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GENERALIZED PRIME IDEAL FACTORIZATION OF SUBMODULES

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Abstract. In this article, we introduce generalized prime ideal factorization for all proper submodules of a finitely generated module over a Noetherian ring. We show that the generalized prime ideal factorization of a product of two coprime ideals is the product of the generalized prime ideal factorization of the ideals. We find conditions under which the generalized prime ideal factorization of a product of prime ideals is equal to the product of the prime ideals. We show that if R is a Dedekind domain, the generalized prime ideal factorization of an ideal $\mathfrak{a}$ in R is exactly the prime ideal factorization of $\mathfrak{a}$.

1. Introduction

Factorization of ideals into a product of prime ideals plays an important role in commutative algebra. For Noetherian rings, this was generalized into primary decomposition by Lasker (1905) and Noether (1921). In 1871, Dedekind had found conditions under which ideals of a ring can be uniquely expressed as a product of prime ideals. Rings with this property are now called Dedekind domains. But in general, the ideals of a Noetherian ring cannot be written as a product of prime ideals. In this article, we extend the notion of prime ideal factorization to submodules of a finitely generated module over a Noetherian ring

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which has the uniqueness property. We call it generalized prime ideal factorization.

In this section, we give the definitions and results which will be used in our article. Throughout this article R will denote a commutative Noetherian ring with identity and all R-modules will be assumed to be finitely generated and unitary. For standard terminology and notations, the reference will be [4] and [5]. Let R be a ring and M be an R-module. Then there exists a filtration

$$\mathcal{F} : 0 = M_0 \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n = M$$

of M. The filtration $\mathcal{F}$ is called a prime filtration of M if $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$ in R for $1 \leq i \leq n$ and the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ is denoted as $\mathrm{Supp}(\mathcal{F})$. It is well known that $\mathrm{Ass}(M) \subseteq \mathrm{Supp}(\mathcal{F}) \subseteq \mathrm{Supp}(M)$. In [1], Dress defined weak prime decomposition of a module M as a filtration $\mathcal{F} : 0 = M_0 \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n = M$ such that for every $i \in \{1, \ldots, n\}$, there exists a prime ideal $\mathfrak{p}_i \in \mathrm{Spec}(R)$ with $\mathfrak{p}_i = (M_{i-1} \colon x)$ for every $x \in M_i \setminus M_{i-1}$. Clearly every prime filtration is a weak prime decomposition. Dress proved that every Noetherian module M admits a weak prime decomposition with $\mathrm{Supp}(\mathcal{F}) = \mathrm{Ass}(M)$ [1, PD7].

Prime extension filtration for a finitely generated module over a Noetherian ring is defined in [2]. A proper submodule N of an R-module M is said to be prime if for any $a \in R$ and $x \in M$, $ax \in N$ implies $a \in (N : M)$ or $x \in N$. We say a submodule K of M is a $\mathfrak{p}$-prime extension of a proper submodule N of M, if N is a prime submodule of K with $(N \colon K) = \mathfrak{p}$ and is denoted as $N \overset{\mathfrak{p}}{\subset} K$. In [2], it is defined that a filtration $\mathcal{F} : N = M_0 \overset{\mathfrak{p}_1}{\subset} M_1 \subset \cdots \overset{\mathfrak{p}_n}{\subset} M_n = M$ is a prime extension filtration of M over N if M_i is a $\mathfrak{p}_i$-prime extension of M_{i-1} in M for $1 \leq i \leq n$. Further, if each M_i is a maximal $\mathfrak{p}_i$-prime extension of M_{i-1} in M, then the filtration is called a maximal prime extension (MPE) filtration of M over N. MPE filtrations exist for finitely generated modules over a Noetherian ring [2, Remark 7].

Proposition 1.1. [2, Proposition 14] *Let N be a proper submodule of M. If $N = M_0 \overset{\mathfrak{p}_1}{\subset} M_1 \subset \cdots \overset{\mathfrak{p}_n}{\subset} M_n = M$ is an MPE filtration of M over N, then $\mathrm{Ass}(M/M_i) \subseteq \mathrm{Ass}(M/N)$ for $1 \leq i \leq n$, and $\mathrm{Ass}(M/N) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$.*

In [2] it is defined that a submodule K is said to be a regular prime extension of N if K is a maximal $\mathfrak{p}$-prime extension of N in M and $\mathfrak{p}$ is a maximal element in $\mathrm{Ass}(M/N)$ and a prime extension filtration $N = M_0 \overset{\mathfrak{p}_1}{\subset} M_1 \subset \cdots \overset{\mathfrak{p}_n}{\subset} M_n = M$ is called a regular prime extension

(RPE) filtration of M over N, if M_i is a regular $\mathfrak{p}_i$-prime extension of M_{i-1} in M for $1 \leq i \leq n$. For a finitely generated module over a Noetherian ring, RPE filtration exists [2, Theorem 11].

We have that $\mathrm{Ass}(M/N)$ is precisely the set of prime ideals occurring in any RPE filtration of M over N. We need the following results.

Lemma 1.2. [3, Lemma 2.8] *Let N be a proper submodule of an R-module M. If K is a regular $\mathfrak{p}$-prime extension of N in M, then for any submodule L of M either $L \cap K = L \cap N$ or $L \cap K$ is a regular $\mathfrak{p}$-prime extension of $L \cap N$ in L.*

The occurrence of prime ideals in an RPE filtration can be interchanged provided it satisfies some conditions.

Proposition 1.3. [2, Corollary 17] *Let N be a proper submodule of M and $N = M_0 \subset \cdots M_{i-1} \overset{\mathfrak{p}_i}{\subset} M_i \overset{\mathfrak{p}_{i+1}}{\subset} M_{i+1} \cdots \subset M_n = M$ be an MPE filtration of M over N. If $\mathfrak{p}_i$ and $\mathfrak{p}_{i+1}$ are distinct maximal elements in $\mathrm{Ass}(M/M_{i-1})$, then there exists a submodule K_i of M, such that $N = M_0 \subset \cdots M_{i-1} \overset{\mathfrak{p}_{i+1}}{\subset} K_i \overset{\mathfrak{p}_i}{\subset} M_{i+1} \cdots \subset M_n = M$ is an MPE filtration of M over N.*

Proposition 1.4. [3, Remark 2.5] *Let N be a proper submodule of M and $N = M_0 \overset{\mathfrak{p}_1}{\subset} M_1 \subset \cdots \overset{\mathfrak{p}_n}{\subset} M_n = M$ be an RPE filtration of M over N. If $\mathfrak{p}$ is a minimal element in $\mathrm{Ass}(M/N)$ and r is the number of times $\mathfrak{p}$ occurs in an RPE filtration of M over N, then we can have an RPE filtration $N = M'_0 \overset{\mathfrak{p}_{i_1}}{\subset} M'_1 \subset \cdots \overset{\mathfrak{p}_{i_n}}{\subset} M'_n = M$, where $\mathfrak{p}_{i_j} = \mathfrak{p}$ for $j = n-r+1, \ldots, n$.*

Proposition 1.5. [2, Theorem 22] *Let N be a proper submodule of M. Then the number of times a prime ideal $\mathfrak{p}$ of R occurs in any two RPE filtrations of M over N are equal, and hence, any two RPE filtrations of M over N have the same length.*

Hence, for every proper submodule N of M, RPE filtration exists and the set of prime ideals and number of occurrences of each prime ideal do not depend on any particular RPE filtration.

2. Generalized Prime Ideal Factorization of Submodules

Definition 2.1. Let N be a proper submodule of M and $N = M_0 \overset{\mathfrak{p}_1}{\subset} M_1 \subset \cdots \overset{\mathfrak{p}_n}{\subset} M_n = M$ be an RPE filtration of M over N. Then we say the product $\mathfrak{p}_1 \cdots \mathfrak{p}_n$ is the generalized prime ideal factorization of N in M and we denote it as $\mathcal{P}_M(N)$.

Example 2.2. For a prime ideal $\mathfrak{p}$ in R, $\mathcal{P}_R(\mathfrak{p}) = \mathfrak{p}$ as $\mathfrak{p} \overset{\mathfrak{p}}{\subset} R$ is the RPE filtration of R over $\mathfrak{p}$.

Example 2.3. $\mathcal{P}_M(0) = 0$ if and only if M is a torsion-free module over an integral domain R.

Example 2.4. [3, Example 2.7] If $n = {p_1}^{r_1} \cdots {p_k}^{r_k}$ is the prime factorization of an integer n, then $\mathcal{P}_{\mathbb{Z}}(n\mathbb{Z}) = (p_1\mathbb{Z})^{r_1} \cdots (p_k\mathbb{Z})^{r_k}$ since

$$n\mathbb{Z} \overset{p_1\mathbb{Z}}{\subset} {p_1}^{r_1-1}{p_2}^{r_2} \cdots {p_k}^{r_k}\mathbb{Z} \overset{p_1\mathbb{Z}}{\subset} \cdots \overset{p_1\mathbb{Z}}{\subset} p_1{p_2}^{r_2} \cdots {p_k}^{r_k}\mathbb{Z} \overset{p_1\mathbb{Z}}{\subset}$$
$${p_2}^{r_2} \cdots {p_k}^{r_k}\mathbb{Z} \subset \cdots \subset {p_k}^{r_k}\mathbb{Z} \overset{p_k\mathbb{Z}}{\subset} {p_k}^{r_k-1}\mathbb{Z} \overset{p_k\mathbb{Z}}{\subset} \cdots \overset{p_k\mathbb{Z}}{\subset} p_k\mathbb{Z} \overset{p_k\mathbb{Z}}{\subset} \mathbb{Z}$$

is an RPE filtration of $\mathbb{Z}$ over $n\mathbb{Z}$.

Example 2.5. We have RPE filtrations

$$(x^2, y) \overset{(x,y)}{\subset} (x, y) \overset{(x,y)}{\subset} k[x, y],$$
$$(x, y^2) \overset{(x,y)}{\subset} (x, y) \overset{(x,y)}{\subset} k[x, y], \text{ and}$$
$$(x^2, xy, y^2) \overset{(x,y)}{\subset} (x, y) \overset{(x,y)}{\subset} k[x, y]$$

in $k[x, y]$. So, we get

$$\mathcal{P}_{k[x,y]}((x^2, y)) = \mathcal{P}_{k[x,y]}((x, y^2)) = \mathcal{P}_{k[x,y]}((x^2, xy, y^2)) = (x, y)^2$$

and therefore, distinct submodules may have the same generalized prime ideal factorization.

Example 2.6. Let N be a $\mathfrak{p}$-primary submodule of M. Then $\mathrm{Ass}(M/N) = \{\mathfrak{p}\}$ and by Proposition 1.1, $\mathcal{P}_M(N) = \mathfrak{p}^r$ for some integer r.

Note that if L is a submodule of both K and M, then $\mathcal{P}_K(L)$ need not be equal to $\mathcal{P}_M(L)$ in general. In example 2.5, we see that $\mathcal{P}_{k[x,y]}((x^2, y)) = (x, y)^2$, and (x^2, y) as a submodule of (x, y) has $\mathcal{P}_{(x,y)}((x^2, y)) = (x, y)$. Now we give a sufficient condition for $\mathcal{P}_K(L) = \mathcal{P}_M(L)$ when $L \subset K \subset M$.

Proposition 2.7. *Let K be a submodule of M. For any submodule L of M, $\mathcal{P}_K(K \cap L) = \mathcal{P}_M(L)$ whenever $(K : M) \nsubseteq \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M/L)} \mathfrak{p}$.*

Proof. Let $L = M_0 \overset{\mathfrak{p}_1}{\subset} M_1 \subset \cdots \overset{\mathfrak{p}_n}{\subset} M_n = M$ be an RPE filtration of M over L. Then $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} = \mathrm{Ass}(M/L)$ by Proposition 1.1. Intersecting with K, we get $K \cap L \subseteq M_1 \cap K \subseteq \cdots \subseteq M_n \cap K = K$. Since $(K : M) \nsubseteq \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M/L)} \mathfrak{p}$, we have $a \in (K : M) \setminus \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M/L)} \mathfrak{p}$.

Suppose $M_{i-1} \cap K = M_i \cap K$ for some i. Since $M_{i-1} \subsetneq M_i$, there exists $x \in M_i \setminus M_{i-1}$. Then $ax \in M_i \cap K = M_{i-1} \cap K \subseteq M_{i-1}$ and $x \notin M_{i-1}$ implies $a \in (M_{i-1} : M_i) = \mathfrak{p}_i \in \mathrm{Ass}(M/L)$, a contradiction. Therefore, $M_{i-1} \cap K \subsetneq M_i \cap K$ for all i. Then by Lemma 1.2, $M_{i-1} \cap K \subset M_i \cap K$ is a regular $\mathfrak{p}_i$-prime extension for all i. This implies $K \cap L \overset{\mathfrak{p}_1}{\subset} M_1 \cap K \subset \cdots \overset{\mathfrak{p}_n}{\subset} M_n \cap K = K$ is an RPE filtration of K over $K \cap L$. Hence, $\mathcal{P}_K(K \cap L) = \mathcal{P}_M(L)$. □

Corollary 2.8. *If $L \subset K \subset M$ and $(K : M)$ contains a non-zero-divisor of M/L, then $\mathcal{P}_K(L) = \mathcal{P}_M(L)$.*

Corollary 2.9. *If $a \in R$ is a non-zero-divisor of M, then $\mathcal{P}_{aM}(0) = \mathcal{P}_M(0)$.*

Proof. $a \in (aM \colon M) \setminus \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}$, since a is a non-zero-divisor of M. □

Theorem 2.10. *Let N and K be submodules of M such that $\mathcal{P}_M(N) = \mathfrak{p}_1 \cdots \mathfrak{p}_n$ and $\mathcal{P}_M(K) = \mathfrak{q}_1 \cdots \mathfrak{q}_k$. If $\mathfrak{p}_i \nsubseteq \mathfrak{q}_j$ and $\mathfrak{q}_j \nsubseteq \mathfrak{p}_i$ for every i and j, then $\mathcal{P}_M(N)\mathcal{P}_M(K) = \mathcal{P}_M(N \cap K)$.*

Proof. We can have RPE filtrations $N = N_0 \overset{\mathfrak{p}_1}{\subset} N_1 \subset \cdots \overset{\mathfrak{p}_n}{\subset} N_n = M$ and $K = K_0 \overset{\mathfrak{q}_1}{\subset} K_1 \subset \cdots \overset{\mathfrak{q}_k}{\subset} K_k = M$ for N and K respectively, since $\mathfrak{p}_i \nsubseteq \mathfrak{q}_j$ and $\mathfrak{q}_j \nsubseteq \mathfrak{p}_i$ for every i and j. Consider the chain

$$N \cap K \subseteq N_1 \cap K \subseteq N_2 \cap K \subseteq \cdots \subseteq N_n \cap K = K \overset{\mathfrak{q}_1}{\subset} K_1 \subset \cdots \overset{\mathfrak{q}_k}{\subset} K_k = M \quad (2.1)$$

Suppose there exists i such that $N_{i-1} \cap K = N_i \cap K$. Since $(K : M)N_i \subseteq N_i \cap K = N_{i-1} \cap K \subseteq N_{i-1}$, $(K : M) \subseteq (N_{i-1} : N_i) = \mathfrak{p}_i \in \mathrm{Ass}(M/N)$. Also, $(K : M) \subseteq \mathfrak{p}_i$ implies $\mathfrak{p}_i \in \mathrm{Supp}(M/K)$. Then $\mathfrak{p}_i$ contains a minimal element of $\mathrm{Supp}(M/K)$ which is also an element of $\mathrm{Ass}(M/K)$. That is, $\mathfrak{p}_i \supseteq \mathfrak{q}_j$ for some j, which is a contradiction.

Therefore, no equality occurs in (2.1) and by Lemma 1.2, (2.1) becomes an RPE filtration. Hence, $\mathcal{P}_M(N \cap K) = \mathfrak{p}_1\mathfrak{p}_2 \cdots \mathfrak{p}_n\mathfrak{q}_1\mathfrak{q}_2 \cdots \mathfrak{q}_k = \mathcal{P}_M(N)\mathcal{P}_M(K)$. □

Corollary 2.11. *Let $N = N_1 \cap \cdots \cap N_r$ be a minimal primary decomposition of N in M where N_i is $\mathfrak{p}_i$-primary for every i. If $\mathfrak{p}_i$ is a minimal associated prime of M/N for every i, then $\mathcal{P}_M(N) = \mathcal{P}_M(N_1) \cdots \mathcal{P}_M(N_r)$.*

Proof. Since N_i is $\mathfrak{p}_i$-primary for every i, we have $\mathcal{P}_M(N_i) = {\mathfrak{p}_i}^{r_i}$ for some integer r_i. Also, we have $\mathfrak{p}_i \nsubseteq \mathfrak{p}_j$ and $\mathfrak{p}_j \nsubseteq \mathfrak{p}_i$, $j \neq i$, since every $\mathfrak{p}_i$ in $\mathrm{Ass}(M/N) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ is minimal. Therefore, using Theorem

2.10 repeatedly, we get $\mathcal{P}_M(N_1)\cdots\mathcal{P}_M(N_r) = \mathcal{P}_M(N_1 \cap \cdots \cap N_r) = \mathcal{P}_M(N)$. $\square$

The above result need not be true if all $\mathfrak{p}_i$ in $\mathrm{Ass}(M/N)$ are not minimal. For example, let $N = (x^2, xy)$ and $M = k[x, y]$. Then $N = N_1 \cap N_2$ where $N_1 = (x)$ and $N_2 = (x^2, y)$. So, the prime ideals are $\mathfrak{p}_1 = (x)$, $\mathfrak{p}_2 = (x, y)$, with $\mathfrak{p}_1 \subset \mathfrak{p}_2$. We have $\mathcal{P}_M(N) = \mathfrak{p}_1\mathfrak{p}_2$ from the RPE filtration

$$(x^2, xy) \overset{(x,y)}{\subset} (x) \overset{(x)}{\subset} k[x, y],$$

and we have $\mathcal{P}_M(N_1) = \mathfrak{p}_1$, $\mathcal{P}_M(N_2) = {\mathfrak{p}_2}^2$ from the RPE filtrations

$$(x) \overset{(x)}{\subset} k[x, y] \quad \text{and} \quad (x^2, y) \overset{(x,y)}{\subset} (x, y) \overset{(x,y)}{\subset} k[x, y]$$

respectively. So, $\mathcal{P}_M(N) \neq \mathcal{P}_M(N_1)\mathcal{P}_M(N_2)$.

Next, we show that the generalized prime ideal factorization of a product of two coprime ideals is the product of the generalized prime ideal factorization of the ideals. More generally, we prove the following.

Corollary 2.12. *Let N, K be submodules of M such that $(N : M)$ and $(K : M)$ are coprime. Then $\mathcal{P}_M(N)\mathcal{P}_M(K) = \mathcal{P}_M(N \cap K)$. In particular, if $\mathfrak{a}$ and $\mathfrak{b}$ are coprime ideals in R, then $\mathcal{P}_R(\mathfrak{a})\mathcal{P}_R(\mathfrak{b}) = \mathcal{P}_R(\mathfrak{a}\mathfrak{b})$.*

Proof. Let $N = N_0 \overset{\mathfrak{p}_1}{\subset} N_1 \subset \cdots \overset{\mathfrak{p}_r}{\subset} N_r = M$ and $K = K_0 \overset{\mathfrak{q}_1}{\subset} K_1 \subset \cdots \overset{\mathfrak{q}_s}{\subset} K_s = M$ be RPE filtrations of M over N and M over K respectively. Then $\mathcal{P}_M(N) = \mathfrak{p}_1 \cdots \mathfrak{p}_r$ and $\mathcal{P}_M(K) = \mathfrak{q}_1 \cdots \mathfrak{q}_s$. Suppose $\mathfrak{p}_i \subseteq \mathfrak{q}_j$ for some i, j. Then, since $\mathfrak{p}_i \in \mathrm{Supp}(M/N)$ and $\mathfrak{q}_j \in \mathrm{Supp}(M/K)$, we have $(N : M) \subseteq \mathfrak{p}_i \subseteq \mathfrak{q}_j$ and $(K : M) \subseteq \mathfrak{q}_j$. This implies $(N : M) + (K : M) \subseteq \mathfrak{q}_j$, i.e., $R \subseteq \mathfrak{q}_j$, a contradiction. Therefore, $\mathfrak{p}_i \nsubseteq \mathfrak{q}_j$ for all i, j. Similarly, $\mathfrak{q}_j \nsubseteq \mathfrak{p}_i$ for all i, j. So, by Theorem 2.10, we get $\mathcal{P}_M(N)\mathcal{P}_M(K) = \mathcal{P}_M(N \cap K)$. $\square$

Example 2.5 shows us that for an ideal $\mathfrak{a}$ in R, the product $\mathcal{P}_R(\mathfrak{a})$ is not equal to $\mathfrak{a}$ in general. Even if $\mathfrak{a}$ is a power of a prime ideal, $\mathcal{P}_R(\mathfrak{a})$ need not be equal to $\mathfrak{a}$. For example, let $R = k[x, y, z]/(xy - z^2)$ and let $\overline{x}, \overline{y}, \overline{z}$ denote the images of x, y, z respectively in R. Then $\mathfrak{p} = (\overline{x}, \overline{z})$ is a prime ideal and $\mathfrak{p}^2$ has the RPE filtration

$$\mathfrak{p}^2 = (\overline{x}^2, \overline{xy}, \overline{xz}, \overline{z}^2) \overset{(\overline{x},\overline{y},\overline{z})}{\subset} (\overline{x}, \overline{z}^2) \overset{(\overline{x},\overline{z})}{\subset} (\overline{x}, \overline{z}) \overset{(\overline{x},\overline{z})}{\subset} R.$$

So, $\mathcal{P}_R(\mathfrak{p}^2) = (\overline{x}, \overline{y}, \overline{z})(\overline{x}, \overline{z})^2 \neq \mathfrak{p}^2$.

Definition 2.13. Let M be an R-module and $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ be prime ideals in R. Then ${\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_k}^{r_k} M$ is called a minimal prime product representation in M if ${\mathfrak{p}_1}^{r_1}\cdots{\mathfrak{p}_i}^{r_i-1}\cdots{\mathfrak{p}_k}^{r_k}M \neq {\mathfrak{p}_1}^{r_1}\cdots{\mathfrak{p}_k}^{r_k}M$ and $\mathfrak{p}_i$ is maximal in $\mathrm{Ass}({\mathfrak{p}_1}^{r_1}\cdots{\mathfrak{p}_{i-1}}^{r_{i-1}}M/{\mathfrak{p}_1}^{r_1}\cdots{\mathfrak{p}_k}^{r_k}M)$ for $i = 1,\ldots,k$.

Next we give a sufficient condition for $\mathcal{P}_R(\mathfrak{a}) = \mathfrak{a}$. More generally, we prove the following theorem.

Theorem 2.14. *Let $N = {\mathfrak{p}_1}^{r_1}\cdots{\mathfrak{p}_k}^{r_k}M$ be a minimal prime product representation in M. Then $\mathcal{P}_M(N) = {\mathfrak{p}_1}^{r_1}\cdots{\mathfrak{p}_k}^{r_k}$.*

Proof. Let $\mathfrak{a}_{i_j}$ denote ${\mathfrak{p}_1}^{r_1}\cdots{\mathfrak{p}_{i-1}}^{r_{i-1}}{\mathfrak{p}_i}^{j}$ for $1 \leq j \leq r_i$ and $1 \leq i \leq k$. We show that

$$N \overset{\mathfrak{p}_1}{\subset} (N:\mathfrak{a}_{1_1}) \overset{\mathfrak{p}_1}{\subset} (N:\mathfrak{a}_{1_2}) \subset \cdots \overset{\mathfrak{p}_1}{\subset} (N:\mathfrak{a}_{1_{r_1}}) \overset{\mathfrak{p}_2}{\subset} (N:\mathfrak{a}_{2_1}) \subset \cdots$$
$$\overset{\mathfrak{p}_i}{\subset} (N:\mathfrak{a}_{i_{r_i}}) \overset{\mathfrak{p}_{i+1}}{\subset} (N:\mathfrak{a}_{(i+1)_1}) \subset \cdots \overset{\mathfrak{p}_k}{\subset} (N:\mathfrak{a}_{k_{r_k}}) = M$$

is an RPE filtration of M over N which would imply that $\mathcal{P}_M(N) = {\mathfrak{p}_1}^{r_1}\cdots{\mathfrak{p}_k}^{r_k}$. So, it is enough to show that $(N:\mathfrak{a}_{i_{j-1}}) \subset (N:\mathfrak{a}_{i_j})$ is a regular prime extension for $1 \leq j \leq r_i$ and $1 \leq i \leq k$.

Clearly $(N:\mathfrak{a}_{i_{j-1}}) \subseteq (N:\mathfrak{a}_{i_j})$, as $\mathfrak{a}_{i_{j-1}} \supset \mathfrak{a}_{i_j}$. Suppose equality holds, since $\mathfrak{a}_{i_j}{\mathfrak{p}_i}^{r_i-j}{\mathfrak{p}_{i+1}}^{r_{i+1}}\cdots{\mathfrak{p}_k}^{r_k}M = N$, we have ${\mathfrak{p}_i}^{r_i-j}{\mathfrak{p}_{i+1}}^{r_{i+1}}\cdots{\mathfrak{p}_k}^{r_k}M \subseteq (N:\mathfrak{a}_{i_j}) = (N:\mathfrak{a}_{i_{j-1}})$. This implies ${\mathfrak{p}_1}^{r_1}\cdots{\mathfrak{p}_i}^{r_i-1}\cdots{\mathfrak{p}_k}^{r_k}M = N$, a contradiction. Therefore, $(N:\mathfrak{a}_{i_{j-1}}) \subsetneq (N:\mathfrak{a}_{i_j})$.

Since for every $x \in (N:\mathfrak{a}_{i_j}) \setminus (N:\mathfrak{a}_{i_{j-1}})$, there exists $a \in \mathfrak{a}_{i_{j-1}} \setminus \mathfrak{a}_{i_j}$ such that $ax \notin N$, the set

$$\mathcal{S} = \big\{(N:ax) \mid x \in (N:\mathfrak{a}_{i_j}) \setminus (N:\mathfrak{a}_{i_{j-1}}),\ a \in \mathfrak{a}_{i_{j-1}} \text{ and } ax \notin N\big\}$$

is non-empty. We claim that $\mathfrak{p}_i$ is a maximal element in $\mathcal{S}$. By maximal condition, $\mathcal{S}$ has a maximal element, say $\mathfrak{q} = (N:by)$. Then $\mathfrak{q}$ is a prime ideal. For if $cd \in \mathfrak{q}$ and $c \notin \mathfrak{q}$ for some $c, d \in R$, then $cby \notin N$, and $cb \in \mathfrak{a}_{i_{j-1}}$ implies $(N:cby) \in \mathcal{S}$ with $\mathfrak{q} \subseteq (N:cby)$. Then by maximality of $\mathfrak{q}$, $\mathfrak{q} = (N:cby)$. Since $cd \in \mathfrak{q}$, $d \in (N:cby) = \mathfrak{q}$. Now, $by \in \mathfrak{a}_{i_{j-1}}M \setminus N$ implies $\mathfrak{q} = (N:by) \in \mathrm{Ass}({\mathfrak{p}_1}^{r_1}\cdots{\mathfrak{p}_{i-1}}^{r_{i-1}}M/N)$. Clearly $\mathfrak{p}_i \subseteq \mathfrak{q}$, and since $\mathfrak{p}_i$ is maximal in $\mathrm{Ass}({\mathfrak{p}_1}^{r_1}\cdots{\mathfrak{p}_{i-1}}^{r_{i-1}}M/N)$, $\mathfrak{p}_i = \mathfrak{q}$. Hence, $\mathfrak{p}_i$ is maximal in $\mathcal{S}$.

Clearly $\mathfrak{p}_i \subseteq ((N:\mathfrak{a}_{i_{j-1}}):(N:\mathfrak{a}_{i_j}))$. Let $a \in ((N:\mathfrak{a}_{i_{j-1}}):(N:\mathfrak{a}_{i_j}))$. Then $\mathfrak{a}_{i_{j-1}}ay \subseteq N$ which implies $aby \in N$. That is, $a \in \mathfrak{q} = \mathfrak{p}_i$, and hence, $((N:\mathfrak{a}_{i_{j-1}}):(N:\mathfrak{a}_{i_j})) = \mathfrak{p}_i$.

Suppose $c \in R$, $x \in (N:\mathfrak{a}_{i_j}) \setminus (N:\mathfrak{a}_{i_{j-1}})$ with $cx \in (N:\mathfrak{a}_{i_{j-1}})$. Then there exists $a \in \mathfrak{a}_{i_{j-1}} \setminus \mathfrak{a}_{i_j}$ such that $ax \notin N$ and $cax \in N$. Let $\mathfrak{b} = (N:ax)$. Then $c \in \mathfrak{b}$ and $\mathfrak{b} \in \mathcal{S}$. Since $\mathfrak{p}_i ax \subseteq N$, $\mathfrak{p}_i \subseteq (N:ax) = \mathfrak{b}$ and by claim, $\mathfrak{p}_i = \mathfrak{b}$. Hence, $c \in \mathfrak{p}_i$ and this implies $(N:\mathfrak{a}_{i_{j-1}}) \overset{\mathfrak{p}_i}{\subset} (N:\mathfrak{a}_{i_j})$ is a $\mathfrak{p}_i$-prime extension in M.

Let K be a maximal $\mathfrak{p}_i$-prime extension of $(N : \mathfrak{a}_{i_{j-1}})$ in M. Then $\mathfrak{p}_i K \subseteq (N : \mathfrak{a}_{i_{j-1}})$, which implies $\mathfrak{a}_{i_j} K \subseteq N$, and therefore, $K \subseteq (N : \mathfrak{a}_{i_j})$. That is, $(N : \mathfrak{a}_{i_j})$ is a maximal $\mathfrak{p}_i$-prime extension of $(N : \mathfrak{a}_{i_{j-1}})$ in M.

Let $\mathfrak{p}' \in \operatorname{Ass}(M/(N : \mathfrak{a}_{i_{j-1}}))$ such that $\mathfrak{p}_i \subseteq \mathfrak{p}'$. Then $\mathfrak{p}' = ((N : \mathfrak{a}_{i_{j-1}}) : x)$ for some $x \in M$. Since $\mathfrak{p}_i x \in (N : \mathfrak{a}_{i_{j-1}})$, $x \in (N : \mathfrak{a}_{i_j}) \setminus (N : \mathfrak{a}_{i_{j-1}})$. Then there exists $a \in \mathfrak{a}_{i_{j-1}}$ such that $ax \notin N$, and therefore, $(N : ax) \in \mathcal{S}$ with $\mathfrak{p}_i \subseteq \mathfrak{p}' \subseteq (N : ax)$. Then by claim, $\mathfrak{p}_i = \mathfrak{p}'$. Therefore, $\mathfrak{p}_i$ is maximal in $\operatorname{Ass}(M/(N : \mathfrak{a}_{i_{j-1}}))$. Hence, $(N : \mathfrak{a}_{i_{j-1}}) \overset{\mathfrak{p}_i}{\subset} (N : \mathfrak{a}_{i_j})$ is a regular $\mathfrak{p}_i$-prime extension in M.

Similarly, $(N : \mathfrak{a}_{i_{r_i}}) \overset{\mathfrak{p}_{i+1}}{\subset} (N : \mathfrak{a}_{(i+1)_1})$ is also a regular $\mathfrak{p}_{i+1}$-prime extension in M for $1 \leq i \leq k-1$. This completes the proof. $\square$

Corollary 2.15. *Let M be an R-module and $\mathfrak{m}_1, \dots, \mathfrak{m}_k$ be maximal ideals in R. If ${\mathfrak{m}_1}^{r_1} \cdots {\mathfrak{m}_i}^{r_i - 1} \cdots {\mathfrak{m}_k}^{r_k} M \neq {\mathfrak{m}_1}^{r_1} \cdots {\mathfrak{m}_k}^{r_k} M$ for every $1 \leq i \leq k$, then $\mathcal{P}_M({\mathfrak{m}_1}^{r_1} \cdots {\mathfrak{m}_k}^{r_k} M) = {\mathfrak{m}_1}^{r_1} \cdots {\mathfrak{m}_k}^{r_k}$.*

Taking $M = R$ in Theorem 2.14, we have the following corollary.

Corollary 2.16. *Let R be a Noetherian ring and ${\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_k}^{r_k}$ be a minimal prime product representation in R. Then $\mathcal{P}_R({\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_k}^{r_k}) = {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_k}^{r_k}$.*

Corollary 2.17. *Let R be an integral domain.*

(i) If $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ are prime ideals in R such that $\mathfrak{p}_i$ is maximal in $\operatorname{Ass}({\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_{i-1}}^{r_{i-1}} / \ {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_k}^{r_k})$ for every $1 \leq i \leq k$, then $\mathcal{P}_R({\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_k}^{r_k}) = {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_k}^{r_k}$.

(ii) If $\mathfrak{m}_1, \dots, \mathfrak{m}_k$ are maximal ideals in R, then $\mathcal{P}_R({\mathfrak{m}_1}^{r_1} \cdots {\mathfrak{m}_k}^{r_k}) = {\mathfrak{m}_1}^{r_1} \cdots {\mathfrak{m}_k}^{r_k}$.

Now we get a necessary and sufficient condition for $\mathcal{P}_R(\mathfrak{a}) = \mathfrak{a}$ in a domain R.

Corollary 2.18. *Let R be an integral domain. Then R is a Dedekind domain if and only if $\mathcal{P}_R(\mathfrak{a}) = \mathfrak{a}$ for every non-zero ideal $\mathfrak{a}$.*

Proof. If R is a Dedekind domain, for any non-zero ideal $\mathfrak{a}$, $\mathfrak{a} = {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_k}^{r_k}$, where $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ are non-zero maximal ideals in R. Then by Corollary 2.17, $\mathcal{P}_R({\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_k}^{r_k}) = {\mathfrak{p}_1}^{r_1} \cdots {\mathfrak{p}_k}^{r_k}$. Hence, $\mathcal{P}_R(\mathfrak{a}) = \mathfrak{a}$.

Conversely, if $\mathcal{P}_R(\mathfrak{a}) = \mathfrak{a}$ for every non-zero ideal $\mathfrak{a}$, this implies that every non-zero ideal of R can be written as a product of a finite number of prime ideals, and hence, R is Dedekind. $\square$

So, for an ideal in a Dedekind domain, the generalized prime ideal factorization coincides with its prime ideal factorization.

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