Journal of Algebra and Related Topics
Vol. 9, No 2, (2021), pp 23-28

ON CHARACTERISTIC IDEAL BUNDLES OF A LIE ALGEBRA BUNDLE

R. KUMAR

ABSTRACT. We show that, every derivation of a Lie algebra bundle can be viewed as an inner derivation if one embeds the Lie algebra bundle into a larger Lie algebra bundle. We define, the radical and the nilradical bundle of a Lie algebra bundle and prove, both are characteristic ideal bundles of Lie algebra bundle.

1. INTRODUCTION

A formal definition of Lie algebra bundle is found in [3]; A Lie algebra bundle ξ is a vector bundle $\xi = (\xi, p, X)$ in which each fibre is a Lie algebra and for each $x \in X$, there exists an open neighborhood U of $x \in X$, a Lie algebra L and a homeomorphism $\phi : U \times L \to p^{-1}(U)$ such that ϕ restricted to each fibre is a Lie algebra isomorphism.

Douady and Lazard's definition [2] is more general; a Lie algebra bundle is a vector bundle $\xi = (\xi, p, X)$ together with a vector bundle morphism $\theta : \xi \oplus \xi \to \xi$ inducing a Lie algebra structure on fibre ξ_x, $\forall\ x \in X$. They have constructed a Lie group bundle (not necessarily Hausdorff) whose Lie algebra bundle is isomorphic to a given Lie algebra bundle [2, p.148]. They then go on to show that a certain $\mathcal{C}^\infty$ bundle of semidirect product Lie algebras admits no Hausdorff Lie group bundles. They conclude by asking whether there is an analytic example. Further, Coppersmith [1] constructed an analytic family of 4-dimensional Lie algebras parametrized by $\mathbb{R}^2$ which cannot be integrated to a Hausdorff family of Lie groups. The fibre g_ϵ over

MSC(2010): 17B20, 17B60, 17B99, 55R25, 55R99
Keywords: Characteristic ideal bundle, Lie algebra bundle, Radical bundle, Vector bundle.
Received: 10 August 2020, Accepted: 20 June 2021.

$\epsilon = (\epsilon_1, \epsilon_2) \in \mathbb{R}^2$ has basis $x_\epsilon, y_\epsilon, z_\epsilon, w_\epsilon$ and bracket operation is given by

$$\begin{aligned} [w_\epsilon, x_\epsilon]_\epsilon = [w_\epsilon, y_\epsilon]_\epsilon &= [w_\epsilon, z_\epsilon]_\epsilon = 0, \\ [x_\epsilon, z_\epsilon]_\epsilon = -y_\epsilon, \quad [y_\epsilon, z_\epsilon]_\epsilon &= x_\epsilon \quad [x_\epsilon, y_\epsilon]_\epsilon = \epsilon_1 z_\epsilon + \epsilon_2 w_\epsilon. \end{aligned}$$

Lie algebra bundles [5–10] over a field of characteristic zero and Lie algebra bundles in terms characteristic ideal bundles [11] over arbitrary characteristic have been studied. In this paper we study, every derivation of a Lie algebra bundle can be viewed as an inner derivation if one embeds the Lie algebra bundle into a larger Lie algebra bundle. As a consequence we show both the radical bundle and the nilradical bundles are characteristic ideal bundles.

2. Preliminaries

A bilinear form β on ξ is a continuous morphism $\beta : \xi \oplus \xi \to F$ such that for each $x \in X$, $\beta \mid_{\xi_x \times \xi_x}$ is a bilinear form on ξ_x. Put $\beta(u, v) = Tr(ad_u ad_v)$ and $T(u, v, w) = Tr(ad_{[u,v]}, ad_w)$. Then β is a symmetric bilinear form on ξ, while $T(u, v, w)$ is a skew symmetric trilinear form. We can easily verified that they are both invariant under all derivations of ξ, that is, for all derivations D of ξ and $u, v, w \in \xi_x$ and $x \in X$ we have

$$\begin{aligned} \beta(Du, v) + \beta(u, Dv) &= 0 \\ T(Du, v, w) + T(u, Dv, w) + T(u, v, Dw) &= 0. \end{aligned}$$

An ideal bundle μ of ξ is called a characteristic ideal bundle if $D(\mu) \subseteq \mu$ for all derivation D of ξ [11]. We define derived series of a Lie algebra bundle ξ by, $\xi^0 = \xi,\ \xi^{(k)} = [\xi^{(k-1)},\ \xi^{(k-1)}]$ for $k \geq 1$. We call ξ is solvable if $\xi^{(m)} = 0$ for some positive integer m. Similarly define the central series ξ by $\xi^0 = \xi, \xi^k = [\xi,\ \xi^{k-1}]$ for $k \geq 1$. We call ξ is nilpotent if there exists positive integer n such that $\xi^n = 0$.

Notations and Terminology

All the underlying vector spaces are real. All Lie algebra bundles, subbundles and ideal bundles are locally trivial over the same base space unless otherwise mentioned. We let, underlying field is of characteristic zero unless otherwise mentioned.

3. Main Theorem

Theorem 3.1. *Let ξ be a Lie algebra bundle. Then there exists a Lie algebra bundle $\xi^{'}$, containing ξ as a Lie algebra bundle, such that every derivation of ξ is the restriction of an inner derivation of $\xi^{'}$.*

Proof. Let D be any derivation of ξ. Consider $\xi^* = \bigcup_{x\in X}\xi_x^*$, where ξ_x^* is a Lie algebra obtained by embedding Lie algebra ξ_x [12]. Let $\phi : U\times L \to \bigcup_{x\in U}\xi_x$ be a local triviality of ξ. Define $\phi^* : U\times L^* \to \bigcup_{x\in U}\xi_x^*$ by $\phi^*(a,(u,\alpha)) = \phi((a,u),\alpha)$, where α is in the underlying field of L. Define topology on $\bigcup_{x\in X}\xi_x^*$ such that ϕ^* is a homeomorphism. Hence ξ^* is a Lie algebra bundle.

Consider $\xi\oplus\xi^*$ as the vector bundle underlying ξ' and define multiplication in ξ_x' by the formula

$$[(u,a)(v,b)] = ([u,v] + aD(v) - bD(u), 0)$$

Then ξ' is a Lie algebra bundle. For, ξ being Lie algebra bundle we have $\phi : U\times L \to \bigcup_{x\in U}\xi_x$. Then we get $\phi^* : U\times L^* \to \bigcup_{x\in U}\xi_x^*$. Define

$$\psi : U\times (L\oplus L^*) \to \bigcup_{x\in U}(\xi_x\oplus\xi_x^*)$$

by

$$\psi(x,u,u^*) = \phi(x,u)\oplus\phi^*(x,u^*)$$

is a homeomorphism. We identify the subalgebra bundle $(\xi,0)$ with ξ. Let $w=(0,1)$. Then, consider the inner derivation $ad_w : \xi' \to \xi'$ by $ad_w(s) = [w,s]$ for $s\in\xi_x'$. When, $u\in\xi$, we get

$$ad_w(u,0) = [w,(u,0)] = [(0,1),(u,0)] = ([0,u] + 1D(u) - 0D(0), 0) = (D(u),0)$$

Therefore, the restriction of the inner derivation ad_w to ξ is equal to the original derivation. □

Definition 3.2. Let $\phi : U\times L \to \bigcup_{x\in U}\xi_x$ be a local triviality of ξ, where L is a Lie algebra. Let R be the radical of L, $\mathtt{Rad}(\xi_x)$ be the radical of ξ_x. Then $\phi|_{U\times R} : U\times R \to \bigcup_{x\in U}\mathtt{Rad}(\xi_x)$ is an isomorphism. We call $\mathtt{Rad}(\xi) = \bigcup_{x\in X}\mathtt{Rad}(\xi_x)$, the maximal solvable ideal bundle as the radical bundle of ξ.

Definition 3.3. Let N is the nilradical of L, $\mathtt{Nil}(\xi_x)$ be the nilradical of ξ_x. Then $\phi|_{U\times N} : U\times N \to \bigcup_{x\in U}\mathtt{Nil}(\xi_x)$ is an isomorphism. We call $\mathtt{Nil}(\xi) = \bigcup_{x\in X}\mathtt{Nil}(\xi_x)$, the maximal nilpotent ideal bundle as the nilradical bundle of ξ.

Lemma 3.4. *If μ is an ideal bundle of ξ, then $\mathtt{Rad}(\mu) = \mathtt{Rad}(\xi)\cap\mu$.*

Proof. Since $\mathtt{Rad}(\mu)$ is a characteristic ideal bundle of μ, $\mathtt{Rad}(\mu)$ is an ideal bundle of ξ. Since, $\mathtt{Rad}(\mu)$ is solvable, $\mathtt{Rad}(\mu)\subseteq\mathtt{Rad}(\xi)$ implying that $\mathtt{Rad}(\mu)\subseteq\mathtt{Rad}(\xi)\cap\mu$. For the reverse inclusion, $\mathtt{Rad}(\xi)\cap\mu$ is a solvable ideal bundle of μ, showing that $\mathtt{Rad}(\xi)\cap\mu\subseteq\mathtt{Rad}(\mu)$, and hence $\mathtt{Rad}(\xi)\cap\mu = \mathtt{Rad}(\mu)$. □

Theorem 3.5. *Suppose ξ is a Lie algebra bundle, and D is a derivation of ξ. Then $D(\mathtt{Rad}(\xi))$ is contained in a nilpotent ideal bundle of ξ.*

Proof. Apply Theorem (3.1) to get an embedding ξ into $\xi^{'}$. We make the identification ξ with $(\xi, 0)$ of $\xi^{'}$, then ξ is an ideal bundle of $\xi^{'}$ and $D = ad_w|_{\xi}$. Using Lemma (3.4) and $\mathtt{Rad}(\xi) \subseteq \mathtt{Rad}(\xi^{'})$, we deduce that

$$D(\mathtt{Rad}(\xi)) = [w, \mathtt{Rad}(\xi)] = [w, \mathtt{Rad}(\xi^{'}) \cap \xi] \subseteq [\xi^{'}, \mathtt{Rad}(\xi^{'})] \cap \xi$$

But $[\xi^{'}, \mathtt{Rad}(\xi^{'})]$ is a nilpotent ideal bundle of $\xi^{'}$, and so $[\xi^{'}, \mathtt{Rad}(\xi^{'})] \cap \xi$ is nilpotent ideal bundle of $\xi^{'}$.

□

4. Applications

Theorem 4.1. *For any Lie algebra bundle ξ, both the radical bundle $\mathtt{Rad}(\xi)$ and the nilradical bundle $\mathtt{Nil}(\xi)$ are characteristic ideal bundles.*

Proof. First, we shall prove that $\beta(s, u) = 0$, $\forall\ s \in \mathtt{Nil}(\xi_x)$, $\forall\ u \in \xi_x$. Since $\mathtt{Nil}^k(\xi)$ is an ideal bundle in ξ and therefore $(ad_s ad_u)\xi \subseteq \mathtt{Nil}(\xi)$ and $(ad_s ad_u)\mathtt{Nil}^k(\xi_x) \subseteq \mathtt{Nil}^{k+1}(\xi_x)$. Hence $(ad_s ad_u)^{k+1}\xi \subseteq \mathtt{Nil}^k(\xi_x)$. But $\mathtt{Nil}(\xi)$ is nilpotent ideal bundle and therefore $\mathtt{Nil}^k(\xi) = 0$ for some positive integer k. Hence $(ad_s ad_u)^{k+1} = 0$. Therefore $ad_s ad_u$ is nilpotent and $Tr(ad_s ad_u) = \beta(s, u) = 0$.

Let $\phi : U \times L \to \bigcup_{x \in U} \xi_x$ be a local triviality of ξ. Put

$$M = \bigcup_{x \in X} M_x, \text{ where } M_x = \{u \in \xi_x |\ T(u, v, w) = 0\ \forall\ v, w \in \xi_x\}.$$

Then M_x is a solvable characteristic ideal in ξ_x [4]. Let $\phi^{-1}(M_x) = I$, For any derivation d of L,

$$\begin{aligned} d(I) &= \phi^{-1}(\phi d \phi^{-1})\phi(I) \\ &\subseteq \phi^{-1}(\phi d \phi^{-1})M_x \subseteq \phi^{-1}M_x = I. \end{aligned}$$

So I is a characteristic ideal in L. Solvability of I follows from solvability of M_x and isomorphism ϕ. Hence I is a solvable characteristic ideal in L. Now we will show that I is the radical of L. Let $r \in R$, then $\phi(r) \in \mathtt{Rad}(\xi_x)$. Using Theorem (3.5), for any $u \in \xi_x$, $[\phi(r), u] = -ad_u(\phi(r)) \in \mathtt{Nil}(\xi_x)$. Hence

$$T(\phi(r), u, w) = \beta([\phi(r), u], w) = 0\ \forall\ u, w \in \xi_x.$$

Thus $\phi(r) \in M_x$, $r \in \phi^{-1}M_x = I$. Since $r \in R$ is arbitrary we have $I = R$, the radical of L. Hence $\mathtt{Rad}(\xi) = \bigcup_{x \in X} M_x$ and $\phi : U \times R \to \bigcup_{x \in U} M_x$ gives the local triviality of M. Since M is a characteristic ideal bundle same is true for $\mathtt{Rad}(\xi)$. Thus the radical bundle is a characteristic ideal bundle of ξ.

If D is derivation of ξ, using Theorem (3.5), we have $D(\mathtt{Rad}(\xi)) \subseteq \mathtt{Nil}(\xi)$ and so $D(\mathtt{Nil}(\xi)) \subseteq D(\mathtt{Rad}(\xi)) \subseteq \mathtt{Nil}(\xi)$. Hence $\mathtt{Nil}(\xi)$ is also a characteristic ideal bundle in ξ. □

A trivial Lie algebra bundle is a semidirect product of its radical bundle and a semisimple subalgebra bundle [8]. We write $\xi = \mathtt{Rad}(\xi) \oplus \mathtt{S}$, where $\mathtt{S}$ is a semisimple subalgebra bundle of ξ.

Theorem 4.2. *Every derivation of ξ is the sum of an inner derivation and a derivation which annuls $\mathtt{S}$.*

Proof. Let D be any derivation of ξ we have $D(\mathtt{Rad}(\xi)) \subseteq \mathtt{Rad}(\xi)$. Thus D induces derivation of $\mathtt{Rad}(\xi)$. On the other hand we know that there exists an element $u_0 \in \xi$ such that $D(s) = [s, u_0]$, $\forall\ s \in \mathtt{S}$. Let consider inner derivation ad_{u_0} and if we put $D^{'} = D - ad_{u_0}$ we have $D^{'}(\mathtt{S}) = 0$. Then $D^{'}[r, s] = [D^{'}(r), s]$. Conversely, if D is any derivation $\mathtt{Rad}(\xi)$ with above condition gives derivation of ξ if we define $D(\mathtt{S}) = 0$ □

Acknowledgments

Author would like to thank the anonymous referee for constructive remarks that improve the presentation of the paper.

References

1. D. Coppersmith, *A familly of Lie algebras not extendible to a family of Lie groups*, Proc. Amer. Math. Soc.,(2) **66** (1977), 365–366.
2. Douady, A. and Lazard, M., *Espace fibrés algèbres de Lie et en groupes*, Invent. Math., **1** (1966), 133–151.
3. W. Greub, S. Halperin and R. Vanstone, *Connections, Curvature and Cohmology*, Academic Press, New York, **2** (1973).
4. Harish-Chandra, *On the Radical of a Lie Algebra*, Proc. Amer. Math. Soc., (1) **1** (1950), 14–17.
5. B. S. Kiranagi, *Lie algebra bundles*, Bull. Sci. Math., 2^e serie, **102** (1978), 57–62.
6. B. S. Kiranagi, *Semisimple Lie alegebra bundles*, Bull. Math. de la Soc. Sci. Math de la R.S. de Roumanie., (75)**27** (1983), 253–257.
7. B. S. Kiranagi and G. Prema, *Cohomology of Lie algebra bundles and its applications*, Indian J.Pure Appl. Math., (7) **16** (1985), 731–735.
8. B. S. Kiranagi and G. Prema, *A decomposition theorem of Lie algebra bundles*, Comm. Algebra., (6) **18** (1990), 869–877.
9. B. S. Kiranagi and G. Prema, *Rigidity theorem for Lie algebra bundles*, Comm. Algebra., (6) **20** (1992), 1549–1556.
10. Kiranagi B.S, G. Prema, Ranjitha Kumar, *On the Radical Bundle of a Lie Algebra Bundle*, Proc.Jangjeon Math. Soc., (4) **15** (2012), 447–453.
11. Kiranagi B.S, G. Prema, Ranjitha Kumar, "On Completely Semsimple Lie Algebra Bundles", *J. Algebra Appl.,* (2) **14** (2014), 1–11.

12. Shamil Asgarli, *Ado's Theorem*, www.math.ubc.ca/reichst/Ado's-Theorem.pdf.

Ranjitha Kumar
Department of Mathematics, School of Applied Sciences
REVA University, Bengaluru-560064, India.
Email: ranju286math@gmail.com, ranjitha.kumar@reva.edu.in