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BZS RINGS, II

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ABSTRACT. An associative ring R, not necessarily commutative and not necessarily with identity, is called *Boolean-zero square* or *BZS* if every element of R is either idempotent or nilpotent of index 2. We continue our investigation of the structure of finite BZS rings.

1. INTRODUCTION

This paper continues the study of *Boolean-zero square* or *BZS* rings, those rings in which every element is either idempotent or nilpotent of index 2, initiated by the authors in [4]. BZS rings generalize both the well-known class of Boolean rings, in which all elements are idempotent, and zero square rings, in which every nonzero element is nilpotent of index two. See [7] and [8] for more information about Boolean rings and zero square rings, respectively. BZS rings are a special case of BZS near-rings, which are studied in [3], and BZS near-rings are in turn a special case of the Malone trivial near-rings introduced in [6] and studied, *inter alia*, in [1]. A *proper* BZS ring is a BZS ring that is neither a Boolean ring nor a zero square ring.

Throughout the paper, R denotes a BZS ring, N denotes the set of nilpotent elements of R, and E denotes the set of idempotent elements of R. We focus mainly on the case when R is a finite, proper BZS ring. In the next section, we summarize some relevant semigroup definitions and results. Then we count the number of isomorphism classes of finite BZS rings of a given size, and we illustrate our results when R has sizes

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4 and 8. Next we show that $N^2 = \{0\}$ when R is finite, and use this to obtain results related to one-sided cyclic ideals generated by a nilpotent element of R. Finally, we prove some results on one-sided cyclic ideals generated by an idempotent element of R.

2. Semigroup Preliminaries

The following definitions and results are standard. They can be found in [2] or [5].

Definition 2.1. A semigroup consisting of idempotents is a *band*.

Definition 2.2. A semigroup S is *completely simple* if S is simple and all idempotents are minimal under the partial order $e \leq f$ if and only if $e = ef$.

Definition 2.3. A semigroup S is *left zero* if $ab = a$ for all $a, b \in S$. A semigroup S is *right zero* if $ab = b$ for all $a, b \in S$.

Definition 2.4. A *rectangular band* S is a direct product of a left zero semigroup LZ_S and a right zero semigroup RZ_S. We denote this direct product by $S = LZ_S \times RZ_S$, or simply $S = LZ \times RZ$ if S is understood.

Note that if S is a rectangular band and $(x, y), (z, w) \in LZ \times RZ$, then $(x, y)(z, w) = (x, w)$.

Theorem 2.5.
(a) A finite simple semigroup is completely simple.
(b) A band S is completely simple if and only S is a rectangular band.

3. Counting Finite BZS Rings

In this section we count the number of non-isomorphic proper BZS rings R of any finite size. The key idea is that the idempotents completely determine the structure of R.

The next Lemma contains results from [4] that will be used in the sequel. We denote the size of a set X by $|X|$.

Lemma 3.1. *Let R be a BZS ring. Let $e, f \in E, x \in N$. Then*

(1) $2r = 0$ for all $r \in R$:
(2) $|E| = |N|$;
(3) $e + f \in N$;
(4) $e + x \in E$;
(5) $x = ex + xe$;
(6) $exe = 0$;
(7) $xex = 0$;
(8) if R is a proper finite BZS ring, $|R| = 2^n$ for some n.

Lemma 3.2. *In a proper BZS ring, the idempotents form a completely simple semigroup under multiplication.*

Proof. Let $e, f \in E$. We show that $efe = e$. From Lemma 3.1(3) we have that $(e + f)^2 = 0$. From Lemma 3.1(1) it follows that $e + f = ef + fe$. Multiply on the left by e to get $e + ef = ef + efe$. Hence $e = efe$. □

Lemma 3.3. *In a proper finite BZS ring, every nilpotent element is the sum of two idempotents.*

Proof. Let $E = \{e_1, e_2, \cdots, e_k\}$. Then the elements $\{e_1 + e_1,\ e_1 + e_2, \cdots, e_1 + e_k\}$ are all nilpotent and unique. Since $|N| = |E|$, this yields all the nilpotents. □

Theorem 3.4. *There are n non-isomorphic finite proper BZS rings of size 2^n.*

Proof. From Lemma 3.3 we know that E completely determines the structure of R. To prove the theorem we need only count the number of completely simple semigroups E of a given size.

Recall that $E = LZ \times RZ$ and that $|E| = 2^{n-1}$. Hence the possibilities for $|LZ|$ and $|RZ|$ are
$|LZ| = 1$ and $|RZ| = 2^{n-1}$;
$|LZ| = 2$ and $|RZ| = 2^{n-2}$;
$\vdots$
$|LZ| = 2^{n-1}$ and $|RZ| = 1$.
There are n possibilities. This completes the proof. □

Theorem 3.5. *Let R and S be two finite proper BZS rings of the same size. Let $E_R = LZ_R \times RZ_R$ be the idempotents in R and $E_S = LZ_S \times RZ_S$ be the idempotents in S. If $|LZ_R| = |RZ_S|$, then R and S are anti-isomorphic.*

Proof. Let $(a, p), (b, q) \in R$. Let $S = \{(p, a)|(a, p) \in R\}$. Define an additive map $f : R \to S$ by $f((a, p)(b, q)) = (q, b)(p, a)$ and $f(a, p) = (p, a)$. Then $f((a, p), (b, q)) = f(a, q) = (q, a)$ and $f((a, p), (b, q)) = f(b, q)f(a, p) = (q, b)(p, a) = (q, a)$. Since the rings are of the same size and $E_S = LZ_S \times RZ_S$, this map is a 1-1 correspondence. This completes the proof. □

We use these results to give the structure of all proper BZS rings of size 4 and 8.

3.1. **Proper BZS Rings of Size 4.** By Theorem 3.4 there are 2 proper BZS rings of size 4. Since $|E| = 2$ the set E is either a left zero semigroup or a right zero semigroup.

Case I. Assume E is left zero. Write $E = \{e, f\}$. Letting $x = e + f$. we write $N = \{0, x\}$. The additive and multiplicative structures of R are given by

$+$	0	e	f	x
0	0	e	f	x
e	e	0	x	f
f	f	x	0	e
x	x	f	e	0

$\times$	0	e	f	x
0	0	0	0	0
e	0	e	e	0
f	0	f	f	0
x	0	x	x	0

Case II. E is right zero. In this case the additive structure is the same. The multiplicative structure is given by the transpose of the table in Case I. That is, the multiplicative structure of R is

$\times$	0	e	f	x
0	0	0	0	0
e	0	e	f	x
f	0	e	f	x
x	0	0	0	0

3.2. **Proper BZS Rings of Size 8.** By Theorem 3.4 there are 3 proper BZS rings of size 8. Since $|E| = 4$ the set E is either a left zero semigroup, a right zero semigroup, or a product $LZ \times RZ$ where $|LZ| = |RZ| = 2$ and LZ, RZ are, respectively, a left zero semigroup and a right zero semigroup.

Case I. E is left zero. Let $E = \{e, f, g, h\}$. Then $N = \{0, e + f, e + g, e + h\}$. Label the nonzero elements of N as x, y, z.

Note that $f + g$ is nilpotent. Then $f + g \neq e + f$, $f + g \neq e + g$, so $f + g = e + h$. Similar arguments yield the following identities.

$x = e + f = g + h$, $y = e + g = f + h$, $z = e + h = f + g$.

The additive and multiplicative structures of R are given by

$+$	0	e	f	g	h	x	y	z
0	0	e	f	g	h	x	y	z
e	e	0	x	y	z	f	g	h
f	f	x	0	z	y	e	h	g
g	g	y	z	0	x	h	e	f
h	h	z	y	x	0	g	f	e
x	x	f	e	h	g	0	z	y
y	y	g	h	e	f	z	0	x
z	z	h	g	f	e	y	x	0

$\times$	0	e	f	g	h	x	y	z
0	0	0	0	0	0	0	0	0
e	0	e	e	e	e	0	0	0
f	0	f	f	f	f	0	0	0
g	0	g	g	g	g	0	0	0
h	0	h	h	h	h	0	0	0
x	0	x	x	x	x	0	0	0
y	0	y	y	y	y	0	0	0
z	0	z	z	z	z	0	0	0

Case II. $E = LZ \times RZ$ where $|LZ| = |RZ| = 2$ and LZ, RZ are, respectively, a left zero semigroup and a right zero semigroup.

Let $E = \{e, f, g, h\}$. Write $e = (a, p)$, $f = (a, q)$, $g = (b, p)$, $h = (b, q)$.

Let $N = \{0, x, y, z\}$. Each element of N is the sum of two idempotents. Using the argument from the previous case, we can write the following:
$x = (a, p) + (a, q) = (b, p) + (b, q)$, $y = (a, p) + (b, p) = (a, q) + (b, q)$, $z = (a, p) + (b, q) = (a, q) + (b, p)$.

The additive structure of R is the same as that in the previous case, since $(R, +)$ is a direct sum of copies of $\mathbb{Z}_2$. The multiplicative structure of R is

$\times$	0	e	f	g	h	x	y	z
0	0	0	0	0	0	0	0	0
e	0	e	f	e	f	x	0	x
f	0	e	f	e	f	x	0	x
g	0	g	h	g	h	x	0	x
h	0	g	h	g	h	x	0	x
x	0	0	0	0	0	0	0	0
y	0	y	y	y	y	0	0	0
z	0	y	y	y	y	0	0	0

Case III. E is right zero. The ring in this case is anti-isomorphic to that of case I.

The preceding examples serve as motivation for and illustrate many of the results in the next two sections.

4. Properties of N

Proposition 4.1. *If R is a proper, finite BZS ring, then $N^2 = \{0\}$.*

Proof. Let $x, y \in N$. By Lemma 3.3 we can write $x = (a, p) + (b, q)$ and $y = (a, p) + (c, t)$. Then

$$xy = [(a, p) + (b, q)][(a, p) + (c, t)] = (a, p) + (a, t) + (b, p) + (b, t).$$

Now write $y' = (a, p) + (b, t)$, and note that $y' \in N$ by Lemma 3.1(3) since it is a sum of two idempotents. So

$$xy' = [(a, p) + (b, q)][(a, p) + (b, t)] = (a, p) + (a, t) + (b, p) + (b, t) = xy$$

and since $(y')^2 = 0$, we also have

$$\begin{aligned} xy' &= (x + y')y' = [(b,q) + (b,t)][(a,p) + (b,t)] \\ &= (b,p) + (b,t) + (b,p) + (b,t) = 0. \end{aligned}$$

Thus, $xy = 0$, as required to show $N^2 = \{0\}$. □

Corollary 4.2. *$RNR = \{0\}$.*

Proof. By Proposition 4.1 we know that $N^2 = \{0\}$. Hence we need only show that $ENE = \{0\}$.

Let $(a,p), (b,q) \in E$ and let $(c,t) + (d,v) \in N$. Then $(a,p)[(c,t) + (d,v)](b,q) = [(a,t) + (a,v)](b,q) = (a,q) + (a,q) = 0$. □

Proposition 4.3. *Let R be a proper, finite BZS ring. For $x, y \in N$, suppose $xR \neq \{0\}$. Then $xR = yR$ if and only if $x = f + g$ and $y = f + g'$ for some $f, g, g' \in E$, with $\pi_1(g) = \pi_1(g')$ (here π_1 represents the projection onto the first factor when E is written as $LZ \times RZ$).*

Proof. First suppose $xR = yR$. Then $xR = xE \cup \{0\} = yE \cup \{0\} = yR$ by Proposition 4.1. Writing $x = (a,p) + (b,q)$ and $y = (a,p) + (c,t)$, any nonzero $z \in xR = yR$ is expressible as $z = xe = ye'$ for some $e, e' \in E$. Put $e = (\alpha, \beta)$ and $e' = (\alpha', \beta')$. Then

$$z = [(a,p) + (b,q)](\alpha,\beta) = [(a,p) + (c,t))](\alpha',\beta')$$

implying $(a,\beta) + (b,\beta) = (a,\beta') + (c,\beta')$. Multiply both sides of this last equality by e on the right to get

$$(a,\beta) + (b,\beta) = (a,\beta) + (c,\beta)$$

from which it follows that $(b,\beta) = (c,\beta)$, so that $b = c$; i.e., $\pi_1(g) = \pi_1(g')$.

Now suppose $x = f + g$ and $y = f + g'$ for some $f, g, g' \in E$ with $\pi_1(g) = \pi_1(g')$. Write $f = (a,p), g = (b,q), g' = (b,t)$. Then for any $e = (\alpha,\beta) \in E$,

$$xe = [(a,p) + (b,q)](\alpha,\beta) = (a,\beta) + (b,\beta)$$

while

$$ye = [(a,p) + (b,t)](\alpha,\beta) = (a,\beta) + (b,\beta) = xe$$

Thus, $xE = yE$, whence $xR = xE \cup \{0\} = yE \cup \{0\} = yR$, as required. □

Proposition 4.4. *Let R be a proper, finite BZS ring, with $x, y \in N$. Then $Rx = yR$ if and only if $Rx = yR = \{0\}$.*

Proof. Write $x = (a,p) + (b,q)$ and $y = (a,p) + (c,t)$, and if $Rx = yR$ note that $Rx = Ex \cup \{0\} = yE \cup \{0\} = yR$ since $N^2 = \{0\}$. Now

$$(a,p)x = (a,p)[(a,p) + (b,q)] = (a,p) + (a,q) \in yR$$

If $(a,p)x = 0$, then we must have $p = q$. Otherwise, $(a,p)x \in yE$, so there exists $(\alpha, \beta) \in E$ such that

$$(a,p) + (a,q) = [(a,p) + (c,t)](\alpha, \beta) = (a, \beta) + (c, \beta)$$

Thus,

$$(a,p)[(a,p) + (a,q)] = (a,p)[(a,\beta) + (c,\beta)]$$

or

$$(a,p) + (a,q) = (a,\beta) + (a,\beta) = 0$$

It follows again that $p = q$, so that in any case $x = (a,p) + (b,p)$. Now given any $(\alpha', \beta') \in E$,

$$(\alpha', \beta')x = (\alpha', \beta')[(a,p) + (b,p)] = (\alpha', p) + (\alpha', p) = 0$$

so that $Ex = \{0\}$. It follows that $Rx = yR = \{0\}$. The other direction is clear. □

Lemma 4.5. *Let R be a proper, finite BZS ring, and suppose $e = (a,p) \in E$ and $x = (b_1, q_1) + (b_2, q_2) \in N$. Then $ex = 0$ if and only if $q_1 = q_2$,*

Proof. Let $ex = 0$. Then $0 = (a,p)[(b_1,q_1) + (b_2,q_2)] = (a,q_1) + (a,q_2)$. This is true if and only $q_1 = q_2$.

□

Lemma 4.6. *Let R be a proper, finite BZS ring, and suppose $e \in E$, and $x \in N$. If $ex = 0$, then $fx = 0$ for all $f \in E$.*

Proof. By Lemma 4.5 we can write x as $x = (b_1, q) + (b_2, q)$. Let $f = (a,p) \in E$. Then $fx = (a,p)[(b_1,q) + (b_2,q)] = (a,q) + (a,q) = 0$. □

Theorem 4.7. *Let R be a proper, finite BZS ring, and suppose $e \in E$, and $x \in N$. The following are equivalent:*

(1) $ex = 0$*;*
(2) $Rx = \{0\}$.

Proof. Let $ex = 0$. Then $fx = 0$ for all $f \in E$ by Lemma 4.6.

By Proposition 4.1 for any $y \in N$, we have $yx = 0$, so that $Rx = \{0\}$ follows.

The other direction is trivial. □

The dual of the previous theorem is also true and can be proved with the same argument.

Theorem 4.8. *Let R be a proper, finite BZS ring, and suppose $e \in E$, and $x \in N$. The following are equivalent:*

(1) $xe = 0$;
(2) $xR = \{0\}$.

5. Cyclic One-sided Ideals Generated by Idempotents

Proposition 5.1. *Let R be a proper, finite BZS ring. For any $e, f \in E$ we have $eR \cap N = fR \cap N$.*

Proof. Let $x \in eR \cap N$. Then $x = xe + ex = fx + xf$. Since $x \in eR$ we have that $x = ex$, and hence $xe = 0$. By Theorem 4.7 we have $xR = \{0\}$. In particular $xf = 0$. Hence $x = xf + fx = 0 + fx = fx \in fR$. □

Proposition 5.2. *Let R be a proper, finite BZS ring with $e, f \in E$. If $g \in eR \cap fR$ for some $g \in E$, then $eR = fR$.*

Proof. Write $e = (a, p), f = (b, q), g = (c, t)$. Then $(c, t) = g = eg = (a, p)(c, t) = (a, t)$ which implies that $c = a$. So $(a, t) = g = fg = (b, q)(a, t) = (b, t)$ which implies that $a = b$.

Now eR, fR contain the same idempotents, namely those of the form (a, x). Also, eR, fR contain the same elements of N by Proposition 5.1. Hence $eR = fR$. □

Proposition 5.3. *Let R be a proper, finite BZS ring with $e, f \in E$. Then $eR \cap Rf = \{ef, 0\}$.*

Proof. Let $x \in eR \cap Rf \cap N$. Then $x = ex + xe = ex$, so that again $xR = \{0\}$. In particular $xf = 0$. Since $x \in Rf$ this implies that $x = 0$.

Now let $g \in eR \cap Rf \cap E$. Write $e = (a, p), f = (b, q), g = (c, t)$. Then since $g \in eR$ we can write $(c, t) = g = eg = (a, p)(c, t) = (a, t)$ so that $c = a$. Since $g \in Rf$ we can write $(a, t) = g = gf = (a, t)(b, q) = (a, q) = (a, p)(b, q) = ef$. This completes the proof. □

We can obtain sharper results if we assume that E is left zero.

Proposition 5.4. *Let R be a proper, finite BZS ring with $e \in E$. The following are equivalent.*

(1) *E is left zero;*
(2) $eN = \{0\}$;
(3) $eR = \{e, 0\}$;
(4) $Re = R$.

Proof. (1) ⇒ (2) Let $x = f + g \in N$, where $f, g \in E$. Then $ex = ef + eg = e + e = 0$.

(2) $\Rightarrow$ (1) Let $e+f \in N$, where $f \in E$. Then $0 = e(e+f) = e+ef$ so that $e = ef$ and E is left zero.
(1), (2) $\Rightarrow$ (3) Since E is left zero, then $eE = \{e\}$. Also $eN = \{0\}$. The result now follows.
(3) $\Rightarrow$ (1) This is immediate.
(1), (2) $\Rightarrow$ (4) Let $f \in E$. Since E is left zero, then $f = fe \in Re$. Also, recall that if $x \in N$ then $x = ex + xe$ for any $e \in E$. Since $eN = \{0\}$ then $x = xe \in Re$.
(4) $\Rightarrow$ (2) Let $x \in N$. Then $x = ex + xe$ But $x \in Re$ implies that $x = ye$ for some $y \in N$. Hence $ex = eye = 0$. Since x is arbitrary, then $eN = \{0\}$. $\square$

As before, we can obtain the dual of this result.

Proposition 5.5. *Let R be a proper, finite BZS ring with $e \in E$. The following are equivalent.*

(1) *E is right zero;*
(2) *$Ne = \{0\}$;*
(3) *$Re = \{e, 0\}$;*
(4) *$eR = R$.*

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