Journal of Algebra and Related Topics
Vol. 9, No 2, (2021), pp 83-95

ON CLOSED HOMOTYPICAL VARIETIES OF SEMIGROUPS

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ABSTRACT. It is known that all subvarieties of the variety of all semigroups are not absolutely closed. So, we determine some closed homotypical varieties of semigroups determined by the identities $axy = x^2ayx$, $axy = xa^2ya$, $axy = yay^2x$, $axy = xaya^2$, $axy = y^2ayx$ and $axy = xayx^2$.

1. INTRODUCTION AND PRELIMINARIES

Let U be a subsemigroup of a semigroup S. Following Isbell [4], we say that U dominates an element d of S if for every semigroup T and for all homomorphisms $\beta, \gamma : S \longrightarrow T$ and $u\beta$=$u\gamma$ for every u in U implies $d\beta$=$d\gamma$. The set of all elements of S dominated by U is called the *dominion* of U in S and we denote it by $Dom(U, S)$. It can be easily verified that $Dom(U, S)$ is a subsemigroup of S containing U. A subsemigroup U of semigroup S is called closed if $Dom(U, S) = U$. A semigroup is called absolutely closed if it is closed in every containing semigroup. Let $\mathcal{C}$ be a class of semigroups. A semigroup U is said to be $\mathcal{C}$-closed if $Dom(U, S) = U$ for all $S \in \mathcal{C}$ such that $U \subseteq S$. Let $\mathcal{B}$ and $\mathcal{C}$ be the classes of semigroups such that $\mathcal{B}$ is a subclass of $\mathcal{C}$. We say that $\mathcal{B}$ is $\mathcal{C}$-closed if every member of $\mathcal{B}$ is $\mathcal{C}$-closed. A class $\mathcal{C}$ of semigroups is said to be closed if $Dom(U, S) = U$ for all $U, S \in \mathcal{C}$ with U as a subsemigroup of S. Let $\mathcal{A}$ and $\mathcal{D}$ be two categories of semigroups with $\mathcal{A}$ is a subcategory of $\mathcal{D}$. It can be easily verified that a semigroup U is $\mathcal{A}$-closed if it is $\mathcal{D}$-closed. For any word u, the content

AMS Subject Classification: 20M07.
Keywords: Zigzag equations; Homotypical; Variety; Identity and Closed.
Received: 16 February 2021, Accepted: 5 June 2021.
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of u (necessarily finite) is the set of all distinct variables appearing in u and is denoted by $C(u)$. The identity $u = v$ is said to be homotypical if $C(u) = C(v)$; otherwise heterotypical. A variety $\mathcal{V}$ of semigroups is said to be homotypical if it admits a homotypical identity.

The following result provided by Isbell [4], known as Isbell's zigzag theorem, is a most useful characterization of semigroup dominions and is of basic importance to our investigations.

Theorem 1.1. *([4], Theorem 2.3) Let U be a subsemigroup of a semigroup S and let $d \in S$. Then $d \in Dom(U, S)$ if and only if $d \in U$ or there exists a series of factorizations of d as follows:*

$$d = a_0t_1 = y_1a_1t_1 = y_1a_2t_2 = y_2a_3t_2 = \cdots = y_ma_{2m-1}t_m = y_ma_{2m} \quad (1.1)$$

where $m \geq 1$, $a_i \in U$ $(i = 0, 1, \ldots, 2m)$, $y_i, t_i \in S$ $(i = 1, 2, \ldots, m)$, and

$$a_0 = y_1a_1, \qquad a_{2m-1}t_m = a_{2m},$$
$$a_{2i-1}t_i = a_{2i}t_{i+1}, \qquad y_ia_{2i} = y_{i+1}a_{2i+1} \qquad (1 \leq i \leq m-1).$$

Such a series of factorization is called a zigzag in S over U with value d, length m and spine $a_0, a_1, \ldots, a_{2m}$.

The following result is also necessary for our investigations.

Theorem 1.2. *([5], Result 3) Let U and S be semigroups with U as a subsemigroup of S. Take any $d \in S \backslash U$ such that $d \in Dom(U, S)$. If (1.1) is a zigzag of minimal length m over U with value d, then $t_j, y_j \in S \backslash U$ for all $j = 1, 2, \ldots, m$.*

Semigroup theoretic notations and conventions of Clifford and Preston [1] and Howie [3] will be used throughout without explicit mention.

2. Closedness and Varieties of semigroups

Higgins [2, Chapter 4] has shown that not all varieties of semigroups are absolutely closed by giving examples of a rectangular band and a normal band etc. that are not absolutely closed. So, it is worthy of attention to find out the varieties of semigroups which are closed in itself. In this section, we have been able to show that some homotypical varieties of semigroups are closed, but to find out the complete list of varieties that are closed still remains an open problem.

Theorem 2.1. *The variety* $\mathcal{V} = [axy = x^2ayx]$ *of semigroups is closed.*

Proof. Take any U, $S \in \mathcal{V}$ with U as a subsemigroup of S such that $d \in Dom(U, S) \backslash U$. Let d has zigzag of type (1.1) in S over U of minimal length m. In order to prove the theorem, we first prove the following lemma.

Lemma 2.2. *For each* $k = 1, 2, ..., m$.

$$d = (\prod_{i=1}^{k} a_{2i-1}^{3}) y_k a_{2k-1} t_k (\prod_{i=1}^{k} a_{2k-(2i-1)}).$$

Proof. For any $x, y, z \in S$, we have

$$\begin{aligned} xyz &= y^2xzy \text{ (as } S \in \mathcal{V}\text{)} \\ &= (yyx)zy \\ &= yyyxyzy \text{ (as } S \in \mathcal{V}\text{)} \\ &= y^3xyzy. \end{aligned} \tag{2.1}$$

To prove the lemma we use induction on k. For $k = 1$, we have

$$\begin{aligned} d &= y_1a_1t_1 \text{ (by zigzag equations)} \\ &= a_1^3y_1a_1t_1a_1 \text{ (by equation (2.1)).} \end{aligned}$$

Thus the result holds for $k = 1$. Assume inductively that the result holds for $k = j < m$. We will show that it also holds for $k = j + 1$. Now

$$\begin{aligned} d &= (\prod_{i=1}^{j} a_{2i-1}^{3}) y_j a_{2j-1} t_j (\prod_{i=1}^{j} a_{2j-(2i-1)}) \text{ (by inductive hypothesis)} \\ &= (\prod_{i=1}^{j} a_{2i-1}^{3}) y_{j+1} a_{2j+1} t_{j+1} (\prod_{i=1}^{j} a_{2j-(2i-1)}) \text{ (by zigzag equations)} \\ &= (\prod_{i=1}^{j} a_{2i-1}^{3}) a_{2j+1}^{3} y_{j+1} a_{2j+1} t_{j+1} a_{2j+1} (\prod_{i=1}^{j} a_{2j-(2i-1)}) \text{ (by equation (2.1))} \\ &= (\prod_{i=1}^{j+1} a_{2i-1}^{3}) y_{j+1} a_{2j+1} t_{j+1} (\prod_{i=1}^{j+1} a_{2(j+1)-(2i-1)}), \end{aligned}$$

as required. □

Now to complete the proof of theorem, letting $k = m$ in Lemma 2.2, we get

$$\begin{aligned}
d &= (\prod_{i=1}^{m} a_{2i-1}^3) y_m a_{2m-1} t_m (\prod_{i=1}^{m} a_{2m-(2i-1)}) \\
&= (\prod_{i=1}^{m-1} a_{2i-1}^3) a_{2m-1}^3 y_m a_{2m} (\prod_{i=1}^{m} a_{2m-(2i-1)}) \text{ (by zigzag equations)} \\
&= (\prod_{i=1}^{m-1} a_{2i-1}^3)(a_{2m-1}^2 (a_{2m-1} y_m) a_{2m} a_{2m-1})(\prod_{i=2}^{m} a_{2m-(2i-1)}) \\
&= (\prod_{i=1}^{m-1} a_{2i-1}^3) a_{2m-1} y_m a_{2m-1} a_{2m} (\prod_{i=2}^{m} a_{2m-(2i-1)}) \text{ (as } S \in \mathcal{V}\text{)} \\
&= (\prod_{i=1}^{m-1} a_{2i-1}^3) a_{2m-1} y_{m-1} a_{2m-2} a_{2m} (\prod_{i=2}^{m} a_{2m-(2i-1)}) \text{ (by zigzag equations)} \\
&= (\prod_{i=1}^{m-2} a_{2i-1}^3)(a_{2m-3}^2 (a_{2m-3} a_{2m-1} y_{m-1})(a_{2m-2} a_{2m}) a_{2m-3})(\prod_{i=3}^{m} a_{2m-(2i-1)}) \\
&= (\prod_{i=1}^{m-2} a_{2i-1}^3) a_{2m-3} a_{2m-1} y_{m-1} a_{2m-3} a_{2m-2} a_{2m} (\prod_{i=3}^{m} a_{2m-(2i-1)}) \text{ (as } S \in \mathcal{V}\text{)} \\
&= (\prod_{i=1}^{m-2} a_{2i-1}^3) a_{2m-3} a_{2m-1} y_{m-2} a_{2m-4} a_{2m-2} a_{2m} (\prod_{i=3}^{m} a_{2m-(2i-1)}) \\
&\quad \text{(by zigzag equations)} \\
&\vdots \\
&= (\prod_{i=1}^{1} a_{2i-1}^3) a_3 \cdots a_{2m-3} a_{2m-1} y_1 a_2 a_4 \cdots a_{2m-2} a_{2m} (\prod_{i=m}^{m} a_{2m-(2i-1)}) \\
&= a_1^2 (a_1 a_3 \cdots a_{2m-3} a_{2m-1} y_1)(a_2 a_4 \cdots a_{2m-2} a_{2m}) a_1 \\
&= a_1 a_3 \cdots a_{2m-3} a_{2m-1} y_1 a_1 a_2 a_4 \cdots a_{2m-2} a_{2m} \text{ (as } S \in \mathcal{V}\text{)} \\
&= a_1 a_3 \cdots a_{2m-3} a_{2m-1} a_0 a_2 a_4 \cdots a_{2m-2} a_{2m} \text{ (by zigzag equations)} \\
&= \prod_{i=1}^{m} a_{2i-1} \prod_{i=0}^{m} a_{2i}.
\end{aligned}$$

Thus $d \in U$, a contradiction as required.

□

Theorem 2.3. *The variety* $\mathcal{V} = [axy = xa^2ya]$ *of semigroups is closed.*

Proof. Take any U, $S \in \mathcal{V}$ with U as a subsemigroup of S such that $d \in Dom(U,S)\backslash U$. Let d has zigzag of type (1.1) in S over U of minimal length m. In order to prove the theorem, we first prove the following lemma.

Lemma 2.4. *For each* $k = 1, 2, ..., m$.

$$d = (\prod_{i=1}^{k} a_{2i-1}^2)y_k a_{2k-1} t_k.$$

Proof. For any $x, y, z \in S$, we have

$$\begin{aligned}
xyz &= yx^2zx \text{ (as } S \in \mathcal{V}\text{)}\\
&= (yxx)zx\\
&= xy^2xyzx \text{ (as } S \in \mathcal{V}\text{)}\\
&= ((xy^2x)(yz)x)\\
&= yz(xy^2x)^2xxy^2x \text{ (as } S \in \mathcal{V}\text{)}\\
&= y(z(xy^2x)^2x(xy^2x))\\
&= yxy^2xzx \text{ (as } S \in \mathcal{V}\text{)}\\
&= y(x(y^2x)z)x\\
&= yy^2xx^2zxx \text{ (as } S \in \mathcal{V}\text{)}\\
&= yy^2(xx^2zx)x\\
&= y^3xxzx \text{ (as } S \in \mathcal{V}\text{)}\\
&= y^2(yx^2zx)\\
&= y^2xyz \text{ (as } S \in \mathcal{V}\text{).}
\end{aligned} \tag{2.2}$$

To prove the lemma we use induction on k. For $k = 1$, we have

$$\begin{aligned}
d &= y_1a_1t_1 \text{ (by zigzag equations)}\\
&= a_1^2y_1a_1t_1 \text{ (by equation (2.2)).}
\end{aligned}$$

Thus the result holds for $k = 1$. Assume inductively that the result holds for $k = j < m$. We will show that it also holds for $k = j + 1$.

Now

$$
\begin{aligned}
d &= (\prod_{i=1}^{j} a_{2i-1}^2) y_j a_{2j-1} t_j \text{ (by inductive hypothesis)}\\
&= (\prod_{i=1}^{j} a_{2i-1}^2) y_{j+1} a_{2j+1} t_{j+1} \text{ (by zigzag equations)}\\
&= (\prod_{i=1}^{j} a_{2i-1}^2) a_{2j+1}^2 y_{j+1} a_{2j+1} t_{j+1} \text{ (by equation (2.2))}\\
&= (\prod_{i=1}^{j+1} a_{2i-1}^2) y_{j+1} a_{2j+1} t_{j+1},
\end{aligned}
$$

as required. □

Now to complete the proof of theorem, letting $k = m$ in Lemma 2.4, we get

$$
\begin{aligned}
d &= (\prod_{i=1}^{m} a_{2i-1}^2) y_m a_{2m-1} t_m\\
&= (\prod_{i=1}^{m-1} a_{2i-1}^2)(a_{2m-1}^2 (y_m a_{2m-1}) t_m)\\
&= (\prod_{i=1}^{m-1} a_{2i-1}^2) y_m a_{2m-1} (a_{2m-1}^2)^2 t_m a_{2m-1}^2 \text{ (as } S \in \mathcal{V} \text{)}\\
&= (\prod_{i=1}^{m-1} a_{2i-1}^2) y_m (a_{2m-1} (a_{2m-1}^2)^2 t_m a_{2m-1}^2)\\
&= (\prod_{i=1}^{m-1} a_{2i-1}^2) y_m a_{2m-1} a_{2m-1} a_{2m-1} t_m \text{ (as } S \in \mathcal{V} \text{)}\\
&= (\prod_{i=1}^{m-2} a_{2i-1}^2) a_{2m-3}^2 y_{m-1} a_{2m-2} a_{2m-1} a_{2m} \text{ (by zigzag equations)}\\
&= (\prod_{i=1}^{m-2} a_{2i-1}^2)(a_{2m-3}^2 (y_{m-1} a_{2m-2}) a_{2m-1}) a_{2m}\\
&= (\prod_{i=1}^{m-2} a_{2i-1}^2) y_{m-1} a_{2m-2} (a_{2m-3}^2)^2 a_{2m-1} a_{2m-3}^2 a_{2m} \text{ (as } S \in \mathcal{V} \text{)}
\end{aligned}
$$

$$= (\prod_{i=1}^{m-2} a_{2i-1}^2) y_{m-1}(a_{2m-2}(a_{2m-3}^2)^2 a_{2m-1} a_{2m-3}^2) a_{2m}$$

$$= (\prod_{i=1}^{m-2} a_{2i-1}^2) y_{m-1} a_{2m-3} a_{2m-3} a_{2m-2} a_{2m-1} a_{2m} \text{ (as } S \in \mathcal{V}\text{)}$$

$$= (\prod_{i=1}^{m-2} a_{2i-1}^2) y_{m-2} a_{2m-4} a_{2m-3} a_{2m-2} a_{2m-1} a_{2m} \text{ (by zigzag equations)}$$

$$\vdots$$

$$= (\prod_{i=1}^{1} a_{2i-1}^2) y_1 a_2 a_3 a_4 a_5 a_6 \cdots a_{2m-1} a_{2m}$$

$$= (a_1^2 (y_1 a_2) a_3) a_4 a_5 a_6 \cdots a_{2m-1} a_{2m}$$

$$= y_1 a_2 (a_1^2)^2 a_3 a_1^2 a_4 a_5 a_6 \cdots a_{2m-1} a_{2m} \text{ (as } S \in \mathcal{V}\text{)}$$

$$= y_1 (a_2 (a_1^2)^2 a_3 a_1^2) a_4 a_5 a_6 \cdots a_{2m-1} a_{2m}$$

$$= y_1 a_1 a_1 a_2 a_3 a_4 a_5 a_6 \cdots a_{2m-1} a_{2m} \text{ (as } S \in \mathcal{V}\text{)}$$

$$= a_0 a_1 a_2 a_3 a_4 a_5 a_6 \cdots a_{2m-1} a_{2m} \text{ (by zigzag equations).}$$

Thus $d \in U$, a contradiction as required. □

Dually, we can prove the following results.

Theorem 2.5. *The variety $\mathcal{V} = [axy = yay^2x]$ of semigroups is closed.*

Theorem 2.6. *The variety $\mathcal{V} = [axy = xaya^2]$ of semigroups is closed.*

Proof. Take any U, $S \in \mathcal{V}$ with U as a subsemigroup of S such that $d \in Dom(U, S) \backslash U$. Let d has zigzag of type (1.1) in S over U of minimal length m. In order to prove the theorem, we first prove the following lemma.

Lemma 2.7. *For each $k = 1, 2, ..., m$.*

$$d = (\prod_{i=1}^{k} a_{2i-1}^2) y_k a_{2k-1} t_k.$$

Proof. For any $x, y, z \in S$, we have

$$\begin{aligned} xyz &= yxzx^2 \text{ (as } S \in \mathcal{V}\text{)} \\ &= (yxz)xx \\ &= xyzyyxx \text{ (as } S \in \mathcal{V}\text{)} \\ &= ((xyz)(yyx)x) \\ &= yyxxyzx(xyz)^2 \text{ (as } S \in \mathcal{V}\text{)} \\ &= yy(x(xyz)x(xyz)^2) \\ &= yyxyzxx \text{ (as } S \in \mathcal{V}\text{)} \\ &= yy(x(yz)(xx)) \\ &= yyyzxxxxx \text{ (as } S \in \mathcal{V}\text{)} \\ &= yyy(zx(xx)x^2) \\ &= yyyxzxx \text{ (as } S \in \mathcal{V}\text{)} \\ &= yy(yxzx^2) \\ &= y^2xyz \text{ (as } S \in \mathcal{V}\text{).} \end{aligned} \tag{2.3}$$

To prove the lemma we use induction on k. For $k = 1$, we have

$$\begin{aligned} d &= y_1a_1t_1 \text{ (by zigzag equations)} \\ &= a_1^2y_1a_1t_1 \text{ (by equation (2.3)).} \end{aligned}$$

Thus the result holds for $k = 1$. Assume inductively that the result holds for $k = j < m$. We will show that it also holds for $k = j + 1$. Now

$$\begin{aligned} d &= (\prod_{i=1}^{j} a_{2i-1}^2)y_ja_{2j-1}t_j \text{ (by inductive hypothesis)} \\ &= (\prod_{i=1}^{j} a_{2i-1}^2)y_{j+1}a_{2j+1}t_{j+1} \text{ (by zigzag equations)} \\ &= (\prod_{i=1}^{j} a_{2i-1}^2)a_{2j+1}^2y_{j+1}a_{2j+1}t_{j+1} \text{ (by equation (2.3))} \\ &= (\prod_{i=1}^{j+1} a_{2i-1}^2)y_{j+1}a_{2j+1}t_{j+1}, \end{aligned}$$

as required. □

Now to complete the proof of theorem, letting $k = m$ in Lemma 2.7, we get

$$
\begin{aligned}
d &= (\prod_{i=1}^{m} a_{2i-1}^2) y_m a_{2m-1} t_m \\
&= (\prod_{i=1}^{m-1} a_{2i-1}^2)(a_{2m-1}^2 (y_m a_{2m-1}) t_m) \\
&= (\prod_{i=1}^{m-1} a_{2i-1}^2) y_m a_{2m-1} a_{2m-1}^2 t_m (a_{2m-1}^2)^2 \text{ (as } S \in \mathcal{V} \text{)} \\
&= (\prod_{i=1}^{m-1} a_{2i-1}^2) y_m (a_{2m-1} a_{2m-1}^2 t_m (a_{2m-1}^2)^2) \\
&= (\prod_{i=1}^{m-1} a_{2i-1}^2) y_m a_{2m-1} a_{2m-1} a_{2m-1} t_m \text{ (as } S \in \mathcal{V} \text{)} \\
&= (\prod_{i=1}^{m-2} a_{2i-1}^2) a_{2m-3}^2 y_{m-1} a_{2m-2} a_{2m-1} a_{2m} \text{ (by zigzag equations)} \\
&= (\prod_{i=1}^{m-2} a_{2i-1}^2)(a_{2m-3}^2 (y_{m-1} a_{2m-2}) a_{2m-1}) a_{2m} \\
&= (\prod_{i=1}^{m-2} a_{2i-1}^2) y_{m-1} a_{2m-2} a_{2m-3}^2 a_{2m-1} (a_{2m-3}^2)^2 a_{2m} \text{ (as } S \in \mathcal{V} \text{)} \\
&= (\prod_{i=1}^{m-2} a_{2i-1}^2) y_{m-1} (a_{2m-2} a_{2m-3}^2 a_{2m-1} (a_{2m-3}^2)^2) a_{2m} \\
&= (\prod_{i=1}^{m-2} a_{2i-1}^2) y_{m-1} a_{2m-3} a_{2m-3} a_{2m-2} a_{2m-1} a_{2m} \text{ (as } S \in \mathcal{V} \text{)} \\
&= (\prod_{i=1}^{m-2} a_{2i-1}^2) y_{m-2} a_{2m-4} a_{2m-3} a_{2m-2} a_{2m-1} a_{2m} \text{ (by zigzag equations)} \\
&\;\vdots \\
&= (\prod_{i=1}^{1} a_{2i-1}^2) y_1 a_2 a_3 a_4 a_5 a_6 \cdots a_{2m-1} a_{2m} \\
&= (a_1^2 (y_1 a_2) a_3) a_4 a_5 a_6 \cdots a_{2m-1} a_{2m} \\
&= y_1 a_2 a_1^2 a_3 (a_1^2)^2 a_4 a_5 a_6 \cdots a_{2m-1} a_{2m} \text{ (as } S \in \mathcal{V} \text{)}
\end{aligned}
$$

$$
\begin{aligned}
&= y_1(a_2a_1^2a_3(a_1^2)^2)a_4a_5a_6\cdots a_{2m-1}a_{2m}\\
&= y_1a_1a_1a_2a_3a_4a_5a_6\cdots a_{2m-1}a_{2m} \text{ (as } S\in\mathcal{V}\text{)}\\
&= a_0a_1a_2a_3a_4a_5a_6\cdots a_{2m-1}a_{2m} \text{ (by zigzag equations).}
\end{aligned}
$$

Thus $d\in U$, a contradiction as required. □

Dually, we can prove the following results.

Theorem 2.8. *The variety* $\mathcal{V}=[axy=y^2ayx]$ *of semigroups is closed.*

Theorem 2.9. *The variety* $\mathcal{V}=[axy=xayx^2]$ *of semigroups is closed.*

Proof. Take any U, $S\in\mathcal{V}$ with U as a subsemigroup of S such that $d\in Dom(U,S)\backslash U$. Let d has zigzag of type (1.1) in S over U of minimal length m. In order to prove the theorem, we first prove the following lemma.

Lemma 2.10. *For each* $k=1,2,...,m$.

$$d=(\prod_{i=1}^{k}a_{2i-1})y_ka_{2k-1}t_k(\prod_{i=1}^{k}a^3_{2k-(2i-1)}).$$

Proof. For any $x,y,z\in S$, we have

$$
\begin{aligned}
xyz &= yxzy^2 \text{ (as } S\in\mathcal{V}\text{)}\\
&= yx(zyy)\\
&= yxyzyyy \text{ (as } S\in\mathcal{V}\text{)}\\
&= yxyzy^3. \qquad (2.4)
\end{aligned}
$$

To prove the lemma we use induction on k. For $k=1$, we have

$$
\begin{aligned}
d &= y_1a_1t_1 \text{ (by zigzag equations)}\\
&= a_1y_1a_1t_1a_1^3 \text{ (by equation (2.4)).}
\end{aligned}
$$

Thus the result holds for $k=1$. Assume inductively that the result holds for $k=j<m$. We will show that it also holds for $k=j+1$.

Now

$$\begin{aligned}
d &= (\prod_{i=1}^{j} a_{2i-1}) y_j a_{2j-1} t_j (\prod_{i=1}^{j} a^3_{2j-(2i-1)}) \text{ (by inductive hypothesis)}\\
&= (\prod_{i=1}^{j} a_{2i-1}) y_{j+1} a_{2j+1} t_{j+1} (\prod_{i=1}^{j} a^3_{2j-(2i-1)}) \text{ (by zigzag equations)}\\
&= (\prod_{i=1}^{j} a_{2i-1}) a_{2j+1} y_{j+1} a_{2j+1} t_{j+1} a^3_{2j+1} (\prod_{i=1}^{j} a^3_{2j-(2i-1)}) \text{ (by equation (2.4))}\\
&= (\prod_{i=1}^{j+1} a_{2i-1}) y_{j+1} a_{2j+1} t_{j+1} (\prod_{i=1}^{j+1} a^3_{2(j+1)-(2i-1)}),
\end{aligned}$$

as required. □

Now to complete the proof of theorem, letting $k = m$ in Lemma 2.10, we get

$$\begin{aligned}
d &= (\prod_{i=1}^{m} a_{2i-1}) y_m a_{2m-1} t_m (\prod_{i=1}^{m} a^3_{2m-(2i-1)})\\
&= (\prod_{i=1}^{m-1} a_{2i-1}) a_{2m-1} y_m a_{2m} a^3_{2m-1} (\prod_{i=2}^{m} a^3_{2m-(2i-1)}) \text{ (by zigzag equations)}\\
&= (\prod_{i=1}^{m-1} a_{2i-1}) (a_{2m-1} y_m a_{2m} a^2_{2m-1}) a_{2m-1} (\prod_{i=2}^{m} a^3_{2m-(2i-1)})\\
&= (\prod_{i=1}^{m-1} a_{2i-1}) y_m a_{2m-1} a_{2m} a_{2m-1} (\prod_{i=2}^{m} a^3_{2m-(2i-1)}) \text{ (as } S \in \mathcal{V}\text{)}\\
&= (\prod_{i=1}^{m-1} a_{2i-1}) y_{m-1} a_{2m-2} a_{2m} a_{2m-1} (\prod_{i=2}^{m} a^3_{2m-(2i-1)}) \text{ (by zigzag equations)}\\
&= (\prod_{i=1}^{m-2} a_{2i-1}) (a_{2m-3} y_{m-1} (a_{2m-2} a_{2m} a_{2m-1}) a^2_{2m-3}) a_{2m-3} (\prod_{i=3}^{m} a^3_{2m-(2i-1)})\\
&= (\prod_{i=1}^{m-2} a_{2i-1}) y_{m-1} a_{2m-3} a_{2m-2} a_{2m} a_{2m-1} a_{2m-3} (\prod_{i=3}^{m} a^3_{2m-(2i-1)}) \text{ (as } S \in \mathcal{V}\text{)}
\end{aligned}$$

$$= (\prod_{i=1}^{m-2} a_{2i-1})y_{m-2}a_{2m-4}a_{2m-2}a_{2m}a_{2m-1}a_{2m-3}(\prod_{i=3}^{m} a^3_{2m-(2i-1)})$$

(by zigzag equations)

$$\vdots$$

$$= (\prod_{i=1}^{1} a_{2i-1})y_1a_2a_4\cdots a_{2m-2}a_{2m}a_{2m-1}a_{2m-3}\cdots a_3(\prod_{i=m}^{m} a^3_{2m-(2i-1)})$$

$$= (a_1y_1(a_2a_4\cdots a_{2m-2}a_{2m}a_{2m-1}a_{2m-3}\cdots a_3)a_1^2)a_1$$

$$= y_1a_1a_2a_4\cdots a_{2m-2}a_{2m}a_{2m-1}a_{2m-3}\cdots a_3a_1 \text{ (as } S\in\mathcal{V}\text{)}$$

$$= a_0a_2a_4\cdots a_{2m-2}a_{2m}a_{2m-1}a_{2m-3}\cdots a_3a_1 \text{ (by zigzag equations)}$$

$$= (\prod_{i=0}^{m} a_{2i})a_{2m-1}\cdots a_3a_1.$$

Thus $d\in U$, a contradiction as required. □

In view of Theorems 2.1, 2.3, 2.5, 2.6, 2.8 and 2.9 we are posing some open problems which are the generalization of these Theorems :

Problem 1. Is the variety $\mathcal{V}$ =[$axy = x^nayx$] ($n \in \mathbb{N}$) of semigroups closed?
Problem 2. Is the variety $\mathcal{V}$ = [$axy = xa^nya$] ($n \in \mathbb{N}$) of semigroups closed?
Problem 3. Is the variety $\mathcal{V}$ =[$axy = yay^nx$] ($n \in \mathbb{N}$) of semigroups closed?
Problem 4. Is the variety $\mathcal{V}$ = [$axy = xaya^n$] ($n \in \mathbb{N}$) of semigroups closed?
Problem 5. Is the variety $\mathcal{V}$ = [$axy = y^nayx$] ($n \in \mathbb{N}$) of semigroups closed?
Problem 6. Is the variety $\mathcal{V}$ = [$axy = xayx^n$] ($n \in \mathbb{N}$) of semigroups closed?

Acknowledgment

I thank referee for giving us good suggestions to improve the paper.

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