Journal of Algebra and Related Topics
Vol. 10, No 1, (2022), pp 95-111

ON DOMINIONS AND DETERMINATION OF CLOSED VARIETIES OF SEMIGROUPS

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Abstract. It is known that all subvarieties of variety of all semigroups are not absolutely closed. So, it is a natural question to find out those subvarieties of variety of all semigroups that are closed in itself or close in larger subvarieties of variety of all semigroups. We have gone through this open problem and able to determine some closed varieties of semigroups defined by the identities $axy = yxax$ $[axy = xyxa]$ and $axy = yxxa$ by using Isbell's zigzag theorem as an essential tool. Further, we partially generalize a result of Isbell on semigroup dominions from the class of commutative semigroups to some generalized classes of commutative semigroups by showing that dominions of such semigroups belong to the same class.

1. Introduction and Preliminaries

Let U be a subsemigroup of a semigroup S. Following Isbell [10], we say that U dominates an element d of S if for every semigroup T and for all homomorphisms $\beta, \gamma : S \longrightarrow T$ and $u\beta = u\gamma$ for every u in U implies $d\beta = d\gamma$. The set of all elements of S dominated by U is called dominion of U in S and we denote it by $Dom(U, S)$. It can be easily verified that $Dom(U, S)$ is a subsemigroup of S containing U. A subsemigroup U of a semigroup S is called closed in S if $Dom(U, S) = U$. A semigroup U is called absolutely closed if it is closed in every containing semigroup. Let $\mathcal{C}$ be a class of semigroups. A semigroup U is said to be $\mathcal{C}$-closed

AMS Subject Classification: 20M07.
Keywords: Zigzag equations; dominion; identity; varieties and Closed.
Received: 10 March 2021, Accepted: 4 June 2021.
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if $Dom(U,S) = U$ for all $S \in \mathcal{C}$ such that $U \subseteq S$. Let $\mathcal{B}$ and $\mathcal{C}$ be classes of semigroups such that $\mathcal{B}$ is a subclass of $\mathcal{C}$. We say that $\mathcal{B}$ is $\mathcal{C}$-closed if every member of $\mathcal{B}$ is $\mathcal{C}$-closed. A class $\mathcal{C}$ of semigroups is said to be closed if $Dom(U,S) = U$ for all $U, S \in \mathcal{C}$ with U as a subsemigroup of S. Let $\mathcal{A}$ and $\mathcal{D}$ be two categories of semigroups with $\mathcal{A}$ as a subcategory of $\mathcal{D}$. Then it can be easily verified that a semigroup U is $\mathcal{A}$-closed if it is $\mathcal{D}$-closed.

The following theorem provided by Isbell [10], known as Isbell's zigzag theorem, is a most useful characterization of semigroup dominions and is of basic importance to our investigations.

Theorem 1.1. *([10], Theorem 2.3) Let U be a subsemigroup of a semigroup S and let $d \in S$. Then $d \in Dom(U,S)$ if and only if $d \in U$ or there exists a series of factorizations of d as follows:*

$$d = a_0t_1 = y_1a_1t_1 = y_1a_2t_2 = y_2a_3t_2 = \cdots = y_ma_{2m-1}t_m = y_ma_{2m} \quad (1.1)$$

where $m \geq 1$, $a_i \in U$ $(i = 0, 1, \ldots, 2m)$, $y_i, t_i \in S$ $(i = 1, 2, \ldots, m)$, and

$$a_0 = y_1a_1, \qquad a_{2m-1}t_m = a_{2m},$$
$$a_{2i-1}t_i = a_{2i}t_{i+1}, \qquad y_ia_{2i} = y_{i+1}a_{2i+1} \qquad (1 \leq i \leq m-1).$$

Such a series of factorization is called a zigzag in S over U with value d, length m and spine $a_0, a_1, \ldots, a_{2m}$.

The following result is from Khan [11] and is also necessary for our investigations.

Theorem 1.2. *([11], Result 3) Let U and S be semigroups with U as a subsemigroup of S. Take any $d \in S \backslash U$ such that $d \in Dom(U,S)$. If (1.1) is a zigzag of shortest possible length m over U with value d, then $t_j, y_j \in S \backslash U$ for all $j = 1, 2, \ldots, m$.*

Definition 1.3. Let

$$x_1x_2x_3 = x_{i_1}x_{i_2}x_{i_3} \qquad (1.2)$$

and

$$x_1x_2x_3x_4 = x_{j_1}x_{j_2}x_{j_3}x_{j_4} \qquad (1.3)$$

be permutation identities, where i and j are nontrivial permutations of the sets $\{1, 2, 3\}$ and $\{1, 2, 3, 4\}$ respectively. Then a semigroup satisfying (1.2) is called

(i) an externally commutative semigroup if $i_1 = 3$ and $i_3 = 1$;
(ii) a right commutative semigroup if $i_2 = 3$ and $i_3 = 2$;
(iii) a left commutative semigroup if $i_1 = 2$ and $i_2 = 1$;
(iv) a Cyclic commutative semigroup if $i_1 = 2$, $i_2 = 3$ and $i_3 = 1$;
(v) a Dual-Cyclic commutative semigroup if $i_1 = 3$, $i_2 = 1$ and $i_3 = 2$;

while satisfying (1.3) is called

(i) a paramedial semigroup if $j_1 = 4$ and $j_4 = 1$;
(ii) a right semicommutative semigroup if $j_3 = 4$ and $j_4 = 3$;
(iii) a left cyclic commutative semigroup if $j_1 = 2$, $j_2 = 3$ and $j_3 = 1$;
(iv) a paracyclic commutative semigroup if $j_1 = 2$, $j_2 = 3$, $j_3 = 4$ and $j_4 = 1$;
(v) a dual paracyclic commutative semigroup if $j_1 = 4$, $j_2 = 1$, $j_3 = 2$ and $j_4 = 3$;

The semigroup theoretic notations and conventions of Clifford and Preston [7] and Howie [9] will be used throughout without explicit mention.

2. Closedness and Varieties of semigroups

In general varieties of bands containing the varieties of rectngular and normal bands are not absolutely closed as Higgins [8, Chapter 4] had given examples of a rectangular band and a normal band that were not absolutely closed. So, it is worthy of mention to find out the varieties which are closed in itself. Scheiblich [13], in this direction, had shown that the variety of all normal bands was closed, Alam and Khan in [3, 4, 5] had shown that the variety of left [right] regular bands, left [right] quasi-normal bands and left [right] semi-normal bands were closed. In [2], Ahanger and Shah had proved a stronger fact that the variety of left [right] regular bands was closed in the variety of all bands and, recently, Abbas and Ashraf [1] had shown that a variety of left [right] normal bands was closed in some containing homotypical varieties (varieties admitting an identity containing same variables on both sides) of semigroups.
In this section, we have shown that some more varieties of semigroups are closed in itself, but finding out a complete list of closed varieties of semigroups still remains an open problem.

Lemma 2.1. *Let U be a subsemigroup of semigroup S such that S satisfies the identity $axy = yxax$ and let $d \in Dom(U, S)\backslash U$ having* *zigzag of type (1.1) in S over U with value d of shortest possible length m. Then $(\prod_{i=0}^{m-1} a_{2i})tm = (\prod_{i=0}^{m} a_{2i})$*

Proof.

$$(\prod_{i=0}^{m-1} a_{2i})t_m$$
$= a_0a_2a_4 \cdots a_{2m-4}a_{2m-2}t_m$
$= (y_1a_1(a_2a_4 \cdots a_{2m-4}a_{2m-2}))t_m$ (by zigzag equations)
$= ((a_2a_4 \cdots a_{2m-4}a_{2m-2}a_1)y_1a_1)t_m$
(since S satisfies the identity $axy = yxax$)
$= (a_1y_1(a_2a_4 \cdots a_{2m-4}a_{2m-2}a_1)y_1)t_m$
(since S satisfies the identity $axy = yxax$)
$= a_1(y_2a_3(a_4 \cdots a_{2m-4}a_{2m-2}a_1))y_1t_m$ (by zigzag equations)
$= a_1((a_4 \cdots a_{2m-4}a_{2m-2}a_1a_3)y_2a_3)y_1t_m$
(since S satisfies the identity $axy = yxax$)
$= a_1(a_3y_2(a_4 \cdots a_{2m-4}a_{2m-2}a_1a_3)y_2)y_1t_m$
(since S satisfies the identity $axy = yxax$)

$\vdots$

$= a_1a_3 \cdots a_{2m-3}y_{m-1}a_{2m-2}a_1a_3 \cdots a_{2m-3}y_{m-1} \cdots y_2y_1t_m$
$= ((a_1a_3 \cdots a_{2m-3})(y_ma_{2m-1})(a_1a_3 \cdots a_{2m-3}y_{m-1} \cdots y_2y_1))t_m$
(by zigzag equations)
$= ((a_1a_3 \cdots a_{2m-3}y_{m-1} \cdots y_2y_1)(y_ma_{2m-1})(a_1a_3 \cdots a_{2m-3})(y_ma_{2m-1}))t_m$
(since S satisfies the identity $axy = yxax$)
$= a_1a_3 \cdots a_{2m-3}((y_{m-1} \cdots y_2y_1y_m)a_{2m-1}(a_1a_3 \cdots a_{2m-3}))y_ma_{2m}$
(by zigzag equations)
$= a_1a_3 \cdots a_{2m-3}a_1a_3 \cdots a_{2m-3}((a_{2m-1}y_{m-1} \cdots y_2y_1)y_ma_{2m-1}y_m)a_{2m}$
(since S satisfies the identity $axy = yxax$)
$= a_1a_3 \cdots a_{2m-3}((a_1a_3 \cdots a_{2m-3})a_{2m-1}y_ma_{2m-1})y_{m-1} \cdots y_2y_1a_{2m}$
(since S satisfies the identity $axy = yxax$)
$= a_1a_3 \cdots a_{2m-3}(y_ma_{2m-1}(a_1a_3 \cdots a_{2m-3}))y_{m-1} \cdots y_2y_1a_{2m}$
(since S satisfies the identity $axy = yxax$)

$= a_1a_3\cdots a_{2m-5}(a_{2m-3}y_{m-1}(a_{2m-2}a_1a_3\cdots a_{2m-5}a_{2m-3})y_{m-1})y_{m-2}\cdots y_2y_1a_{2m}$
(by zigzag equations)
$= a_1a_3\cdots a_{2m-5}((a_{2m-2}a_1a_3\cdots a_{2m-5})a_{2m-3}y_{m-1}a_{2m-3})y_{m-2}\cdots y_2y_1a_{2m}$
(since S satisfies the identity $axy = yxax$)
$= a_1a_3\cdots a_{2m-5}(y_{m-1}a_{2m-3}(a_{2m-2}a_1a_3\cdots a_{2m-5}))y_{m-2}\cdots y_2y_1a_{2m}$
(since S satisfies the identity $axy = yxax$)
$= a_1a_3\cdots a_{2m-5}y_{m-2}a_{2m-4}a_{2m-2}a_1a_3\cdots a_{2m-5}y_{m-2}\cdots y_2y_1a_{2m}$
(by zigzag equations)

$\vdots$

$= (a_1y_1(a_2a_4\cdots a_{2m-4}a_{2m-2}a_1)y_1)a_{2m}$
$= ((a_2a_4\cdots a_{2m-4}a_{2m-2})a_1y_1a_1)a_{2m}$
(since S satisfies the identity $axy = yxax$)
$= (y_1a_1(a_2a_4\cdots a_{2m-4}a_{2m-2}))a_{2m}$
(since S satisfies the identity $axy = yxax$)
$= a_0a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m}$ (by zigzag equations)
$= (\prod_{i=0}^{m} a_{2i}),$

as required. □

Theorem 2.2. *The variety $\mathcal{V} = [axy = yxax]$ of semigroups, i.e. the class of all semigroups satisfying the identity $axy = yxax$, is closed.*

Proof. Take any U, $S \in \mathcal{V}$ with U a subsemigroup of S and let $d \in Dom(U, S)\backslash U$. Suppose that d has zigzag of type (1.1) in S over U with value d of shortest possible length m. Now

$d = a_0t_1$ (by zigzag equations)
$= y_1a_1t_1$ (by zigzag equations)
$= (t_1a_1y_1)a_1$ (since S satisfies the identity $axy = yxax$)
$= (y_1a_1t_1a_1)a_1$ (since S satisfies the identity $axy = yxax$)
$= y_1a_2t_2a_1a_1$ (by zigzag equations)
$= (y_2a_3t_2)a_1a_1$ (by zigzag equations)
$= (t_2a_3y_2a_3)a_1a_1$ (since S satisfies the identity $axy = yxax$)

$= (t_2a_3y_1)a_2a_1a_1$ (by zigzag equations)
$= y_1((a_3t_2a_3)(a_2a_1)a_1)$ (since S satisfies the identity $axy = yxax$)
$= y_1(a_1(a_2a_1)(a_3t_2a_3)(a_2a_1))$ (since S satisfies the identity $axy = yxax$)
$= a_0(a_2a_1(a_3t_2a_3a_2)a_1)$ (by zigzag equations)
$= a_0a_3t_2(a_3a_2a_1a_2)$ (since S satisfies the identity $axy = yxax$)
$= (a_0a_3(t_2a_1a_2)a_3)$ (since S satisfies the identity $axy = yxax$)
$= ((t_2a_1a_2)a_3a_0)$ (since S satisfies the identity $axy = yxax$)
$= (t_2a_1(a_2a_3y_1)a_1)$ (by zigzag equations)
$= ((a_2a_3y_1)a_1t_2)$ (since S satisfies the identity $axy = yxax$)
$= (a_2a_3(a_0t_2))$ (by zigzag equations)
$= a_0(t_2a_3a_2a_3)$ (since S satisfies the identity $axy = yxax$)
$= a_0(a_2a_3t_2)$ (since S satisfies the identity $axy = yxax$)
$= (\prod_{i=0}^{1} a_{2i})(a_3t_2)$
$\vdots$
$= (\prod_{i=0}^{m-2} a_{2i})(a_{2m-3}t_{m-1})$
$= a_0a_2a_4 \cdots a_{2m-4}a_{2m-2}t_m$ (by zigzag equations)
$= a_0a_2a_4 \cdots a_{2m-4}a_{2m-2}a_{2m}$ (by Lemma 2.1)
$= (\prod_{i=0}^{m} a_{2i})$
$\in U$

$\Rightarrow Dom(U, S) = U$.
Thus the proof of the theorem is complete. □

Dually, we may prove the following result.

Theorem 2.3. *The variety $\mathcal{V} = [axy = xyxa]$ of semigroups, i.e. the class of all semigroups satisfying the identity $axy = xyxa$, is closed.*

Lemma 2.4. *Let U be a subsemigroup of semigroup S such that S satisfies the identity $axy = yxxa$ and let $d \in Dom(U, S)\setminus U$ having zigzag of type (1.1) in S over U with value d of shortest possible length* *m. Then*

$$(\prod_{i=0}^{m-1} a_{2i})t_m = (\prod_{i=0}^{m} a_{2i}).$$

Proof.

$(\prod_{i=0}^{m-1} a_{2i})t_m$
$= a_0a_2a_4 \cdots a_{2m-4}a_{2m-2}t_m$
$= y_1(a_1a_2(a_4 \cdots a_{2m-4}a_{2m-2}t_m))$ (by zigzag equations)
$= y_1((a_4 \cdots a_{2m-4}a_{2m-2}t_m)a_2(a_2a_1))$
(since S satisfies the identity $axy = yxxa$)
$= y_1a_2(a_1a_2a_2(a_4 \cdots a_{2m-4}a_{2m-2}t_m))$
(since S satisfies the identity $axy = yxxa$)
$= y_1a_2((a_4 \cdots a_{2m-4}a_{2m-2}t_m)a_2a_1)$
(since S satisfies the identity $axy = yxxa$)
$= y_2(a_3a_4(a_6 \cdots a_{2m-4}a_{2m-2}t_ma_2))a_1$ (by zigzag equations)
$= y_2((a_6 \cdots a_{2m-4}a_{2m-2}t_ma_2)a_4(a_4a_3))a_1$
(since S satisfies the identity $axy = yxxa$)
$= y_2a_4(a_3a_4a_4(a_6 \cdots a_{2m-4}a_{2m-2}t_ma_2))a_1$
(since S satisfies the identity $axy = yxxa$)
$= y_2a_4((a_6 \cdots a_{2m-4}a_{2m-2}t_ma_2)a_4a_3)a_1$
(since S satisfies the identity $axy = yxxa$)
$\vdots$
$= y_{m-1}a_{2m-2}t_ma_2a_4 \cdots a_{2m-2}a_{2m-3} \cdots a_3a_1$
$= (y_ma_{2m-1}t_m)a_2a_4 \cdots a_{2m-2}a_{2m-3} \cdots a_3a_1$ (by zigzag equations)
$= (t_ma_{2m-1}a_{2m-1})y_ma_2a_4 \cdots a_{2m-2}a_{2m-3} \cdots a_3a_1$
(since S satisfies the identity $axy = yxxa$)
$= (a_{2m-1}a_{2m-1}a_{2m-1}t_m)y_ma_2a_4 \cdots a_{2m-2}a_{2m-3} \cdots a_3a_1$
(since S satisfies the identity $axy = yxxa$)
$= (a_{2m-1}a_{2m-1}(a_{2m}y_m))a_2a_4 \cdots a_{2m-2}a_{2m-3} \cdots a_3a_1$ (by zigzag equations)

$= a_{2m}(y_m a_{2m-1} a_{2m-1} a_{2m-1}) a_2 a_4 \cdots a_{2m-2} a_{2m-3} \cdots a_3 a_1$
(since S satisfies the identity $axy = yxxa$)
$= (a_{2m} a_{2m-1} a_{2m-1} y_m) a_2 a_4 \cdots a_{2m-2} a_{2m-3} \cdots a_3 a_1$
(since S satisfies the identity $axy = yxxa$)
$= (y_m a_{2m-1} a_{2m}) a_2 a_4 \cdots a_{2m-2} a_{2m-3} \cdots a_3 a_1$
(since S satisfies the identity $axy = yxxa$)
$= y_{m-1} a_{2m-2}((a_{2m} a_2 a_4 \cdots a_{2m-4}) a_{2m-2} a_{2m-3}) a_{2m-5} \cdots a_3 a_1$
(by zigzag equations)
$= y_{m-1}((a_{2m-2} a_{2m-3}) a_{2m-2} a_{2m-2}(a_{2m} a_2 a_4 \cdots a_{2m-4})) a_{2m-5} \cdots a_3 a_1$
(since S satisfies the identity $axy = yxxa$)
$= y_{m-1}((a_{2m} a_2 a_4 \cdots a_{2m-4}) a_{2m-2} a_{2m-2} a_{2m-3}) a_{2m-5} \cdots a_3 a_1$
(since S satisfies the identity $axy = yxxa$)
$= y_{m-1}(a_{2m-3} a_{2m-2}(a_{2m} a_2 a_4 \cdots a_{2m-4})) a_{2m-5} \cdots a_3 a_1$
(since S satisfies the identity $axy = yxxa$)
$= y_{m-2} a_{2m-4} a_{2m-2} a_{2m} a_2 a_4 \cdots a_{2m-4} a_{2m-5} \cdots a_3 a_1$ (by zigzag equations)
$\vdots$
$= y_1 a_2((a_4 \cdots a_{2m-2} a_{2m}) a_2 a_1)$
$= y_1((a_2 a_1) a_2 a_2(a_4 \cdots a_{2m-2} a_{2m}))$
(since S satisfies the identity $axy = yxxa$)
$= y_1((a_4 \cdots a_{2m-2} a_{2m}) a_2 a_2 a_1)$
(since S satisfies the identity $axy = yxxa$)
$= y_1(a_1 a_2(a_4 \cdots a_{2m-2} a_{2m}))$ (since S satisfies the identity $axy = yxxa$)
$= a_0 a_2 a_4 \cdots a_{2m-2} a_{2m}$ (by zigzag equations)
$= (\prod_{i=0}^{m} a_{2i}),$

as required. □

Theorem 2.5. *The variety $\mathcal{V} = [axy = yxxa]$ of semigroups, i.e. the class of all semigroups satisfying the identity $axy = yxxa$, is closed.*

Proof. Take any U, $S \in \mathcal{V}$ with U a subsemigroup of S and let $d \in Dom(U,S) \backslash U$. Suppose that d has zigzag of type (1.1) in S over U with value d of shortest possible length m. Now

$$
\begin{aligned}
d &= a_0t_1 \text{ (by zigzag equations)}\\
&= y_1a_1t_1 \text{ (by zigzag equations)}\\
&= (t_1(a_1a_1)y_1) \text{ (since } S \text{ satisfies the identity } axy = yxxa)\\
&= (y_1(a_1a_1)(a_1a_1)t_1) \text{ (since } S \text{ satisfies the identity } axy = yxxa)\\
&= y_1((a_1a_1)a_1a_2)t_2 \text{ (by zigzag equations)}\\
&= y_1(a_2a_1a_1(a_1a_1))t_2 \text{ (since } S \text{ satisfies the identity } axy = yxxa)\\
&= (y_2a_3(a_1a_1a_1a_1))t_2 \text{ (by zigzag equations)}\\
&= ((a_1a_1)a_1a_1(a_3a_3))y_2t_2 \text{ (since } S \text{ satisfies the identity } axy = yxxa)\\
&= a_3((a_3a_1)a_1a_1(y_2t_2)) \text{ (since } S \text{ satisfies the identity } axy = yxxa)\\
&= a_3y_2(t_2a_1a_3)a_1 \text{ (since } S \text{ satisfies the identity } axy = yxxa)\\
&= a_3y_2(a_3a_1a_1t_2)a_1 \text{ (since } S \text{ satisfies the identity } axy = yxxa)\\
&= a_3((y_1a_2a_1)a_1t_2)a_1 \text{ (by zigzag equations)}\\
&= a_3t_2a_1a_1(y_1a_2(a_1a_1)) \text{ (since } S \text{ satisfies the identity } axy = yxxa)\\
&= ((a_3t_2)(a_1a_1)(a_1a_1)(a_2a_2y_1))\\
&\text{(since } S \text{ satisfies the identity } axy = yxxa)\\
&= ((a_2a_2y_1)a_1a_1(a_3t_2)) \text{ (since } S \text{ satisfies the identity } axy = yxxa)\\
&= ((a_3t_2a_1)a_2a_2y_1) \text{ (since } S \text{ satisfies the identity } axy = yxxa)\\
&= (y_1a_2(a_3t_2))a_1 \text{ (since } S \text{ satisfies the identity } axy = yxxa)\\
&= ((a_3t_2)a_2a_2y_1)a_1 \text{ (since } S \text{ satisfies the identity } axy = yxxa)\\
&= ((a_3t_2)a_2a_2a_0) \text{ (by zigzag equations)}\\
&= (a_0a_2(a_3t_2)) \text{ (since } S \text{ satisfies the identity } axy = yxxa)\\
&= (\prod_{i=0}^{1} a_{2i})(a_3t_2)\\
&\vdots\\
&= (\prod_{i=0}^{m-2} a_{2i})(a_{2m-3}t_{m-1})\\
&= a_0a_2a_4 \cdots a_{2m-4}a_{2m-2}t_m \text{ (by zigzag equations)}\\
&= a_0a_2a_4 \cdots a_{2m-4}a_{2m-2}a_{2m} \text{ (by Lemma 2.4)}\\
&= (\prod_{i=0}^{m} a_{2i})\\
&\in U
\end{aligned}
$$

$\Rightarrow Dom(U, S) = U$.
Thus the proof of the theorem is complete. □

3. Dominions and some generalized classes of commutative semigroups

Isbell [10, Corollary 2.5] showed that the dominion of a commutative semigroup is commutative. However, Khan [12] gave a counterexample to show that this stronger result is false for each (nontrivial) permutation identity other than commutativity. Recently Alam, Higgins, and Khan [6] generalized Isbell's result from commutative semigroups to $\mathcal{H}$-commutative semigroups. Now, we find some generalized classes of commutative semigroups for which this stronger result is true in some weaker form.

Theorem 3.1. *Let U be an externally commutative subsemigroup of a paramedial semigroup S. Then $Dom(U, S)$ is externally commutative semigroup.*

Proof. Let U be an externally commutative subsemigroup of a paramedial semigroup S. Then we have to show that $Dom(U, S)$ is also externally commutative semigroup.

Case (i): If $d_1, d_2, d_3 \in U$, then the result holds trivially.

Case (ii): $d_1 \in Dom(U, S) \backslash U$ and $d_2, d_3 \in U$.
Then, by Theorem 1.1, d_1 has zigzag equations of type (1.1) in S over U of length m. Now

$$\begin{aligned}
d_1d_2d_3 &= y_m(a_{2m}d_2d_3) \text{ (by zigzag equations)}\\
&= y_m(d_3d_2a_{2m}) \text{ (since U is externally commutative)}\\
&= y_md_3d_2a_{2m-1}t_m \text{ (by zigzag equations)}\\
&= y_m(d_3d_2a_{2m-1})t_m\\
&= y_m(a_{2m-1}d_2d_3)t_m \text{ (since U is externally commutative)}\\
&= y_{m-1}a_{2m-2}d_2d_3t_m \text{ (by zigzag equations)}\\
&= y_{m-1}(a_{2m-2}d_2d_3)t_m\\
&= y_{m-1}(d_3d_2a_{2m-2})t_m \text{ (since U is externally commutative)}
\end{aligned}$$

$$
\begin{aligned}
&= y_{m-1}d_3d_2a_{2m-3}t_{m-1} \text{ (by zigzag equations)}\\
&\vdots\\
&= y_1(d_3d_2a_1)t_1\\
&= y_1(a_1d_2d_3)t_1 \text{ (since U is externally commutative)}\\
&= a_0d_2d_3t_1 \text{ (by zigzag equations)}\\
&= (a_0d_2d_3)t_1\\
&= (d_3d_2a_0)t_1 \text{ (since U is externally commutative)}\\
&= d_3d_2d_1 \text{ (by zigzag equations),}
\end{aligned}
$$

as required.

Case (iii): $d_1, d_2 \in Dom(U, S)\backslash U$ and $d_3 \in U$.
Then, by Theorem 1.1, d_2 has zigzag equations of type (1.1) in S over U of length m. Now

$$
\begin{aligned}
d_1d_2d_3 &= d_1y_ma_{2m}d_3 \text{ (by zigzag equations)}\\
&= d_3y_ma_{2m}d_1 \text{ (since S satisfies the identity } x_1x_2x_3x_4 = x_4x_2x_3x_1)\\
&= d_3d_2d_1 \text{ (by zigzag equations),}
\end{aligned}
$$

as required.

Case (iv): $d_1, d_2, d_3 \in Dom(U, S)\backslash U$.
Then, by Theorem 1.1, d_3 has zigzag equations of type (1.1) in S over U of length m. Now

$$
\begin{aligned}
d_1d_2d_3 &= (d_1d_2a_0)t_1 \text{ (by zigzag equations)}\\
&= (a_0d_2d_1)t_1 \text{ (by Case (iii))}\\
&= y_1a_1d_2d_1t_1 \text{ (by zigzag equations)}\\
&= y_1(a_1d_2d_1)t_1\\
&= y_1(d_1d_2a_1)t_1 \text{ (by case (iii))}\\
&= y_1d_1d_2a_2t_2 \text{ (by zigzag equations)}\\
&= y_1(d_1d_2a_2)t_2\\
&= y_1(a_2d_2d_1)t_2 \text{ (by Case (iii))}\\
&= y_2a_3d_2d_1t_2 \text{ (by zigzag equations)}\\
&\vdots\\
&= y_m(a_{2m-1}d_2d_1)t_m
\end{aligned}
$$

$$= y_m(d_1d_2a_{2m-1})t_m \text{ (by Case (iii))}$$
$$= y_md_1d_2a_{2m} \text{ (by zigzag equations)}$$
$$= y_m(d_1d_2a_{2m})$$
$$= y_m(a_{2m}d_2d_1) \text{ (by Case (iii))}$$
$$= d_3d_2d_1 \text{ (by zigzag equations),}$$

as required. Thus the proof of the theorem is complete. □

Theorem 3.2. *Let U be a right commutative subsemigroup of a right semi-commutative semigroup S. Then $Dom(U, S)$ is right commutative semigroup.*

Proof. Let U be a right commutative subsemigroup of a right semicommutative semigroup S. Then we have to show that $Dom(U, S)$ is also right commutative semigroup.

Case (i): If $d_1, d_2, d_3 \in U$, then the result holds trivially.

Case (ii): $d_1 \in Dom(U, S)\backslash U$ and $d_2, d_3 \in U$.
Then, by Theorem 1.1, d_1 has zigzag equations of type (1.1) in S over U of length m. Now

$$d_1d_2d_3 = y_m(a_{2m}d_2d_3) \text{ (by zigzag equations)}$$
$$= y_m(a_{2m}d_3d_2) \text{ (since U is right commutative)}$$
$$= d_1d_3d_2 \text{ (by zigzag equations),}$$

as required.

Case (iii): $d_1, d_2 \in Dom(U, S)\backslash U$ and $d_3 \in U$.
Then, by Theorem 1.1, d_2 has zigzag equations of type (1.1) in S over U of length m. Now

$$d_1d_2d_3 = d_1a_0t_1d_3 \text{ (by zigzag equations)}$$
$$= d_1a_0d_3t_1 \text{ (since S satisfies the identity } x_1x_2x_3x_4 = x_1x_2x_4x_3)$$
$$= (d_1a_0d_3)t_1$$
$$= (d_1d_3a_0)t_1 \text{ (by Case (ii))}$$
$$= d_1d_3d_2 \text{ (by zigzag equations),}$$

as required.

Case (iv): $d_1, d_2, d_3 \in Dom(U, S)\backslash U$.
Then, by Theorem 1.1, d_3 has zigzag equations of type (1.1) in S over

U of length m. Now

$$\begin{aligned} d_1d_2d_3 &= (d_1d_2a_0)t_1 \text{ (by zigzag equations)} \\ &= (d_1a_0d_2)t_1 \text{ (by Case (iii))} \\ &= d_1a_0t_1d_2 \text{ (since S satisfies the identity } x_1x_2x_3x_4 = x_1x_2x_4x_3) \\ &= d_1d_3d_2 \text{ (by zigzag equations),} \end{aligned}$$

as required. Thus the proof of the theorem is complete. □

Theorem 3.3. *Let U be a left commutative subsemigroup of a left cyclic semigroup S. Then $Dom(U,S)$ is left commutative semigroup.*

Proof. Let U be a left commutative subsemigroup of a left cyclic semigroup S. Then we have to show that $Dom(U,S)$ is also left commutative semigroup.

Case (i): If $d_1, d_2, d_3 \in U$, then the result holds trivially.

Case (ii): $d_1 \in Dom(U,S)\backslash U$ and $d_2, d_3 \in U$.
Then, by Theorem 1.1, d_1 has zigzag equations of type (1.1) in S over U of length m. Now

$$\begin{aligned} d_1d_2d_3 &= y_m(a_{2m}d_2d_3) \text{ (by zigzag equations)} \\ &= y_m(d_2a_{2m}d_3) \text{ (by Case (i))} \\ &= d_2a_{2m}y_md_3 \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_2x_3x_1x_4) \\ &= d_2a_{2m-1}t_my_md_3 \text{ (by zigzag equations)} \\ &= (d_2a_{2m-1}t_my_m)d_3 \\ &= (a_{2m-1}t_md_2y_m)d_3 \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_2x_3x_1x_4) \\ &= ((a_{2m-1}t_m)d_2y_md_3) \\ &= (d_2y_m(a_{2m-1}t_m)d_3) \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_2x_3x_1x_4) \\ &= d_2y_ma_{2m}d_3 \text{ (by zigzag equations)} \\ &= d_2d_1d_3 \text{ (by zigzag equations),} \end{aligned}$$

as required.

Case (iii): $d_1, d_2 \in Dom(U,S)\backslash U$ and $d_3 \in U$.
Then, by Theorem 1.1, d_2 has zigzag equations of type (1.1) in S over U of length m. Now

$$\begin{aligned} d_1d_2d_3 &= d_1a_0t_1d_3 \text{ (by zigzag equations)} \\ &= a_0t_1d_1d_3 \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_2x_3x_1x_4) \\ &= d_2d_1d_3 \text{ (by zigzag equations),} \end{aligned}$$

as required.

Case (iv): $d_1, d_2, d_3 \in Dom(U,S)\backslash U$.
Then, by Theorem 1.1, d_3 has zigzag equations of type (1.1) in S over U of length m. Now

$$\begin{aligned}
d_1d_2d_3 &= d_1d_2y_ma_{2m} \text{ (by zigzag equations)}\\
&= d_2y_md_1a_{2m} \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_2x_3x_1x_4)\\
&= y_md_1d_2a_{2m} \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_2x_3x_1x_4)\\
&= y_m(d_1d_2a_{2m})\\
&= y_m(d_2d_1a_{2m}) \text{ (by Case (iii))}\\
&= d_2d_1y_ma_{2m} \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_2x_3x_1x_4)\\
&= d_2d_1d_3 \text{ (by zigzag equations),}
\end{aligned}$$

as required. Thus the proof of the theorem is complete. □

Theorem 3.4. *Let U be a cyclic commutative subsemigroup of a para-cyclic commutative semigroup S. Then $Dom(U,S)$ is cyclic commutative semigroup.*

Proof. Let U be a cyclic commutative subsemigroup of a para-cyclic commutative semigroup S. Then we have to show that $Dom(U,S)$ is also cyclic commutative semigroup.

Case (i): If $d_1, d_2, d_3 \in U$, then the result holds trivially.

Case (ii): $d_1 \in Dom(U,S)\backslash U$ and $d_2, d_3 \in U$.
Then, by Theorem 1.1, d_1 has zigzag equations of type (1.1) in S over U of length m. Now

$$\begin{aligned}
d_1d_2d_3 &= a_0t_1d_2d_3 \text{ (by zigzag equations)}\\
&= t_1d_2d_3a_0 \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_2x_3x_4x_1)\\
&= d_2d_3a_0t_1 \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_2x_3x_4x_1)\\
&= d_2d_3d_1 \text{ (by zigzag equations),}
\end{aligned}$$

as required.

Case (iii): $d_1, d_2 \in Dom(U,S)\backslash U$ and $d_3 \in U$.
Then, by Theorem 1.1, d_2 has zigzag equations of type (1.1) in S over U of length m. Now

$$\begin{aligned}
d_1d_2d_3 &= d_1y_ma_{2m}d_3 \text{ (by zigzag equations)}\\
&= y_ma_{2m}d_3d_1 \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_2x_3x_4x_1)\\
&= d_2d_3d_1 \text{ (by zigzag equations),}
\end{aligned}$$

as required.

Case (iv): $d_1, d_2, d_3 \in Dom(U,S)\backslash U$.
Then, by Theorem 1.1, d_3 has zigzag equations of type (1.1) in S over U of length m. Now

$$\begin{aligned}
d_1d_2d_3 &= (d_1d_2a_0)t_1 \text{ (by zigzag equations)}\\
&= (d_2a_0d_1)t_1 \text{ (by Case (iii))}\\
&= d_2y_1a_1d_1t_1 \text{ (by zigzag equations)}\\
&= (d_2y_1a_1(d_1t_1))\\
&= (y_1a_1(d_1t_1)d_2)\\
&\text{(since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_2x_3x_4x_1)\\
&= ((y_1a_1)d_1t_1d_2)\\
&= (d_1t_1d_2(y_1a_1))\\
&\text{(since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_2x_3x_4x_1)\\
&= (t_1d_2(y_1a_1)d_1)\\
&\text{(since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_2x_3x_4x_1)\\
&= (t_1d_2y_1a_1)d_1\\
&= (d_2y_1a_1t_1)d_1\\
&\text{(since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_2x_3x_4x_1)\\
&= d_2a_0t_1d_1 \text{ (by zigzag equations)}\\
&= d_2d_3d_1 \text{ (by zigzag equations)},
\end{aligned}$$

as required. Thus the proof of the theorem is complete. □

Theorem 3.5. *Let U be a dual-cyclic commutative subsemigroup of a dual paracyclic semigroup S. Then $Dom(U,S)$ is dual-cyclic commutative semigroup.*

Proof. Let U be a dual-cyclic commutative subsemigroup of a dual paracyclic semigroup S. Then we have to show that $Dom(U,S)$ is also dual-cyclic commutative semigroup.

Case (i): If $d_1, d_2, d_3 \in U$, then the result holds trivially.

Case (ii): $d_1 \in Dom(U,S)\backslash U$ and $d_2, d_3 \in U$.
Then, by Theorem 1.1, d_1 has zigzag equations of type (1.1) in S over

U of length m. Now

$$\begin{aligned} d_1d_2d_3 &= y_m a_{2m} d_2 d_3 \text{ (by zigzag equations)} \\ &= d_3 y_m a_{2m} d_2 \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_4x_1x_2x_3) \\ &= d_3 d_1 d_2 \text{ (by zigzag equations),} \end{aligned}$$

as required.

Case (iii): $d_1, d_2 \in Dom(U,S)\setminus U$ and $d_3 \in U$.
Then, by Theorem 1.1, d_2 has zigzag equations of type (1.1) in S over U of length m. Now

$$\begin{aligned} d_1d_2d_3 &= d_1 a_0 t_1 d_3 \text{ (by zigzag equations)} \\ &= d_3 d_1 a_0 t_1 \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_4x_1x_2x_3) \\ &= d_3 d_1 d_2 \text{ (by zigzag equations),} \end{aligned}$$

as required.

Case (iv): $d_1, d_2, d_3 \in Dom(U,S)\setminus U$.
Then, by Theorem 1.1, d_3 has zigzag equations of type (1.1) in S over U of length m. Now

$$\begin{aligned} d_1d_2d_3 &= d_1 d_2 y_m a_{2m} \text{ (by zigzag equations)} \\ &= a_{2m} d_1 d_2 y_m \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_4x_1x_2x_3) \\ &= y_m a_{2m} d_1 d_2 \text{ (since } S \text{ satisfies the identity } x_1x_2x_3x_4 = x_4x_1x_2x_3) \\ &= d_3 d_1 d_2 \text{ (by zigzag equations),} \end{aligned}$$

as required. Thus the proof of the theorem is complete. □

Acknowledgment

I would like to express my special thanks of gratitude to my supervisor, *Dr. Wajih Ashraf*, who gave me an exposure "On Dominions and Determination of Closed Varieties of Semigroups" and guided me throughout this research paper.

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