k, m, a) = (3, 9, 7) \quad \text{for} \quad 2L_{k+5} + F_m = 2^a.$$

A complete resolution and analysis gives solutions that are already listed in Theorem 1.1. This completes the analysis of the special cases.

Finally let us use (3.12) in order to find an improved upper bound on n_1. Set

$$z_4 := a \log 2 - n_1 \log \alpha + \log \Upsilon_3(n_1 - n_2, n_1 - n_3, n_1 - n_4),$$

where Υ_3 is the function given by the formula

$$\Upsilon_3(t, u, v) := \sqrt{5}\left(1 + \alpha^{-t} + \alpha^{-u} + \alpha^{-v}\right)^{-1}$$

with $t = n_1 - n_2, u = n_1 - n_3$ and $v = n_1 - n_4$. From (3.12), we get

$$|1 - e^{z_3}| < \frac{2}{\alpha^{n_1}}. \tag{4.14}$$

Since $z_3 \neq 0$, as before, two cases arise : $z_4 < 0$ and $z_4 > 0$.
If $z_4 > 0$, then

$$0 < z_4 \leqslant e^{z_4} - 1 < \frac{2}{\alpha^{n_1}}.$$

Suppose now that $z_4 < 0$. We have $2/\alpha^{n_1} < 1/2$ for all $n_1 > 350$. Then, from (4.14), we have

$$|1 - e^{z_4}| < \frac{1}{2}$$

and therefore $e^{|z_3|} < 2$.

Since $z_3 < 0$, we have :

$$0 < |z_3| \leqslant e^{|z_3|} - 1 = e^{|z_3|}\left|e^{|z_3|} - 1\right| < \frac{4}{\alpha^{n_1}}$$

which gives

$$0 < |z_3| < \frac{4}{\alpha^{n_1}}$$

for both cases ($z_3 < 0$ and $z_3 > 0$) and holds for all $n_1 > 350$.

Replacing the expression for z_3 in the above inequality and arguing again as before, we conclude that

$$0 < \left| a\left(\frac{\log 2}{\log \alpha}\right) - n_1 + \frac{\log \Upsilon_3(n_1 - n_2, n_1 - n_3, n_1 - n_4)}{\log \alpha} \right| < 9 \cdot \alpha^{-n_1}. \tag{4.15}$$

Here, we also take, $M := 2.8 \times 10^{58}$ and we apply Lemma 2.1 last time in inequality (4.15) for all choices $n_1 - n_2 \in \{0, 314\}$, $n_1 - n_3 \in \{0, 314\}$ and $n_1 - n_4 \in \{0, 314\}$ with (n_1, n_2, n_3, n_4, a) a possible solution of equation (1.1), and by omitting study of special cases (because they give a solution presented in Theorem 1.1), we get:

$$n_1 < 320.$$

This is false due to our assumption that $n_1 > 350$.

This ends the proof of our main theorem. □

Remark 4.1. *Note that the computations for this last case took* 2 *hours on an ASUS CORE i*5 8*th Gen. processor.*

5. Comments

In this paper, we found all instances in which a power of two can be expressed as a sum of four Fibonacci numbers. Given the results obtained, we can make the following conjecture.

Conjecture 5.1. *Consider the Diophantine equation*

$$F_{n_1} + F_{n_2} + F_{n_3} + F_{n_4} = p^a, p \geqslant 2, a \geqslant 2 \tag{5.1}$$

where n_1, n_2, n_3, n_4, a are positive integers with $n_1 \geqslant n_2 \geqslant n_3 \geqslant n_4$ and p is prime, then $p = 2, 3, 5, 7$.

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