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ON PRINCIPALLY RIGHT 2-ABSORBING PRIMARY AND WEAKLY 2-ABSORBING PRIMARY IDEALS

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Abstract. Let R be a noncommutative ring. The purpose of this note is to investigate the concept of p right 2-absorbing primary (resp., weakly 2-absorbing) ideals generalizing 2-absorbing (resp., weakly 2-absorbing) ideals of noncommutative rings. From [4, Proposition 1.3] we have that for commutative rings the notions of primary rings (ideals) coincide with p right primary rings (ideals). Hence all results about 2-absorbing and weakly 2-absorbing primary ideals will follow as special cases from results proved in this note. We have $\sqrt{I} = \sum\{V \vartriangleleft R | V^n \subseteq I$ for some positive integer $n\}$ and as in [5] we call $\sqrt{I}$ the pseudo radical of I. Also $\sqrt{0} = \sqrt{R} = \sum\{V \vartriangleleft R | V^n = 0$ for some positive integer $n\}$. A proper ideal I of R is said to be a 2-absorbing (resp., weakly 2-absorbing) primary ideal if whenever $a, b, c \in R$ such that $aRbRc \subseteq I$ (resp.$0 \neq aRbRc \subseteq I$) then either $ab \in I$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$. A number of results concerning p right 2-absorbing primary (resp., weakly 2-absorbing) ideals are given, generalizing the corresponding results from commutative rings to noncommutative rings.

1. introduction

Throughout this paper the rings are associative but not necessarily assumed to have a unity unless indicated otherwise. Also an ideal means a two-sided ideal.

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A. Badawi in [1] introduced a new generalization of prime ideals over a commutative ring. A proper ideal I of a commutative ring R with $1 \neq 0$ is said to be a 2-absorbing ideal if whenever $abc \in I$, then either $ab \in I$ or $ac \in I$ or $bc \in I$. Clearly, every prime ideal is a 2-absorbing ideal. A 2-absorbing (resp., weakly 2-absorbing) ideal of a noncommutative ring was introduced by Groenewald in [10] and [12]. A proper ideal I of a noncommutative ring R is said to be a 2-absorbing (resp., weakly 2-absorbing) ideal if whenever $a, b, c \in R$ such that $aRbRc \subseteq I$ (resp., $0 \neq aRbRc \subseteq I$), then either $ab \in I$ or $ac \in I$ or $bc \in I$. Recently (see [2] and [3]), the concept of 2-absorbing ideal and weakly 2-absorbing ideal was extended to the concept of 2-absorbing primary ideal and weakly 2-absorbing primary ideals. This is a generalization of a primary and a weakly primary ideal. Recall from [2] and [3] that a proper ideal of R is said to be a 2-absorbing primary ideal of R (weakly 2-absorbing primary ideal of R) if whenever $a, b, c \in R$ with $abc \in I$ ($0 \neq abc \in I$), then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$ where $\sqrt{I} = \{a \in R : a^n \in I$ for some positive integer $n\}$. In [9] the notion of right primary (p right primary) was introduced and it was proved in [5] that for commutative rings the notions of primary rings (ideals) coincide with p right primary rings (ideals). Hence all results about 2-absorbing and weakly 2- absorbing primary ideals will follow as special cases from results proved in this note. Let R be a noncommutative ring and I be an ideal of R. I is a p right 2-absorbing (resp., p right weakly 2-absorbing) primary ideal if whenever $a, b, c \in R$ such that $aRbRc \subseteq I$ (resp. $0 \neq aRbRc \subseteq I$), then either $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$ where $\sqrt{I} = \sum \{V \lhd R | V^n \subseteq I$ for some positive integer $n\}$.

Note that a 2-absorbing ideal is a p right 2-absorbing primary ideal. However, these are different concepts. For instance, consider the ideal $I = (12)$ of $\mathbb{Z}$. The ideal $M_2(I)$ of the matrix ring $M_2(\mathbb{Z})$ is not a 2-absorbing ideal, since $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} M_2(\mathbb{Z}) \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} M_2(\mathbb{Z}) \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \subseteq M_2(I)$, but $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \notin M_2(I)$ and $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \notin M_2(I)$. However, $M_2(I)$ is a p right 2-absorbing primary ideal of $M_2(\mathbb{Z})$ since $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \in \sqrt{M_2(I)}$. It is also clear that every p right primary ideal of a ring R is a p right 2-absorbing primary ideal of R. However, the converse is not true. For example, $M_2((6))$ is a p right 2-absorbing primary ideal of $M_2(\mathbb{Z})$, but it is not a p right primary ideal of $M_2(\mathbb{Z})$ since $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \in M_2((6))$ but $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \notin M_2((6))$ and $\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \notin \sqrt{M_2((6))}$.

Definition 1.1. (see [5, Definition 1.1] and [12, Definition 2.1]) Let R be a ring and I an ideal of R.

(1) The ideal I is called a (principally) right primary ideal if whenever A and B are (principal) ideals of R with $AB \subseteq I$, then either $A \subseteq I$ or $B^n \subseteq I$ for some positive integer n depending on A and B.
(2) R is said to be a (principally) right primary ring if the zero ideal is a (principal) right primary ideal of R.
(3) As in [5] $\sqrt{I} = \sum \{V \lhd R | V^n \subseteq I$ for some positive integer $n\}$ is called the pseudo radical of I. Also $\sqrt{0} = \sqrt{R} = \sum \{V \lhd R | V^n = 0$ for some positive integer $n\}$.
(4) The prime radical, $\rho(I)$, of I is the intersection of all prime ideals of R containing I. Thus $\mathcal{P}(R)$ is $\rho(0)$ and $\sqrt{I} \subseteq \rho(I)$.

Right primary ideals and principally right primary ideals were defined in [9] where they are called "generalized right primary" and "principally generalized right primary ideals". Similarly, one defines left primary and principal left primary rings and ideals. Some of the results will be stated for right-sided conditions, with the left-handed analogs being obvious to the reader. We use "p" as an abbreviation for principally. As noted above it follows from [5, Proposition 1.3 (iv)] that if R is a **commutative** ring then an ideal I of R is a **primary ideal if and only if it is a p right primary** ideal of R.

Lemma 1.2. [5, Lemma 1.2.] *Let A, B and I be ideals of a ring R. Then we have the following.*

(1) *$A \subseteq B$ implies $\sqrt{A} \subseteq \sqrt{B}$.*
(2) *Assume that $A \subseteq \sqrt{I}$. If A is finitely generated or $\left(\sqrt{I}\right)^m \subseteq I$ for some positive integer m, then $A^n \subseteq I$ for some positive integer n. In particular, if $\sqrt{I}$ is finitely generated, then $\left(\sqrt{I}\right)^n \subseteq I$ for some positive integer n.*
(3) *If $\left(\sqrt{I}\right)^m \subseteq I$ for some positive integer m, then $\sqrt{I} = \rho(I) = \sqrt{\sqrt{I}}$.*

We adopt the following notation:

(1) $A \lhd R$, $A \lhd_r R$, $A \lhd_l R$ mean that A is a two-sided, right, left ideal of R, respectively;
(2) for a nonempty subset X of R, we use $\langle X\rangle$, $\langle X\rangle_r$, and $\langle X\rangle_l$ for the two-sided, right, left ideal, respectively, of R generated by X;
(3) $\mathbb{N}$ and $\mathbb{Z}$ are the set of natural numbers and the set of rational integers, respectively.

Lemma 1.3. [8, Lemma 3.1] *Let $I \lhd R$, if $b \in \sqrt{I}$ then there exists a positive integer m such that $(\langle b\rangle)^m \subseteq I$.*

Hence if A and I are ideals of R and $A \nsubseteq \sqrt{I}$ then $(A)^n \nsubseteq I$ for every $n \in \mathbb{N}$.

Proposition 1.4. [9] *Let $I \lhd R$. The following are equivalent:*
(1) *I is a p right primary ideal;*
(2) *if $A \lhd R$ and $b \in R$ such that $A\langle b\rangle \subseteq I$, then either $A \subseteq I$ or $\langle b\rangle^n \subseteq I$, for some $n \in \mathbb{N}$;*
(3) *if $A, B \lhd R$ and B is finitely generated with $AB \subseteq I$, then either $A \subseteq I$ or $B^n \subseteq I$, for some $n \in \mathbb{N}$;*
(4) *if $a, b \in R$ such that $aRb \subseteq I$, then either $a \in I$ or $\langle b\rangle^n \subseteq I$, for some $n \in \mathbb{N}$;*
(5) *if $a, b \in R$ with $\langle a\rangle_r \langle b\rangle_r \subseteq I$, then either $\langle a\rangle_r \subseteq I$ or $(\langle b\rangle_r)^n \subseteq I$, for some $n \in \mathbb{N}$;*
(6) *if $a, b \in R$ with $\langle a\rangle_l \langle b\rangle_l \subseteq I$, then either $\langle a\rangle_l \subseteq I$ or $(\langle b\rangle_l)^n \subseteq I$, for some $n \in \mathbb{N}$.*

Proposition 1.5. *Let R be a ring and $I \lhd R$. If $A, B \lhd R$, then I is a p right primary ideal if and only if $AB \subseteq I$ implies that $A \subseteq I$ or $B \subseteq \sqrt{I}$.*

Proof. Assume that I is a p right primary ideal. Say $AB \subseteq I$, but $A \nsubseteq I$. Then there exists $a \in A$ such that $\langle a\rangle \nsubseteq I$. Let $b \in B$. Since $\langle a\rangle\langle b\rangle \subseteq I$, $\langle b\rangle^n \subseteq I$ for some positive integer n. As $B = \sum_{b \in B} \langle b\rangle$, we have that $B \subseteq \sqrt{I}$. Conversely, let $A = \langle a\rangle$ and $B = \langle b\rangle$ be principal ideals of R such that $AB \subseteq I$. Then $A \subseteq I$ or $B \subseteq \sqrt{I}$ by assumption. If $A \subseteq I$, then we done. So assume $\langle b\rangle \subseteq \sqrt{I}$. By Lemma 1.3, $\langle b\rangle^n \subseteq I$ for some positive integer n. So I is p right primary. □

In [13] the notion of a weakly (principally) right primary ideal was introduced i.e. The ideal I is called a **weakly (principally) right primary ideal** if whenever A and B are (principal) ideals of R with $0 \neq AB \subseteq I$, then either $A \subseteq I$ or $B^n \subseteq I$ for some positive integer n depending on A and B.

Proposition 1.6. [13, Proposition 2.6] *Let $I \lhd R$. The following are equivalent:*

(1) *I is a weakly right primary ideal;*
(2) *if $A, B \lhd_r R$ such that $0 \neq AB \subseteq I$, then either $A \subseteq I$ or $B^n \subseteq I$, for some $n \in \mathbb{N}$;*
(3) *if $A, B \lhd_l R$ such that $0 \neq AB \subseteq I$, then either $A \subseteq I$ or $B^n \subseteq I$, for some $n \in \mathbb{N}$;*
(4) *if $A_1, ..., A_n$ are ideals of R with $0 \neq A_1 \cdots A_n \subseteq I$ and $A_j \nsubseteq I$ for $j = 1, ..., n$, then there exists $m \in \mathbb{N}$ such that $A_k^m \subseteq I$ for at least one $k > 1$.*

Definition 1.7. If P is a weakly p right primary ideal then $a, b \in R$ is a **twin zero** of P if $\langle a \rangle \langle b \rangle = 0$, $\langle a \rangle \nsubseteq P$ and $(\langle b \rangle)^n \nsubseteq P$ for every $n \in \mathbb{N}$.

Note that if I is a weakly p right primary ideal of R that is not a p right primary ideal then I has a twin-zero (a, b) for some $a, b \in R$.

Proposition 1.8. [13, Proposition 2.7] *Let $I \lhd R$. The following are equivalent:*

(1) *I is a weakly p right primary ideal;*
(2) *let A and B be ideals of R such that $0 \neq AB \subseteq I$ then $A \subseteq I$ or $B \subseteq \sqrt{I}$;*
(3) *let A and B be ideals of R. If B is finitely generated and $0 \neq AB \subseteq I$ then $A \subseteq I$ or $B^n \subseteq I$ for some $n \in \mathbb{N}$;*
(4) *if $A \lhd R$ and $b \in R$ such that $0 \neq A \langle b \rangle \subseteq I$, then either $A \subseteq I$ or $(\langle b \rangle)^n \subseteq I$, for some $n \in \mathbb{N}$.*

Remark 1.9. It follows from Proposition 1.8 that if P is a weakly p right primary ideal then $a, b \in R$ is a **twin zero** of P if $\langle a \rangle \langle b \rangle = 0$, $a \notin P$ and $b \notin \sqrt{P}$.

2. P RIGHT 2-ABSORBING PRIMARY IDEALS

In what follows R is a noncommutative ring with a unity unless indicated otherwise.

Definition 2.1. A proper ideal I of R is called a p right 2-absorbing primary ideal of R if whenever $a, b, c \in R$ and $aRbRc \subseteq I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.

Proposition 2.2. *Let I be an ideal of R. If I is a p right primary ideal of R, then I is a p right 2-absorbing primary ideal of R.*

Proof. Let $a, b, c \in R$ such that $aRbRc \subseteq I$. Hence $(RaRbR)(RcR) \subseteq I$. From Proposition 1.4 we have $(RaRbR) \subseteq I$ or $(RcR)^n \subseteq I$ for some positive integer n. Hence $(RaR)(RbR) \subseteq I$ or $c \in \sqrt{I}$. If $c \in \sqrt{I}$, then $ac \in \sqrt{I}$ and we are done. So suppose $(RaR)(RbR) \subseteq I$. Again, by applying Proposition 1.4 we get $(RaR) \subseteq I$ or $(RbR)^n \subseteq I$ for some positive integer n. Hence $a \in I$ or $b \in \sqrt{I}$. Thus $ab \in I$ or $bc \in \sqrt{I}$ and we are done. □

Corollary 2.3. *Let I be an ideal of R. If I is a p right primary ideal of R and $a, b, c \in R$ such that $aRbRc \subseteq I$, then $a \in I$ or $b \in \sqrt{I}$ or $c \in \sqrt{I}$.*

Theorem 2.4. *Let R be a ring with $1 \neq 0$. Suppose that I_1 and I_2 are p right primary ideals of R then, $I_1 \cap I_2$ is a p right 2-absorbing primary ideal of R.*

Proof. Let $H = I_1 \cap I_2$. From [5, Lemma 1.2]$\sqrt{H} = \sqrt{I_1} \cap \sqrt{I_2} = \sqrt{I_1 \cap I_2}$. Suppose that $aRbRc \subseteq H$ for some $a, b, c \in R$ and that $ac \notin \sqrt{H}$ and $bc \notin \sqrt{H}$. Then $a, b, c \notin \sqrt{H} = \sqrt{I_1} \cap \sqrt{I_2}$. We show that $ab \in H$. We can get $aRb \subseteq I_1 \cap I_2$ or $aRb \nsubseteq I_1 \cap I_2$. If $aRb \subseteq I_1 \cap I_2$, then we are done. We show that $aRb \nsubseteq I_1 \cap I_2$ is not possible. Suppose $aRb \nsubseteq I_1 \cap I_2$. We can have $aRb \nsubseteq I_1$ or $aRb \nsubseteq I_2$ or $aRb \nsubseteq I_1$ and $aRb \nsubseteq I_2$. If $aRb \subseteq I_1$ and $aRb \nsubseteq I_2$, then since I_1 and I_2 are p right primary ideals of R we get $a \in I_1$ or $b \in \sqrt{I_1}$ and $c \in \sqrt{I_2}$. Hence $ac \in \sqrt{H}$ or $bc \in \sqrt{H}$ which is not possible. If $aRb \nsubseteq I_1$ and $aRb \subseteq I_2$ then again since I_1 and I_2 are p right primary ideals of R we get $a \in I_2$ or $b \in \sqrt{I_2}$ and $c \in \sqrt{I_1}$. Again, $ac \in \sqrt{H}$ or $bc \in \sqrt{H}$ which is not possible. If $aRb \nsubseteq I_1$ and $aRb \nsubseteq I_2$, we get $c \in \sqrt{I_1} \cap \sqrt{I_2}$ which is a contradiction. Hence $I_1 \cap I_2$ is a p right 2-absorbing primary ideal of R. □

Remark 2.5. In general the intersection of two p right 2-absorbing primary ideals need not be a p right 2-absorbing primary ideal: ([2, Example 2.17] Let $I_1 = 50\mathbb{Z}$ and $I_2 = 75\mathbb{Z}$. Then I_1, I_2 are p right 2-absorbing primary ideals of $\mathbb{Z}$ but $I_1 \cap I_2$ is not a p right 2-absorbing primary ideal. Here $I_1 = 50\mathbb{Z}$ and $I_2 = 75\mathbb{Z}$ are not p right primary ideals of $\mathbb{Z}$.

From [5], let I be proper ideal of R and $A, B \lhd R$. Then I is p-nilary if and only if $AB \subseteq I$ implies that $A \subseteq \sqrt{I}$ or $B \subseteq \sqrt{I}$. I is strongly p-nilary if $\sqrt{I}$ is a prime ideal.

Proposition 2.6. *Let R be a ring with $1 \neq 0$ and let I be an ideal of R. If $\sqrt{I}$ is a prime ideal of R i.e. I is strongly p-nilary then I is a p right 2-absorbing primary ideal of R.*

Proof. Let $a, b, c \in R$ such that $aRbRc \subseteq I$. Suppose that $ab \notin I$. Now $acRbc \subseteq aRbRc \subseteq I \subseteq \sqrt{I}$ and since $\sqrt{I}$ is a prime ideal of R we have $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. □

Following the arguments as in [10, Lemma 1.12 and Theorem 1.14] we have :

Proposition 2.7. *Let P a proper ideal of R. If P is a p right 2-absorbing primary ideal of R and if $a, b \in R$ and K a left ideal of R such that $aRbK \subseteq P$ then $ab \in P$ or $aK \subseteq \sqrt{P}$ or $bK \subseteq \sqrt{P}$.*

Theorem 2.8. *Let P be a proper ideal of the ring R. The following are equivalent:*

(1) *P is a p right 2-absorbing primary ideal of R.*
(2) *If I and J and K are left ideals of R such that $IJK \subseteq P$, then $IK \subseteq P$ or $JK \subseteq \sqrt{P}$ or $IJ \subseteq \sqrt{P}$.*

Lemma 2.9. *Let $f : R \to S$ be a homomorphism of rings R and S. Then the following statements hold:*

(1) If I' is an ideal of S, then $f^{-1}(\sqrt{I'}) = \sqrt{f^{-1}(I')}$;
(2) If f is an epimorphism and I is an ideal of R with $Ker(f) \subseteq I$, then $f(\sqrt{I}) \subseteq \sqrt{f(I)}$.

Proof.

1.
$$\begin{aligned} f^{-1}(\sqrt{I'}) &= f^{-1}\left(\sum \{J' : J' \lhd S \text{ and } (J')^n \subseteq I' \text{ for some } n\}\right) \\ &= \sum \{f^{-1}(J') : J' \lhd S \text{ and } (f^{-1}(J'))^n \subseteq f^{-1}(I') \text{ for some } n\} \\ &= \sqrt{f^{-1}(I')}. \end{aligned}$$

2.
$$\begin{aligned} f(\sqrt{I}) &= f\left(\sum \{J : J \lhd R \text{ and } (J)^n \subseteq I \text{ for some } n\}\right) \\ &= \sum \{f(J) : J \lhd R \text{ and } (f(J))^n \subseteq f(I) \text{ for some } n\} \\ &\subseteq \sqrt{f(I)}. \end{aligned}$$

Theorem 2.10. *Let $f : R \to R'$ be an epimorphism of rings. Then the following statements hold.*

(1) *If I' is a p right 2-absorbing primary ideal of R', then $f^{-1}(I')$ is a p right 2-absorbing primary ideal of R;*
(2) *If I is a p right 2-absorbing primary ideal of R containing $Ker(f)$, then $f(I)$ is a p right 2-absorbing primary ideal of R'.*

Proof. (1) Let $a, b, c \in R$ such that $aRbRc \subseteq f^{-1}(I')$. Then $f(aRbRc) = f(a)R'f(b)R'f(c) \subseteq I'$. Since I' is a p right 2-absorbing primary ideal of R', $f(a)f(b) \in I'$ or $f(b)f(c) \in \sqrt{I'}$ or $f(a)f(c) \in \sqrt{I'}$, and thus $ab \in f^{-1}(I')$ or $bc \in \sqrt{f^{-1}(I')}$ or $ac \in \sqrt{f^{-1}(I')}$. By using the equality in Lemma 2.9 $ab \in f^{-1}(I')$ or $bc \in f^{-1}(\sqrt{I'})$ or $ac \in f^{-1}(\sqrt{I'})$. We conclude that $f^{-1}(I')$ is a p right 2-absorbing primary ideal of R.

(2) Let $a', b', c' \in R'$ and $a'R'b'R'c' \subseteq f(I)$. Then there exist $a, b, c \in R$ such that $f(a) = a'$, $f(b) = b'$, $f(c) = c'$, and $f(aRbRc) = a'R'b'R'c' \subseteq f(I)$. Since $\mathrm{Ker} f \subseteq I$, we have $aRbRc \subseteq I$ and since I is a p right 2-absorbing primary ideal of R, it follows that $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Hence $a'b' \in f(I)$ or $a'c' \in f(\sqrt{I}) \subseteq \sqrt{f(I)}$ or $b'c' \in f(\sqrt{I}) \subseteq \sqrt{f(I)}$. Thus $f(I)$ is a p right 2-absorbing primary ideal of R'.

□

Corollary 2.11. *Let R be a ring with $1 \neq 0$. Suppose that I, J are distinct proper ideals of R. If $J \subseteq I$ and I is a p right 2-absorbing primary ideal of R, then I/J is a p right 2-absorbing primary ideal of R/I.*

Proof. The proof is clear by Theorem 2.10. □

3. p right weakly 2-absorbing primary ideals

Definition 3.1. A proper ideal I of R is called a p right weakly 2-absorbing primary ideal of R if whenever $a, b, c \in R$ and $0 \neq aRbRc \subseteq I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.

Example 3.2. Consider the ring $\mathbb{Z}_{30}$ and the ideal $I = \{0\}$. From the definition $M_2(I)$ is a p right weakly 2-absorbing primary ideal of $M_2(\mathbb{Z}_{30})$. $M_2(I)$ is not a p right 2-absorbing primary ideal of $M_2(\mathbb{Z}_{30})$, since

$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} M_2(\mathbb{Z}_{30}) \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} M_2(\mathbb{Z}_{30}) \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} = \{0\} \subseteq M_2(I),$$

but neither $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \in M_2(I)$ nor $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} \in \sqrt{M_2(I)}$ nor $\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} \in \sqrt{M_2(I)}$.

Proposition 3.3. *If a proper ideal I of R is a p right weakly primary ideal then it is a p right weakly 2-absorbing primary ideal.*

Proof. Let $a, b, c \in R$ and $0 \neq aRbRc \subseteq I$. Now $0 \neq (RaRb)Rc \subseteq I$. From Proposition 1.6 we have $(RaRb) \subseteq I$ or $(Rc)^n \subseteq I$. Hence $0 \neq RaRb \subseteq I$ or $c \in \sqrt{I}$. If $c \in \sqrt{I}$, then $ac \in \sqrt{I}$ and we are done. If $0 \neq RaRb \subseteq I$, then again by Proposition 1.6 we have $Ra \subseteq I$ or $b \in \sqrt{I}$. Hence $ab \in I$ or $bc \in \sqrt{I}$ and we are done. □

Theorem 3.4. *Let I be a proper ideal of R such that $\sqrt{I}$ is a weakly prime ideal of R. Then I is a p right weakly 2-absorbing primary ideal of R*

Proof. Suppose that $0 \neq aRbRc \subseteq I$ for some $a, b, c \in R$ and $ab \notin I$. Now $aRb \nsubseteq I$. Hence $aRb \nsubseteq \sqrt{I}$ or $aRb \subseteq \sqrt{I}$. Suppose $aRb \nsubseteq \sqrt{I}$. Since $0 \neq (RaRb)Rc \subseteq \sqrt{I}$ and $RaRb \nsubseteq \sqrt{I}$ we have $Rc \subseteq \sqrt{I}$ because $\sqrt{I}$ is a weakly prime ideal of R. Hence $ac \in \sqrt{I}$. If $aRb \subseteq \sqrt{I}$ then since $aRbRc \neq \{0\}$ we have $\{0\} \neq aRb \subseteq \sqrt{I}$. Now, $\sqrt{I}$ weakly prime gives $a \in \sqrt{I}$ or $b \in \sqrt{I}$. Hence $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. □

Theorem 3.5. *Let I be a weakly p.right primary of R that is not p.right primary and J be an ideal of R such that $J \subseteq I$. Then J is a p right weakly 2-absorbing primary ideal of R. In particular, if L is an ideal of R, then $A = I \cap L$ and $B = ILI$ are p right weakly 2-absorbing primary ideals of R.*

Proof. Since I is a weakly p right primary ideal of R that is not p.right primary it follows from [13, Theorem 2.21] that $\sqrt{I} = \sqrt{0}$. Now, since $J \subseteq I$ we have $\sqrt{J} \subseteq \sqrt{I} = \sqrt{0} \subseteq \sqrt{J}$. Hence $\sqrt{I} = \sqrt{0} = \sqrt{J}$.Let $0 \neq aRbRc \subseteq J$ for some $a, b, c \in R$ and suppose that $ab \notin J$. Since $J \subseteq I$, we have $0 \neq aRbRc \subseteq I$. We consider two cases.

Case one: $aRb \nsubseteq I$. Now $RaRbR \nsubseteq I$ and $0 \neq (RaRbR)(RcR) \subseteq I$. Since I is a weakly p.right primary ideal of R we get $c \in RcR \subseteq \sqrt{I} = \sqrt{J}$. Thus $ac \in \sqrt{J}$.

Case two: Suppose that $aRb \subseteq I$. Since $aRbRc \neq 0$, we have $\{0\} \neq aRb \subseteq I$ and I a weakly p.right primary ideal of R gives $a \in I \subseteq \sqrt{I} = \sqrt{J}$ or $b \in \sqrt{I} = \sqrt{J}$. Hence $ac \in \sqrt{J}$ or $bc \in \sqrt{J}$. Thus J is a p right weakly 2-absorbing primary ideal of R. □

Definition 3.6. Let I be a p right weakly 2-absorbing primary ideal of R. We say (a, b, c) is a triple-zero of I if $aRbRc = 0$, $ab \notin I$, $bc \notin \sqrt{I}$, and $ac \notin \sqrt{I}$.

Lemma 3.7. *Let I be a p right weakly 2-absorbing primary ideal of a ring R and suppose that (a, b, c) is a triple-zero of I for some $a, b, c \in R$. Then:*

(1) $aRbI = 0$;
(2) $IbRc = 0$;
(3) $aIc = 0$;
(4) $I^2c = 0$;
(5) $aI^2 = 0$;
(6) $IbI = 0$.

Proof. (1) Suppose $aRbI \neq 0$. Hence there exist $r \in R$ and $i \in I$ such that $0 \neq arbi$. Now, $arb(i + c) = arbi + arbc = arbi \neq 0$. Hence $0 \neq aRbR(i + c) \subseteq I$. Since I is a p right weakly 2-absorbing primary ideal, we have $a(i+c) \in \sqrt{I}$ or $b(i+c) \in \sqrt{I}$.

Hence $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$ which contradicts the fact that (a,b,c) is a triple-zero of I.

(2) Suppose $IbRc \neq 0$. Hence there exist $r \in R$ and $i \in I$ such that $0 \neq ibrc$. Now $(a+i)brc = abrc + ibrc = ibrc \neq 0$. Hence $0 \neq (a+i)RbRc \subseteq I$. Since I is a p right weakly 2-absorbing primary ideal, we have $(a+i)b \in \sqrt{I}$ or $(a+i)c \in \sqrt{I}$. Hence $ab \in \sqrt{I}$ or $ac \in \sqrt{I}$ which contradicts the fact that (a,b,c) is a triple-zero of I.

(3) Suppose $aIc \neq 0$. Hence there exits $i \in I$ such that $0 \neq aib$. Now $a(b+i)c = abc+aic = aic \neq 0$. Hence $0 \neq aR(b+i)Rc \subseteq I$. Since I is a p right weakly 2-absorbing primary ideal, we have $a(b+i) \in \sqrt{I}$ or $(b+i)c) \in \sqrt{I}$. Hence $ab \in \sqrt{I}$ or $bc \in \sqrt{I}$ which contradicts the fact that (a,b,c) is a triple-zero of I.

(4) Suppose $I^2c \neq 0$. Hence there exit i and $j \in I$ such that $0 \neq ijc$. Now $(a+i)(b+j)c = abc + ajc + ibc + ijc = ijc \neq 0$ from 2. and 3.. Hence $0 \neq (a+i)R(b+j)Rc \subseteq I$. Since I is a p right weakly 2-absorbing primary ideal, we have $(a+i)c \in \sqrt{I}$ or $(b+j)c \in \sqrt{I}$ or $(a+i)(b+j) \in I$. Hence $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$ or $ab \in I$ which contradicts the fact that (a,b,c) is a triple-zero of I.

(5) Suppose $aI^2 \neq 0$. Hence there exit i and $j \in I$ such that $0 \neq aij$. Now $a(b+i)(c+j) = abc + abj + aic + aij = aij \neq 0$ from 1. and 3.. Hence $0 \neq aR(b+i)R(c+j) \subseteq I$. Since I is a p right weakly 2-absorbing primary ideal, we have $a(b+i) \in I$ or $a(c+j) \in \sqrt{I}$ or $(b+i)(c+j) \in \sqrt{I}$. Hence $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$ which contradicts the fact that (a,b,c) is a triple-zero of I.

(6) Suppose $IbI \neq 0$. Hence there exits $i,j \in I$ such that $0 \neq ibj$. Now $(a+i)b(c+j) = abc+abj+ibc+ibj = ibj \neq 0$ from 1. and 2.. Hence $0 \neq (a+i)RbR(c+j) \subseteq I$. Since I is a p right weakly 2-absorbing primary ideal, we have $(a+i)b \in I$ or $b(c+j) \in \sqrt{I}$ or $(a+i)(c+j) \in \sqrt{I}$. Hence $ab \in I$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$ which contradicts the fact that (a,b,c) is a triple-zero of I.

□

Theorem 3.8. *Let I be an ideal of a ring R such that $I^3 \neq \{0\}$. I is a p right weakly 2-absorbing primary ideal of R if and only if it is a $2-$absorbing ideal.*

Proof. Suppose I is a p right weakly 2-absorbing primary ideal which is not a p right 2-absorbing primary ideal of R. Hence I has a triple-zero (a,b,c) for some $a,b,c \in R$. Suppose that $i_1i_2i_3 \neq 0$ for some

$i_1, i_2, i_3 \in I$. Then by Lemma 3.7 we have $(a+i_1)(b+i_2)(c+i_3) = i_1 i_2 i_3 \neq 0$. Hence $0 \neq (a+i_1)R(b+i_2)R(c+i_3) \subseteq I$. Since I is weakly 2−absorbing p right primary, we have either $(a+i_1)(b+i_2) \in I$ or $(a+i_1)(c+i_3) \in \sqrt{I}$ or $(b+i_2)(c+i_3) \in \sqrt{I}$, and thus either $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$, a contradiction. Hence I is a p right 2-absorbing primary ideal. The converse is clear.

Corollary 3.9. *If I is a p right weakly 2-absorbing primary ideal which is not a p right 2-absorbing primary ideal then $I^3 = 0$.*

□

The following example shows that a proper ideal I of R with the property $I^3 = 0$ need not be a p right weakly 2-absorbing primary ideal of.

Example 3.10. [3, Example 2.13] Let $R = \mathbb{Z}_{90}$. Then $I = \{0, 30, 60\}$ is an ideal of R and clearly $I^3 = 0$. Since $0 \neq 2{\cdot}3{\cdot}5 = 30 \in I$, $2{\cdot}3 = 6 \notin I$, $2{\cdot}5 = 10 \notin \sqrt{I}$ and $3{\cdot}5 = 15 \notin \sqrt{I}$, we conclude that I is not a p right weakly 2-absorbing primary ideal of R.

Corollary 3.11. [13, Theorem 2.1] *If I is a weakly p right primary ideal which is not a p right primary ideal, then $\sqrt{I} = \sqrt{0} = \sum\{J \lhd R : J^m = 0$ for some $m\}$.*

Corollary 3.12. *If $\mathcal{P}(R)$ is the prime radical of R, then if I is a p right weakly 2-absorbing primary ideal of R that is not a p right 2-absorbing primary ideal, then $I \subseteq \mathcal{P}(R)$.*

Proof. This follows from the fact that $\mathcal{P}(R)$ is a semiprime ideal and $I^3 = \{0\}$. □

Definition 3.13. [4, Definition 2.1] Let R be a ring and I an ideal of R. The ring R is 2-primal if the prime radical $\mathcal{P}(R)$ of R is equal to the set of nilpotent elements of R. The ideal I is 2-primal if the factor ring R/I is a 2-primal ring.

Proposition 3.14. *If P is a p right weakly 2-absorbing primary ideal that is not a p right 2-absorbing primary ideal of the ring R, then R is 2-primal if and only if P is a 2-primal ideal.*

Proof. This follows from Corollary 3.11 and [4, Proposition 2.4]. □

Theorem 3.15. *Let $f : R \to R'$ be a homomorphism of rings. Then the following statements hold.*

(1) *If f is a isomorphism and J' is a p right weakly 2-absorbing primary ideal of R', then $f^{-1}(J')$ is a p right weakly 2-absorbing primary ideal of R.*

(2) *If f is an epimorphism and J is a p right weakly 2-absorbing primary ideal of R containing Ker(f), then $f(J)$ is a p right weakly 2-absorbing primary ideal of R'.*

Proof. (1) Let $a, b, c \in R$ such that $0 \neq aRbRc \subseteq f^{-1}(J')$. Since $\mathrm{Ker}(f) = 0$, we get $0 \neq f(aRbRc)$. Since f is also an epimorphism we have $f(R) = R'$ and thus $0 \neq f(a)f(R)f(b)f(R)f(c) = f(a)R'f(b)R'f(c) \subseteq J'$. Since J' is a p right weakly 2-absorbing primary ideal of R' we have $f(a)f(b) \in J'$ or $f(a)f(c) \in \sqrt{J'}$ or $f(b)f(c) \in \sqrt{J'}$ and thus $ab \in f^{-1}(J')$ or $bc \in \sqrt{f^{-1}(J')}$or $ac \in \sqrt{f^{-1}(J')}$. From Lemma 2.9 we have $ab \in f^{-1}(J')$ or $bc \in f^{-1}(\sqrt{J'})$ or $ac \in f^{-1}(\sqrt{J'})$, we conclude that $f^{-1}(J')$ is a p right 2-absorbing primary ideal of R.

(2) Let $a', b', c' \in R'$ and $0 \neq a'R'b'R'c' \subseteq f(J)$. Then there exist $a, b, c \in R$ such that $f(a) = a'$, $f(b) = b'$, $f(c) = c'$ and $0 \neq f(aRbRc) = a'f(R)b'f(R)c' = a'R'b'R'c' \subseteq f(J)$. Since $\mathrm{Ker}(f) \subseteq J$, we have $0 \neq aRbRc \subseteq J$. It implies that $ab \in J$ or $ac \in \sqrt{J}$ or $bc \in \sqrt{J}$. It means that $a'b' \in f(J)$ or $b'c' \in f(\sqrt{J}) \subseteq \sqrt{f(J)}$ or $a'c' \in f(\sqrt{J}) \subseteq \sqrt{f(J)}$. Thus $f(J)$ is a p right weakly 2-absorbing primary ideal of R.

□

Theorem 3.16. *Let I, J be proper ideals of R with $I \subseteq J$. Then the followings statements hold.*

(1) *If J is a p right weakly 2-absorbing primary ideal of R, then J/I is a p right weakly 2-absorbing primary ideal of R of R/I.*
(2) *If I is a p right 2-absorbing primary ideal of R and J/I is a p right 2-absorbing primary ideal of R/I, then J is a p right 2-absorbing primary ideal of R.*
(3) *If I is a p right weakly 2-absorbing primary ideal of R and J/I is a p right weakly 2-absorbing primary ideal of R/I, then J is a p right weakly 2-absorbing primary ideal of R.*

Proof. (1) It is obtained from Theorem 3.15.

(2) Let $a, b, c \in R$ and $aRbRc \subseteq J$. If $aRbRc \subseteq I$, then $ab \in I \subseteq J$ or $bc \in \sqrt{I} \subseteq \sqrt{J}$ or $ac \in \sqrt{I} \subseteq \sqrt{J}$. So we may assume that $aRbRc \nsubseteq I$. Then we have $0 \neq (a+I)R/I(b+I)R/I(c+I) \subseteq J/I$. Since J/I is a p right weakly 2-absorbing primary ideal of R/I, we conclude $(a+I)(b+I) = ab+I \in J/I$ or $(a+I)(c+I) = ac+I \in \sqrt{J/I}$ or $(b+I)(c+I) = bc+I \in \sqrt{J/I}$. It follows that $ab \in J$ or $ac \in \sqrt{J}$ or $bc \in \sqrt{J}$. Thus J is a p right 2-absorbing primary ideal of R.

(3) Let $a, b, c \in R$ and $0 \neq aRbRc \subseteq J$. Then by a similar argument as in 2., J is a p right weakly 2-absorbing primary ideal of R. □

Proposition 3.17. *Let P be a p right weakly 2-absorbing primary ideal of a ring R and suppose that $aRbK \subseteq P$ for some elements $a, b \in R$ and some ideal K of R such that (a, b, c) is not a triple-zero of P for every $c \in K$. If $ab \notin P$, then $aK \subseteq \sqrt{P}$ or $bK \subseteq \sqrt{P}$.*

Proof. Let $aRbK \subseteq P$ and suppose that $aK \nsubseteq \sqrt{P}$ and $bK \nsubseteq \sqrt{P}$. Then there exist k_1, k_2 in K such that $ak_1 \notin \sqrt{P}$ and $bk_2 \notin \sqrt{P}$. Since $aRbRk_1 \subseteq aRbK \subseteq P$ and since (a, b, k_1) is not a triple-zero of P and $ab \notin P, ak_1 \notin \sqrt{P}$, we get $bk_1 \in \sqrt{P}$. Also, since $aRbRk_2 \subseteq aRbK \subseteq P$ and since (a, b, k_2) is not a triple-zero of P and $ab \notin P$, $bk_2 \notin \sqrt{P}$, we get $ak_2 \in \sqrt{P}$. Now, since $aRbR(k_1 + k_2) \subseteq aRbK \subseteq P$ and since $(a, b, (k_1 + k_2))$ is not a triple-zero of P and $ab \notin P$ we have $a(k_1 + k_2) \in \sqrt{P}$ or $b(k_1 + k_2) \in \sqrt{P}$. If $a(k_1 + k_2) \in \sqrt{P}$, i.e. $(ak_1 + ak_2) \in \sqrt{P}$, then since $ak_2 \in \sqrt{P}$ we get $ak_1 \in \sqrt{P}$ which is contradiction. If $b(k_1 + k_2) \in \sqrt{P}$, i.e. $(bk_1 + bk_2) \in \sqrt{P}$ then since $bk_1 \in \sqrt{P}$ we get $bk_2 \in \sqrt{P}$ which is a contradiction. Thus either $aK \subseteq \sqrt{P}$ or $bK \subseteq \sqrt{P}$. □

Definition 3.18. Let P be a weakly $2-$absorbing p right primary deal of R and suppose that $IJK \subseteq P$ for some ideals I, J, and K of R. We say P is free triple-zero with respect to IJK if (a, b, c) is not a triple-zero of P for every $a \in I$, $b \in J$ and $c \in K$.

Remark 3.19. Let P be a p right weakly 2-absorbing primary ideal of R and suppose that $IJK \subseteq P$ for some ideals I, J, and K of R such that P is free triple-zero with respect to IJK. Then if $a \in I, b \in J$, and $c \in K$, $ab \in P$ or $ac \in \sqrt{P}$ or $bc \in \sqrt{P}$.

Theorem 3.20. *Let P be a p right weakly 2-absorbing primary ideal of R and suppose that $0 \neq IJK \subseteq P$ for some ideals I, J and K of R such that P is free triple-zero with respect to IJK. Then $IK \subseteq P$ or $JK \subseteq \sqrt{P}$ or $IJ \subseteq \sqrt{P}$.*

Proof. Suppose $0 \neq IJK \subseteq P$ and $IJ \nsubseteq P$. We show that $IK \subseteq \sqrt{P}$ or $JK \subseteq \sqrt{P}$. Suppose $IK \nsubseteq \sqrt{P}$ and $JK \nsubseteq \sqrt{P}$. Then there exist $a_1 \in I$ and $a_2 \in J$ such that $a_1K \nsubseteq \sqrt{P}$ and $a_2K \nsubseteq \sqrt{P}$. But $a_1Ra_2K \subseteq IJK \subseteq P$. Since $a_1K \nsubseteq \sqrt{P}$ and $a_2K \nsubseteq \sqrt{P}$ it follows from Proposition 3.17 that $a_1a_2 \in P$. Since $IJ \nsubseteq \sqrt{P}$, there exist $b_1 \in I$ and $b_2 \in J$ such that $b_1b_2 \notin \sqrt{P}$. Now, since $b_1Rb_2K \subseteq IJK \subseteq P$

and also $b_1b_2 \notin P$, it follows from Proposition 3.17 that $b_1K \subseteq \sqrt{P}$ or $b_2K \subseteq \sqrt{P}$. We have the following cases:

Case (1) $b_1K \subseteq \sqrt{P}$ and $b_2K \nsubseteq \sqrt{P}$:

Since $a_1Rb_2K \subseteq IJK \subseteq P$ and $a_1K \nsubseteq \sqrt{P}$ and $b_2K \nsubseteq \sqrt{P}$, it follows from Proposition 3.17 that $a_1b_2 \in P$. Since $b_1K \subseteq \sqrt{P}$ and $a_1K \nsubseteq \sqrt{P}$, we conclude $(a_1 + b_1)K \nsubseteq \sqrt{P}$. On the other hand since $(a_1+b_1)Rb_2K \subseteq P$ and neither $(a_1+b_1)K \subseteq \sqrt{P}$ nor $b_2K \subseteq \sqrt{P}$, we get that $(a_1 + b_1)b_2 \in P$ by Proposition 3.17. But, because $(a_1 + b_1)b_2 = (a_1b_2 + b_1b_2) \in P$ and $(a_1 + b_1)b_2 \in P$, we get $b_1b_2 \in P$ which is a contradiction.

Case (2) $b_2K \subseteq \sqrt{P}$ and $b_1K \nsubseteq \sqrt{P}$:

By a similar argument to case (1) we get a contradiction.

Case (3) $b_1K \subseteq \sqrt{P}$ and $b_2K \subseteq \sqrt{P}$:

$b_2K \subseteq \sqrt{P}$ and $a_2K \nsubseteq \sqrt{P}$ gives $(a_2 + b_2)K \nsubseteq \sqrt{P}$. But $a_1R(a_2 + b_2)K \subseteq \sqrt{P}$ and neither $a_1K \subseteq \sqrt{P}$ nor $(a_2 + b_2)K \subseteq \sqrt{P}$, hence $a_1(a_2+b_2) \in P$ by Proposition 3.17. Since $a_1a_2 \in P$ and $(a_1a_2+a_1b_2) \in P$, we have $a_1b_2 \in P$. Since $(a_1+b_1)Ra_2K \subseteq P$ and neither $a_2K \subseteq \sqrt{P}$ nor $(a_1 + b_1)K \subseteq \sqrt{P}$, we conclude $(a_1 + b_1)a_2 \in P$ by Proposition 3.17. But $(a_1 + b_1)a_2 = a_1a_2 + b_1a_2$, so $(a_1a_2 + b_1a_2) \in P$ and since $a_1a_2 \in P$, we get $b_1a_2 \in P$. Now, since $(a_1 + b_1)R(a_2 + b_2)K \subseteq P$ and neither $(a_1 + b_1)K \subseteq \sqrt{P}$ nor $(a_2 + b_2)K \subseteq \sqrt{P}$, we have $(a_1 + b_{1)}(a_2 + b_2) = (a_1a_2 + a_1b_2 + b_1a_2 + b_1b_2) \in P$ by Proposition 3.17. But $a_1a_2, a_1b_2, b_1a_2 \in P$, so $b_1b_2 \in P$ which is a contradiction. Consequently $IK \subseteq \sqrt{P}$ or $JK \subseteq \sqrt{P}$. □

4. Idealization

We now show how to construct p right 2-absorbing primary ideals and p right weakly 2-absorbing primary ideals using the Method of Idealization. In what follows, R is a ring (associative, not necessarily commutative and not necessarily with identity) and M is an $R-R$-bimodule. The idealization of M is the ring $R \boxplus M$ with $(R \boxplus M, +) = (R, +) \oplus (M, +)$ and the multiplication is given by $(r, m)(s, n) = (rs, rn + ms)$. $R \boxplus M$ itself is, in a canonical way, an $R-R$-bimodule and $M \simeq 0 \boxplus M$ is a nilpotent ideal of $R \boxplus M$ of index 2. We also have $R \simeq R \boxplus 0$ and the latter is a subring of $R \boxplus M$. Note also that $R \boxplus M$ is a subring of the Morita ring $\begin{bmatrix} R & M \\ 0 & R \end{bmatrix}$ via the mapping $(r, m) \mapsto \begin{bmatrix} r & m \\ 0 & r \end{bmatrix}$. We will require some knowledge about the ideal structure of $R \boxplus M$. If I is an ideal of R and N is an $R-R$-bi-submodule of M, then $I \boxplus N$ is an ideal of $R \boxplus M$ if and only if $IM + MI \subseteq N$.

Theorem 4.1. [13, Theorem 3.4] *Let R be a ring with identity and M an $R-R$-bimodule, with I a proper ideal of R. Then $I \boxplus M$ is a p right primary ideal of $R \boxplus M$ if and only if I is a p right primary ideal of R.*

Now, we investigate the transfer of a p right 2-absorbing primary ideal in trivial ring extensions.

Theorem 4.2. *Let R be a ring with identity and M an $R-R$-bimodule, with I a proper ideal of R. Then $I \boxplus M$ is a p right 2-absorbing primary ideal of $R \boxplus M$ if and only if I is a p right 2-absorbing primary ideal of R.*

Proof. Assume $I \boxplus M$ is a p right 2-absorbing primary ideal of $R \boxplus M$ and let $aRbRc \subseteq I$ for $a, b, c \in R$. We have $(a, 0)R \boxplus M(b, 0)R \boxplus M(c, 0) \subseteq I \boxplus M$ and $I \boxplus M$ a p right 2-absorbing primary ideal of $R \boxplus M$ gives $(a, 0)(b, 0) = (ab, 0) \in I \boxplus M$ or $(a, 0)(c, 0) = (ac, 0) \in \sqrt{I \boxplus M} = \sqrt{I} \boxplus M$ or $(b, 0)(c, 0) = (bc, 0) \in \sqrt{I \boxplus M} = \sqrt{I} \boxplus M$. Hence $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$ and we are done.

Conversely, let $(a, m)R \boxplus M(b, n)R \boxplus M(c, p) \subseteq I \boxplus M$ for some $(a, m), (b, n)$,
$(c, p) \in R \boxplus M$. Hence $aRbRc \subseteq I$ and since I is a p right 2-absorbing primary ideal of R, we have $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. If for example $ab \in I$, then $(a, m)(b, n) = (ab, mb + an) \in I \boxplus M$ or if $ac \in \sqrt{I}$, then $(a, m)(c, p) = (ac, mc + ap) \in \sqrt{I} \boxplus M = \sqrt{I \boxplus M}$. Similarly, if $bc \in \sqrt{I}$ we get $(b, n)(c, p) \in \sqrt{I \boxplus M}$ which completes the proof. □

Theorem 4.3. *Now let R be a noncommutative ring and M an $R-R$-bimodule. Let I be an ideal of R such that $\sqrt{I}$ is a prime ideal and N a $R-R$-bi submodule such that $IM + MI \subseteq N$then $I \boxplus N$ is a p right 2-absorbing primary ideal of $R \boxplus M$.*

Proof. Since $\sqrt{I}$ is a prime ideal it follows from [14] that $\sqrt{I \boxplus N} = \sqrt{I} \boxplus M$ is a prime ideal of $R \boxplus M$. It now follows from Proposition 2.6 that $I \boxplus N$ is a p right 2-absorbing primary ideal of $R \boxplus M$. □

Theorem 4.4. *Let R be a ring with identity and M an $R-R$-bimodule, with I a proper ideal of R. $I \boxplus M$ is a p right weakly 2-absorbing primary ideal of $R \boxplus M$ if and only if I is a p right weakly 2-absorbing primary ideal of R and for any triple zero (a, b, c) of I we have $aRbRM = MRbRc = aMc = 0$.*

Proof. Suppose $I \boxplus M$ is a p right weakly 2-absorbing primary ideal of $R \boxplus M$. Let $0 \neq aRbRc \subseteq I$ where $a, b, c \in R$. Now $(0,0) \neq (a,0)R \boxplus M(b,0)R \boxplus M(c,0) \subseteq I \boxplus M$ and $I \boxplus M$ a p right weakly 2-absorbing primary ideal gives $(ab,0) = (a,0)(b,0) \in I \boxplus M$ or $(ac,0) = (a,0)(c,0) \in \sqrt{I \boxplus M} = \sqrt{I} \boxplus M$ or $(bc,0) = (b,0)(c,0) \in I \boxplus M = \sqrt{I} \boxplus M$. Hence $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. So I is weakly $2-$absorbing p right primary. Now suppose (a,b,c) is a triple zero of I. We claim that $aRbRM = MRbRc = aMc = 0$. Assume say $aRbRM \neq 0$, so there exist $m \in M$ and $r_1, r_2 \in R$ such that $ar_1br_2m \neq 0$. Now we have $(0,0) \neq (ar_1br_2c, ar_1br_2m) = (a,0)(r_1,0)(b,0)(r_2,0)(c,m) \in (a,0)R \boxplus M(b,0)R \boxplus M(c,m) \subseteq aRbRc \boxplus M = 0 \boxplus M \subseteq I \boxplus M$. But $(a,0)(b,0) \notin I \boxplus M$ and $(a,0)(c,m) \notin \sqrt{I} \boxplus M = \sqrt{I \boxplus M}$ and $(b,0)(c,m) \notin \sqrt{I} \boxplus M = \sqrt{I \boxplus M}$ contradicting the fact that $I \boxplus M$ is a p right 2-absorbing primary ideal. If $MRbRc \neq 0$ then there exist $n \in M$ and $r_1, r_2 \in R$ such that $nr_1br_2c \neq 0$. As above, we have $(0,0) \neq (ar_1br_2c, nr_1br_2c) = (a,n)(r_1,0)(b,0)(r_2,0)(c,0) \in (a,n)R \boxplus M(b,0)R \boxplus M(c,0) \subseteq aRbRc \boxplus M = 0 \boxplus M \subseteq I \boxplus M$. But $(a,n)(b,0) \notin I \boxplus M$ and $(a,n)(c,0) \notin \sqrt{I} \boxplus M = \sqrt{I \boxplus M}$ and $(b,0)(c,0) \notin \sqrt{I} \boxplus M = \sqrt{I \boxplus M}$ and again contradicting the fact that $I \boxplus M$ is a p right 2-absorbing primary ideal. If $aMc \neq 0$, then there exists $t \in M$ such that $atc \neq 0$. Now, $(0,0) \neq (abc, atc) = (a,0)(1,0)(b,t)(1,0)(c,0) \in (a,0)R \boxplus M(b,t)R \boxplus M(c,0) \subseteq aRbRc \boxplus M = 0 \boxplus M \subseteq I \boxplus M$. But $(a,0)(b,t) \notin I \boxplus M$ and $(a,0)(c,0) \notin \sqrt{I} \boxplus M = \sqrt{I \boxplus M}$ and $(b,t)(c,0) \notin \sqrt{I} \boxplus M = \sqrt{I \boxplus M}$ so contradicting the fact that $I \boxplus M$ is a p right 2-absorbing primary ideal.

Conversely, let I be a p right weakly 2-absorbing primary ideal of R and suppose $(0,0) \neq (a,n)R \boxplus M(b,m)R \boxplus M(c,t) \subseteq I \boxplus M$ for $(a,n), (b,m), (c,t) \in R \boxplus M$. Now $aRbRc \subseteq I$.

Two cases are possible:

Case 1: $0 \neq aRbRc \subseteq I$. Now I a p right weakly 2-absorbing primary ideal of R gives $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Hence $(a,n)(b,m) \in I \boxplus M$ or $(a,n)(c,t) \in \sqrt{I} \boxplus M = \sqrt{I \boxplus M}$ or $(b,m)(c,t) \in \sqrt{I} \boxplus M = \sqrt{I \boxplus M}$ as desired.

Case 2: $0 = aRbRc \subseteq I$. We may assume $ab \notin I$ and $ac \notin \sqrt{I}$ and $bc \notin \sqrt{I}$. Hence (a,b,c) is a triple zero of I and from assumption $aRbRM = MRbRc = aMc = 0$. Now $(a,n)R \boxplus M(b,m)R \boxplus M(c,t) \subseteq (aRbRc, MRbRc + aMc + aRbRM) = (0,0)$ a contradiction. □

Corollary 4.5. *Let R be a semi-prime ring with identity which is not a prime ring and M be an $R-R$-bimodule. Then the unique p right weakly 2-absorbing primary ideal which is not a p right 2-absorbing* *primary ideal of* $R \boxplus M$ *which has the form* $I \boxplus M$ *where* I *is an ideal of* R*, is the ideal* $0 \boxplus M$.

Proof. Let I be an ideal of R and $J := I \boxplus M$ such that J is a p right weakly 2-absorbing primary ideal which is not a p right 2-absorbing primary ideal of R. Then I is a p right weakly 2-absorbing primary ideal of R which is not a $2-$absorbing ideal of R. From Corollary 3.12 $I \subseteq \mathcal{P}(R) = 0$. This means that $J = 0 \boxplus M$. The zero ideal $\{0\}$ is a p right weakly 2-absorbing primary ideal of R. Let (a, b, c) be a triple zero of $\{0\}$. Hence $aRbRc = \{0\}$ with $ab \neq 0$ and $ac \neq 0$ and $bc \neq 0$. We claim that $aRbRM = MRbRc = aMc = 0$. Without loss of generality, we may assume $aRbRM \neq 0$. Then, there exist $n \in M$ and $r_1, r_2 \in R$ such that $ar_1br_2n \neq 0$. Now, $(0, 0) \neq (0, ar_1br_2n) = (a, 0)(r_1, 0)(b, 0)(r_2, 0)(c, n) \in (a, 0)R \boxplus M(b, 0)R \boxplus M(c, n) \subseteq aRbRc \boxplus M = 0 \boxplus M = J$ and neither $(a, 0)(b, 0) \in J$ nor $(a, 0)(c, n) \in \sqrt{J}$ nor $(b, 0)(c, n) \in \sqrt{J}$, a desired contradiction since J is a p right weakly 2-absorbing primary ideal of $R \boxplus M$. As in Theorem 4.4 for $MRbRc \neq 0$ and $aMc \neq 0$ we also get the desired contradiction. On the other hand, by Theorem 4.4 and since $J^3 = 0$, J is not a p right 2-absorbing primary ideal of $R \boxplus M$, which completes the proof. □

Theorem 4.6. *Now let* R *be a noncommutative ring and* M *an* $R-R$*-bimodule. Let* I *be an ideal of* R *such that* $\sqrt{I}$ *is a weakly prime ideal and* N *a* $R-R$*-bi submodule such that* $IM + MI \subseteq N$ *and whenever* $a, b \in R$, $aRb = 0$, $a \notin \sqrt{I}$,$b \notin \sqrt{I}$ *and* $aM = Mb = 0$*then* $I \boxplus N$ *is a p right weakly 2-absorbing primary ideal of* $R \boxplus M$.

Proof. From [12, Theorem 2.1] $\sqrt{I} \boxplus M = \sqrt{I \boxplus N}$ is a weakly prime ideal of $R \boxplus M$. From Theorem 3.4 $I \boxplus N$ is a p right weakly 2-absorbing primary ideal of $R \boxplus M$. □

5. Product of rings

A mapping γ which assigns to each ring R an ideal $\gamma(R)$ is called an ideal- mapping; $\gamma(R)$ is called the radical of the ring R. If $f(\gamma(R)) \subseteq \gamma(S)$ for any surjective homomorphism $f : R \longrightarrow S$, then the ideal-mapping γ is called a preradical.

From [7] we recall: An ideal-mapping γ is a *summable radical* (*s-radical* for short) if there is a homomorphically closed class $\mathcal{M}$ such that $\gamma(R) = \sum\{I \lhd R |\ I \in \mathcal{M}\}$ for all rings R. Any s-radical is an idempotent preradical.

For a preradical γ and an ideal I of a ring R, $\gamma(R/I)$ is an ideal of R/I; hence it is of the from $\gamma(R/I) = \gamma^{\star}(I)/I$ for some uniquely determined ideal $\gamma^{\star}(I)$ of R with $\gamma(I) \subseteq I \subseteq \gamma^{\star}(I)$. This ideal $\gamma^{\star}(I)$

is called the radical of the ideal I and is not to be confused with the radical $\gamma(I)$ of the ring I. Now also $\gamma(R) = \gamma^\star(0)$ and $\gamma^\star(R) = R$.

Let R and S be rings not necessarily commutative and not necessarily with identity. Let $I \lhd R \times S$. Then

$I_R = \{a \in R : \text{there exists } b \in S \text{ such that } (a,b) \in I\}$ is an ideal of R

$I_S = \{b \in S : \text{there exists } a \in R \text{ such that } (a,b) \in I\}$ is an ideal of S. Now $I \subseteq I_R \times I_S \lhd R \times S$.

Theorem 5.1. [15] *Let $\mathcal{M}$ be a homomorphically closed class that satisfies:*

- **M_1:** *For $U \lhd R \times S$, $I \lhd R$ and $J \lhd S$ with $I \times J \subseteq U$ and $U/(I \times J) \in \mathcal{M}$ and also $U_R/I \in \mathcal{M}$ and $U_S/J \in \mathcal{M}$;*
- **M_2:** *$A, B \in \mathcal{M}$ implies $A \times B \in \mathcal{M}$;*
- **M_3:** *if $J_i \in \mathcal{M}$ for $i = 1, 2, \cdots n$, $(n \geqslant 1)$, then $\sum_{i=1}^{n} J_i \in \mathcal{M}$.*

Then $\gamma^\star(I) \times \gamma^\star(J) = \gamma^\star(I \times J)$ for all $I \lhd R$ and $J \lhd S$.

The class $\mathcal{M}$ of of nilpotent rings is homomorphically closed and fulfills conditions M_1,M_2 and M_3. Hence we have the following:

Let $R = R_1 \times R_2$ where each R_i is a ring with identity. If $I_1 \times I_2 \lhd R_1 \times R_2$, then the following hold:

(1) $\sqrt{(I_1 \times R_2)} = \sqrt{I_1} \times R_2$.
(2) $\sqrt{(R_1 \times I_2)} = R_1 \times \sqrt{I_2}$ and
(3) $\sqrt{I_1 \times I_2} = \sqrt{I_1} \times \sqrt{I_2}$.

Theorem 5.2. [13, Theorem] *Let $R = R_1 \times R_2$ where each R_i is a ring with identity. Then the following hold:*

(1) *If $P_1$is a p right primary ideal of R_1, then $P_1 \times R_2$ is a p right primary ideal of R.*
(2) *If P_2 is a p right primary ideal of R_2, then $R_1 \times P_2$ is a p right primary ideal of R.*
(3) *If P is a weakly p right primary ideal of R, then either $P = 0$ or P is p right primary.*

Theorem 5.3. *Let $R = R_1 \times R_2$ where each R_i is a ring with identity. Then the following statements hold:*

(1) *I_1 is a p right 2-absorbing primary ideal of R_1 if and only if $I_1 \times R_2$ is a p right 2-absorbing primary ideal of R;*
(2) *I_2 is a p right 2-absorbing primary ideal of R_2 if and only if $R_1 \times I_2$ is a p right 2-absorbing primary ideal of R.*

Proof.

(1) Let I_1 be a p right 2-absorbing primary ideal of R_1 . Assume that $(a_1, b_1)R(a_2, b_2)R(a_3, b_3) \subseteq (a_1R_1a_2R_1a_3, b_1R_1b_2R_1b_3)$
$\subseteq I_1 \times R_2$ for $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in R = R_1 \times R_2$. Then $a_1R_1a_2R_1a_3 \subseteq I_1$. Now, since I_1 is a p right 2-absorbing primary ideal of R_1 we have $a_1a_2 \in I_1$ or $a_1a_3 \in \sqrt{I_1}$or $a_2a_3 \in \sqrt{I_1}$. Hence $(a_1, b_1)(a_2, b_2) = (a_1a_2, b_1b_2)$
$\in I_1 \times R_2$ or $(a_1, b_1)(a_3, b_3) = (a_1a_3, b_1b_3) \in \sqrt{I_1} \times R_2 = \sqrt{(I_1 \times R_2)}$ or $(a_2, b_2)(a_3, b_3) = (a_2a_3, b_2b_3) \in \sqrt{I_1} \times R_2 = \sqrt{(I_1 \times R_2)}$. Therefore $I_1 \times R_2$ is a p right 2-absorbing primary ideal of R. Conversely, let $a_1R_1a_2R_1a_3 \subseteq I_1$ for $a_1, a_2, a_3 \in R_1$. Now $(a_1, 1)R(a_2, 1)R(a_3, 1) \subseteq (a_1R_1a_2R_1a_3, 1R_21R_21)$
$\subseteq I_1 \times R_2$. Since $I_1 \times R_2$ is a p right 2-absorbing primary ideal of R we have $(a_1, 1), (a_2, 1) = (a_1a_2, 1) \in I_1 \times R_2$ or $(a_1, 1), (a_3, 1) = (a_1a_3, 1) \in \sqrt{(I_1 \times R_2)} = \sqrt{I_1} \times R_2$ or $(a_2, 1), (a_3, 1) = (a_2a_3, 1) \in \sqrt{(I_1 \times R_2)} = \sqrt{I_1} \times R_2$. Hence $a_1a_2 \in I_1$ or $a_1a_3 \in \sqrt{I_1}$or $a_2a_3 \in \sqrt{I_1}$ and $I_1$is a p right 2-absorbing primary ideal.

(2) The proof is similar 1.

□

Proposition 5.4. *Let $R = R_1 \times R_2$ where each R_i is a ring with identity. Let I_1 and I_2 be proper ideals of R_1 and R_2 respectively. If $I = I_1 \times I_2$ is a p right 2-absorbing primary ideal of R then I_1 and I_2 are p right 2-absorbing primary ideals of R_1 and R_2 respectively.*

Proof. Let $aR_1bR_1c \subseteq I_1$for some $a, b, c \in R_1$.Then $(a, x)R(b, x)R(c, x) = (aR_1bR_1c, xR_2xR_2x) \subseteq I_1 \times I_2$ for $x \in I_2$. As $I_1 \times I_2$ is a p right 2-absorbing primary ideal of R either $(a, x)(b, x) \in I_1 \times I_2$ or $(a, x)(c, x) \in \sqrt{I_1 \times I_2} = \sqrt{I_1} \times \sqrt{I_2}or(b, x)(c, x) \in \sqrt{I_1 \times I_2} = \sqrt{I_1} \times \sqrt{I_2}$, that is either $ab \in I_1$ or $ac \in \sqrt{I_1}$ or $bc \in \sqrt{I_1}$. Thus I_1 is a p right 2-absorbing primary ideal of R_1. Similarly, we can show that I_2 is a p right 2-absorbing primary ideal of R_2. □

Proposition 5.5. *Let $R = R_1 \times R_2$ where each R_i is a ring with identity. Then the following are equivalent:*

(1) *Let I_1 (I_2) be a p right 2-absorbing primary ideal of $R_1(R_2)$.*
(2) *$I_1 \times R_2$ $(R_1 \times I_2)$ is a p right 2-absorbing primary ideal of R.*
(3) *$I_1 \times R_2$ $(R_1 \times I_2)$ is a p right weakly 2-absorbing primary ideal of R.*

Proof. 1. $\Longrightarrow$ 2.**:** Follows from Theorem 5.3.
2. $\Longrightarrow$ 3.**:** Clear
3. $\Longrightarrow$ 1.**:** Let $aR_1bR_1c \subseteq I_1$ for some $a, b, c \in R_1$. Then for each $0 \neq r \in R_2$, we have $(0, 0) \neq (a, 1)R(b, 1)R(c, r) \subseteq I_1 \times R_2$.

This gives $(a,1)(b,1) \in I_1 \times R_2$ or $(a,1)(c,r) \subseteq \sqrt{(I_1 \times R_2)} = \sqrt{I_1} \times R_2$ or $(b,1)(c,r) \subseteq \sqrt{(I_1 \times R_2)} = \sqrt{I_1} \times R_2$ since $I_1 \times R_2$ is a p right weakly 2-absorbing primary ideal of R. That is, either $ab \in I_1$ or $ac \in \sqrt{I_1}$or $bc \in \sqrt{I_1}$ and $I_1$is a p right 2-absorbing primary ideal of R_1.

□

Theorem 5.6. *Let $R = R_1 \times R_2$ where each R_i is a ring with identity $\neq 0$ and I_1, I_2 be nonzero ideals of R_1 and R_2, respectively,. If $I_1 \times I_2$ is a proper ideal of R, then the following statements are equivalent:*

(1) *$I_1 \times I_2$ is a p right weakly 2-absorbing primary ideal of R;*
(2) *$I_1 = R_1$ and I_2 is a p right 2-absorbing primary ideal of R_1 or $I_2 = R_2$ and I_1 is a p right 2-absorbing primary ideal of R_2 or I_1, I_2 are p right primary ideals of R_1, R_2, respectively;*
(3) *$I_1 \times I_2$ is a p right 2-absorbing primary ideal of R.*

Proof. 1. $\Rightarrow$ 2.: Assume that $I_1 \times I_2$ is a p right weakly 2-absorbing primary ideal of R. If $I_1 = R_1$ ($I_2 = R_2$), then I_2 is a p right 2-absorbing primary ideal of R_2 (I_1 is a p right 2-absorbing primary ideal of R_1) by Proposition 5.5. So we may assume that $I_1 \neq R_1$ and $I_2 \neq R_2$. Let $a, b \in R_2$ such that $aR_2b \subseteq I_2$ and let $0 \neq x \in I_1$. Then $0 \neq (x,1)R(1,a)R(1,b) = (xR_11R_11, 1R_2aR_2b) \subseteq I_1 \times I_2$. Since I_1 is proper, $(1,a)(1,b) = (1,ab) \notin \sqrt{I_1 \times I_2}$. Hence we have $(x,1)(1,a) = (x,a) \in I_1 \times I_2$ or $(x,1)(1,b) = (x,b) \in \sqrt{I_1 \times I_2} = \sqrt{I_1} \times \sqrt{I_2}$and so $a \in I_2$ or $b \in \sqrt{I_2}$. Thus I_2 is a p right primary ideal of R_2. Similarly, it can be easily shown that I_1 is a p right primary ideal of R_1.

2. $\Rightarrow$ 3.: If $I_1 = R_1$ and I_2 is a p right 2-absorbing primary ideal of R_1 or $I_2 = R_2$ and I_1 is a p right 2-absorbing primary ideal of R_2 Then $I_1 \times I_2$ is a p right 2-absorbing primary ideal of R by Theorem 5.3. Now assume I_1, I_2 are p right primary ideals of R_1, R_2, respectively. Now $I_1' = I_1 \times R_2$ and $I_2' = R_1 \times I_2$ are p right primary ideals. $I_1 \times I_2 = (I_1 \times R_2) \cap I_1 \times R_2$. From Theorem 2.4 $I_1 \times I_2$ is a p right 2-absorbing primary ideal of R.

3. $\Rightarrow$ 1.: This is clear.

□

Example 5.7. Let $R = \mathbb{Z} \times \mathbb{Z}$

(1) We consider $I_1 = 12\mathbb{Z}$ and $I_2 = 6\mathbb{Z}$ which are p right 2-absorbing primary ideals of $\mathbb{Z}$. Then $I = 12\mathbb{Z} \times 6\mathbb{Z}$ is a p right 2-absorbing primary ideal. However, I_1 and I_2 are not p right primary ideals.
(2) Assume that $J = 4\mathbb{Z} \times 6\mathbb{Z}$ is an ideal of R. As we know $4\mathbb{Z}$ is a p right primary ideal. $6\mathbb{Z}$ is not a p right primary ideal

although it is a p right 2-absorbing primary ideal. Then it is easy to see that $J = 4\mathbb{Z} \times 6\mathbb{Z}$ is a p right 2-absorbing primary ideal of R.

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