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FILTRATION, ASYMPTOTIC σ-PRIME DIVISORS AND SUPERFICIAL ELEMENTS

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Abstract. Let $(A, \mathfrak{M})$ be a Noetherian local ring with infinite residue field $A/\mathfrak{M}$ and I be a $\mathfrak{M}$-primary ideal of A. Let $f = (I_n)_{n\in\mathbb{N}}$ be a good filtration on A such that I_1 containing I. Let σ be a semi-prime operation in the set of ideals of A. Let $l \geq 1$ be an integer and $(f^{(l)})_\sigma = \sigma(I_{n+l}) : \sigma(I_n)$ for all large integers n and $\rho^f_\sigma(A) = min\{n \in \mathbb{N} \mid \sigma(I_l) = (f^{(l)})_\sigma, for\ all\ l \geq n\}$. Here we show that, if I contains an $\sigma(f)$-superficial element, then $\sigma(I_{l+1}) : I_1 = \sigma(I_l)$ for all $l \geq \rho^f_\sigma(A)$. We suppose that P is a prime ideal of A and there exists a semi-prime operation $\widehat{\sigma}_P$ in the set of ideals of A_P such that $\widehat{\sigma}_P(JA_P) = \sigma(J)A_P$, for all ideal J of A. Hence $Ass_A\big(A/\sigma(I_l)\big) \subseteq Ass_A\big(A/\sigma(I_{l+1})\big)$, for all $l \geq \rho^f_\sigma(A)$.

1. Introduction

Let A be a Noetherian ring, $k \geq 1$ be an integer, I be an ideal of A and σ be a semi-prime operation in the set of ideals of A such that I is an ideal containing a $A/\sigma(I^{k+1})$-regular element. [2] proves in it's lemma 1 that there exists an integer $m_0 > k$ such that $\sigma(I^{m_0+1}) :_A I = \sigma(I^{m_0})$. However to show that the sequence $\Big\{Ass\big(A/\sigma(I^n)\big)\Big\}_{n\geq 0}$ is increasing from a certain rank, we assumed that the relationship $\sigma(I^{n+1}) :_A I = \sigma(I^n)$ is right for all large n (see [2, Theorem 5]). [4] reveals conditions under which the relationship ($\circledast$) $\sigma(I^{n+1}) :_A I = \sigma(I^n)$

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holds from a certain rank.
The section 3 of this paper is found within the framework when generalizing the condition of the relation $(\circledast)$ to good filtrations. We show that, if $f = (I_n)_{n\in\mathbb{Z}}$ is a good filtration on A, then the sequence $\Big\{Ass\big(A/\sigma(I_n)\big)\Big\}_{n\geq 0}$ is increasing from a certain rank.
In section 4, we suppose that $\big(\sigma(I^n)\big)_{n\in\mathbb{N}}$ is a I-good filtration, $\Delta = \big\{I^n \mid n \in \mathbb{N} - \{0\}\big\}$ and $J_\Delta = \bigcup_{K\in\Delta}(JK : K)$ is the Δ-closure of J, for all ideal J of A, and $(I^k)_\sigma = \sigma(I^{n+k}) : \sigma(I^n)$, with $k \geq 1$ and $n \gg 0$ be two integers. The Proposition 4.1 shows that there exists an integer $n_0 \geq 1$ such that $(I^k)_\sigma = \big(\sigma(I^k)\big)_\Delta$ for all $k \geq n_0$.

2. Preliminary

What we will develop in this paper is an extension of [4]. That's why, we remember some definitions and useful properties of [4] for the rest.
Let A be a commutative and unitary ring.

(1) A filtration on A is a sequence $f = (I_n)_{n\in\mathbb{Z}}$ of ideals of A such that
$I_0 = A$, $I_{n+1} \subseteq I_n$ for all $n \in \mathbb{Z}$ and $I_pI_q \subseteq I_{p+q}$ for all $p, q \in \mathbb{Z}$. It follows that $I_n = A$, for all $n \leq 0$.

(2) Let I be an ideal of A, a filtration $f = (I_n)_{n\in\mathbb{Z}}$ on A is I-good if
i) $I.I_n \subseteq I_{n+1}$ $\forall n \geq 0$ and ii) $\exists n_0 \in \mathbb{Z}$ tel que $\forall n \geq n_0$, $I.I_n = I_{n+1}$. Then $I^nI_{n_0} = I_{n_0+n}$, $\forall n \geq 1$. $f = (I_n)_{n\in\mathbb{Z}}$ is said good if it is I_1-good.

(3) For all integers $l \geq 1$, let $f^{(l)} = (I_{nl})_{n\in\mathbb{Z}}$. $f^{(l)}$ is a filtration on A. It's the extracted filtration from order l of f.

(4) Let $\mathfrak{I}(A)$ be the set of ideals of A. Suppose the map $\sigma : \mathfrak{I}(A) \longrightarrow \mathfrak{I}(A)$ and the following properties:
  (a) $I \subseteq \sigma(I)$,
  (b) $\sigma(\sigma(I)) = \sigma(I)$,
  (c) If $I \subseteq J$, then $\sigma(I) \subseteq \sigma(J)$,
  (d) $\sigma(I)\sigma(J) \subseteq \sigma(IJ)$,
  (e) $\sigma(bI) = b\sigma(I)$,
for all $I, J \in \mathfrak{I}(A)$ and b a regular element of A. Then, σ is a closure operation if (a)-(c) hold for all $I, J \in \mathfrak{I}(A)$. It is a

semi-prime operation if (a)-(d) hold for all $I, J \in \Im(A)$. Finally it is a prime operation if (a)-(e) hold for all $I, J \in \Im(A)$ and b a regular element of A.

(5) If σ is a semi-prime operation, then $\sigma[\sigma(I)\sigma(J)] = \sigma(IJ)$, for all $I, J \in \Im(A)$.

(6) For all $I \in \Im(A)$, $\sigma(I)$ is the σ-closure of I.

(7) If $f = (I_n)_{n\in\mathbb{Z}}$ is a filtration on A and σ is a semi-prime operation in $\Im(A)$, then $\sigma(f) = \big(\sigma(I_n)\big)_{n\in\mathbb{Z}}$ is a filtration on A and is the σ-closure of f.

(8) Suppose that A is a Noetherian ring and the filtration $f = (I_n)_{n\in\mathbb{Z}}$ is good, then:

(a) There exists $n_0 \geq 1$ such that for all $n \geq n_0$, $\Big\{\sigma(I_{n+1}) :_A \sigma(I_n)\Big\}_{n\in\mathbb{N}}$ is an increasing sequence, hence it stabilizes. Let's pose $f_\sigma = \sigma(I_{n+1}) :_A \sigma(I_n) = \sigma(I_{n+1}) : \sigma(I_n)$ for all large n, (see [3, proposition 4.1]).

(b) If $l \geq 1$ is an integer, then it exists $n_0 \geq 1$ such that for all $n \geq n_0$, $\Big\{\sigma(I_{n+l}) : \sigma(I_n)\Big\}_{n\in\mathbb{N}}$ is an increasing sequence, hence it stabilizes and we have $(f^{(l)})_\sigma = \sigma(I_{n+l}) : \sigma(I_n)$ for all large n, (see [3, proposition 4.2]). Indeed, let n and l be two integers and let $a \in \sigma(I_{n+l}) : \sigma(I_n)$. Then $a\sigma(I_n) \subseteq \sigma(I_{n+l})$ and $a\sigma(I_1)\sigma(I_n) \subseteq \sigma(I_1)\sigma(I_{n+l})$. Since σ is semi-prime operation and $\sigma[\sigma(I_1)\sigma(I_n)] = \sigma(I_1 I_n)$, we have $a\sigma(I_1 I_n) \subseteq \sigma(I_{n+1+l})$. We know that f is good, then it exists an integer $n_0 \geq 1$ such that for all $n \geq n_0$, $I_1 I_n = I_{n+1}$. Thus $a\sigma(I_{n+1}) \subseteq \sigma(I_{n+1+l})$ and $a \in \sigma(I_{n+1+l}) : \sigma(I_{n+1})$. Therefore $\Big\{\sigma(I_{n+l}) : \sigma(I_n)\Big\}_{n\in\mathbb{N}}$ is an increasing sequence for all $n \geq n_0$. And it stabilizes for some large enough integers n because A is a Noetherian ring. Let n_1 be this integer such that for all $n \geq n_1$, $\sigma(I_{n+l}) : \sigma(I_n) = \sigma(I_{n_1+l}) : \sigma(I_{n_1})$. For all $n \geq n_1$ we have $ln \geq n_1$, then $\sigma(I_{n+l}) : \sigma(I_n) = \sigma(I_{ln+l}) : \sigma(I_{ln}) = \sigma(I_{l(n+1)}) : \sigma(I_{ln})$. Since $I_{l(n+1)}$ and I_{ln} are two consecutive terms of the filtration $f^{(l)}$, thus we can write $(f^{(l)})_\sigma = \sigma(I_{n+l}) : \sigma(I_n)$ for all $n \geq n_1$.

(9) Let $(A, \mathfrak{M})$ be a Noetherian local ring with infinite residue field $A/\mathfrak{M}$. Let I be a $\mathfrak{M}$-primary ideal of A and $f = (I_n)_{n\in\mathbb{Z}}$ be a I-good filtration on A. An element $x \in I$ is f-superficial if there exists an integer n_0 such that $(I_{n+1} :_A x) \cap I_{n_0} = I_n$, for all $n \geq n_0$.

3. Asymptotic prime σ-divisors and $\sigma(f)$-superficial elements

Throughout this section, A is a Noetherian ring and $f = (I_n)_{n\in\mathbb{N}}$ is a good filtration on A, with I_n a nonzero ideal for all $n \geq 1$. Let σ be a semi-prime operation in the set of ideals of A and $\sigma(f) = \big(\sigma(I_n)\big)_{n\in\mathbb{N}}$ be the σ-closure of f.

Lemma 3.1. *Let $k, r \geq 1$ be two integers. Let $f = (I_n)_{n\in\mathbb{N}}$ be a filtration on A such that I_r containing a $A/\sigma(I_{k+r})$-regular element. Then there exists an integer $m_0 \geq k$ such that $\sigma(I_{m_0+r}) : I_r = \sigma(I_{m_0})$.*

Proof. Let $m \geq 1$ be an integer. Let $a \in \sigma(I_m)$, then $aI_r \subseteq \sigma(I_{m+r})$. Hence $a \in \sigma(I_{m+r}) : I_r$. Thus $\sigma(I_m) \subseteq \sigma(I_{m+r}) : I_r$, for all $m \geq 1$. Conversely, suppose for all $m \geq k$, $\sigma(I_m) \subsetneq \sigma(I_{m+r}) : I_r$. In particular for $m = k + r$ we have $\sigma(I_{k+r}) \subsetneq \sigma(I_{k+2r}) : I_r$. Let $a \in \sigma(I_{k+2r}) : I_r$ and $a \notin \sigma(I_{k+r})$. We have, $a \notin \sigma(I_{k+r})$ implies that $a + \sigma(I_{k+r})$ is different from the class of zero modulo $\sigma(I_{k+r})$. Since $a \in \sigma(I_{k+2r}) : I_r$, we have $aI_r \subseteq \sigma(I_{k+r})$. Thus $\overline{a}I_r = 0 + \sigma(I_{k+r})$. Since I_r contains a $A/\sigma(I_{k+r})$-regular element, then $a \in \sigma(I_{k+r})$, which is absurd. Hence there exists an integer $m_0 \geq k$ such that $\sigma(I_{m_0+r}) : I_r = \sigma(I_{m_0})$. □

Proposition 3.2. *i) f_σ is σ-closed.*
ii) Let l be an large enough integer such that $\sigma(I_{l+r}) : I_r = \sigma(I_l)$, for all $r \geq 1$. Then $\sigma(I_l) = (f^{(l)})_\sigma$.

Proof. i) Since σ is semi-prime operation in the set of ideals of A and f_σ is an ideal of A, then $f_\sigma \subseteq \sigma(f_\sigma)$. Conversely, $\sigma(f_\sigma) = \sigma\big(\sigma(I_{n+1}) : \sigma(I_n)\big) \subseteq \sigma(I_{n+1}) : \sigma(I_n) = f_\sigma$ for all large enough n , (see [2, proposition 3]). Hence $f_\sigma = \sigma(f_\sigma)$.
ii) Suppose that l is a large enough integer such that $\sigma(I_{l+r}) : I_r = \sigma(I_l)$, for all $r \geq 1$. Then $(f^{(l)})_\sigma = \sigma(I_{n+l}) : \sigma(I_n) \subseteq \sigma(I_{n+l}) : I_n = \sigma(I_l)$, for all large enough n. Therefore $(f^{(l)})_\sigma \subseteq \sigma(I_l)$. We also have by [3, proposition 4.3], $\sigma(I_l) \subseteq (f^{(l)})_\sigma$. Thus $\sigma(I_l) = (f^{(l)})_\sigma$.
□

Proposition 3.3. *Let $f = (I_n)_{n\in\mathbb{N}}$ be a good filtration on A. Then $\big\{(f^{(n)})_\sigma\big\}_{n\in\mathbb{N}}$ is a filtration on A.*

Proof. i) $(f^{(0)})_\sigma = \sigma(I_n) : \sigma(I_n) = A$ for all n. ii) Let $n \geq 1$ an integer and $a \in (f^{(n)})_\sigma = \sigma(I_{k+n}) : \sigma(I_k)$ for all large enough k. Therefore $a \in \sigma(I_{k+n-1}) : \sigma(I_k) = (f^{(n-1)})_\sigma$, for all large enough k, we have the decreasing of the sequence. iii) By [3, proposition 4.3], we have $(f^{(p)})_\sigma (f^{(q)})_\sigma \subseteq (f^{(p+q)})_\sigma$, for all integer $p, q \geq 0$. Then $\big\{(f^{(n)})_\sigma\big\}_{n\in\mathbb{N}}$ is a filtration on A.

□

For the rest of this paper we assume that $(A, \mathfrak{M})$ is a Noetherian local ring with infinite residue field $A/\mathfrak{M}$ and I is a $\mathfrak{M}$-primary ideal of A. Let $f = (I_n)_{n\in\mathbb{N}}$ be a good filtration on A such that I_1 containing I.

Definition 3.4. An element x of I is said to be $\sigma(f)$-superficial, if there exists an integer $n_0 \geq 0$ such that $\big(\sigma(I_{n+1}) :_A x\big) \cap \sigma(I_{n_0}) = \sigma(I_n)$, for all $n \geq n_0$.

Proposition 3.5. *Let $x \in I$ a $\sigma(f)$-superficial element. For all integer $n \geq 1$ we have:*
i) $(f^{(n+1)})_\sigma : x = (f^{(n)})_\sigma$.
ii) $(x) \cap (f^{(n+1)})_\sigma = x(f^{(n)})_\sigma$.

Proof. Let $n \geq 1$ be an integer.
i) If $a \in (f^{(n+1)})_\sigma : x$, then $ax \in (f^{(n+1)})_\sigma = \sigma(I_{k+n+1}) : \sigma(I_k)$, for all $k \gg 0$. therefore $a\sigma(I_k) \subseteq \sigma(I_{k+n+1}) : x$. Since x is $\sigma(f)$-superficial, then there exists k_0 such that $\big(\sigma(I_{m+1}) : x\big) \cap \sigma(I_{k_0}) = \sigma(I_m)$, for all $m \geq k_0$. Therefore for some large enough k we have $a\sigma(I_k) \subseteq \sigma(I_{k+n+1}) : x$ and $a\sigma(I_k) \subseteq \sigma(I_{k_0})$. Thus $a\sigma(I_k) \subseteq \big(\sigma(I_{k+n+1}) : x\big) \cap \sigma(I_{k_0}) = \sigma(I_{k+n})$ with $k + n \geq k_0$. Then $a\sigma(I_k) \subseteq \sigma(I_{k+n})$, thus $a \in \sigma(I_{k+n}) : \sigma(I_k) = (f^{(n)})_\sigma$, for all large enough k. Conversely, suppose $I \subseteq I_1$, therefore $x(f^{(n)})_\sigma \subseteq I_1(f^{(n)})_\sigma \subseteq \sigma(I_1)(f^{(n)})_\sigma \subseteq (f^{(1)})_\sigma (f^{(n)})_\sigma \subseteq (f^{(n+1)})_\sigma$ (See [3, proposition 4.3]). Then $(f^{(n)})_\sigma \subseteq (f^{(n+1)})_\sigma : x$. It follows that $(f^{(n+1)})_\sigma : x = (f^{(n)})_\sigma$ for all integer $n \geq 1$.

ii) We have a consequence of i). Indeed, let $n \geq 1$ be an integer and $z \in (x) \cap (f^{(n+1)})_\sigma$. Then there exists $a \in A$ such that $z = ax \in (f^{(n+1)})_\sigma$. It follows that $a \in (f^{(n+1)})_\sigma : x = (f^{(n)})_\sigma$ and $z = ax \in x(f^{(n)})_\sigma$. Conversely $x(f^{(n)})_\sigma \subseteq I_1(f^{(n)})_\sigma \subseteq (f^{(n+1)})_\sigma$ and $x(f^{(n)})_\sigma \subseteq (x)$. Then $x(f^{(n)})_\sigma \subseteq (x) \cap (f^{(n+1)})_\sigma$. Hence $(x) \cap (f^{(n+1)})_\sigma = x(f^{(n)})_\sigma$, for all integer $n \geq 1$.

□

By the Proposition 3.2, under certain conditions, we have shown that for some large enough integer l, $\sigma(I_l) = (f^{(l)})_\sigma$. Let $\rho^f_\sigma(A) = min\big\{n \in$

$\mathbb{N} \mid \quad \sigma(I_l) = (f^{(l)})_\sigma, for\ all\ l \geq n\}$. The existence of such $\rho_\sigma^f(A)$ is proved in [8].

Proposition 3.6. *Let $x \in I$ be a $\sigma(f)$-superficial element. Then for all $l \geq \rho_\sigma^f(A)$, $\sigma(I_{l+1}) : x = \sigma(I_l)$ and $(x) \cap \sigma(I_{l+1}) = x\sigma(I_l)$.*

Proof. The proof is easy by Proposition 3.5.

□

Proposition 3.7. *Let $l \geq 1$ be an integer. If $x \in I$ is $A/\sigma(I_{l+1})$-regular element, then $\sigma(I_{l+r}) : x^r = \sigma(I_{l+1}) : x$, for all $r \geq 1$.*

Proof. Let $l \geq 1$ et $r > 1$ be two integers such that $a \in \sigma(I_{l+r}) : x^r$, then $ax^r \in \sigma(I_{l+r}) \subseteq \sigma(I_{l+1})$. hence $x(ax^{r-1} + \sigma(I_{l+1})) = \overline{0}$ (class of zero modulo $\sigma(I_{l+1})$). Since x is $A/\sigma(I_{l+1})$-regular, then $ax^{r-1} \in \sigma(I_{l+1})$. By taking the process, we show that $a \in \sigma(I_{l+1}) : x$. Inversely, if $a \in \sigma(I_{l+1}) : x$, $ax^r \in I_1^{r-1}\sigma(I_{l+1}) \subseteq \sigma(I_{l+r})$. Thus $a \in \sigma(I_{l+r}) : x^r$. It follows that $\sigma(I_{l+r}) : x^r = \sigma(I_{l+1}) : x$, for all $r \geq 1$.

□

Corollary 3.8. *Let $x \in I$ be a $\sigma(f)$-superficial element and let $l \geq \rho_\sigma^f(A)$ be an integer. If x is $A/\sigma(I_{l+1})$-regular, then $\sigma(I_{l+r}) : x^r = \sigma(I_l)$ for all integer $r \geq 1$.*

Proof. It's a consequence of Propositions 3.6 and 3.7.

□

Let P be a prime ideal of the Noetherian ring A. Let $\sum_A$ be the set of all semi-prime operations σ for which $\widehat{\sigma}_P$, such that $\widehat{\sigma}_P(JA_P) = \sigma(J)A_P$ for all ideal J of A, are well-defined et are semi-prime operations on the set of all ideals of A_P. The example of section 3 of [3] shows that $\sum_A$ is not empty. Then for the hypothesis of theorem 3.9-ii) we will restrict us on the set $\sum_A$.

Theorem 3.9. *Let $x \in I$ be a $\sigma(f)$-superficial element and let $l \geq \rho_\sigma^f(A)$ be an integer. We have:*
i) $\sigma(I_{l+1}) : I_1 = \sigma(I_l)$.
ii) Suppose that for a prime ideal P of A, there exists a semi-prime operation $\widehat{\sigma}_P$ in the set of ideals of A_P such that $\widehat{\sigma}_P(JA_P) = \sigma(J)A_P$, for all ideal J of A. Then $Ass_A\big(A/\sigma(I_l)\big) \subseteq Ass_A\big(A/\sigma(I_{l+1})\big)$.

Proof. i) Let $l \geq \rho_\sigma^f(A)$ and $a \in \sigma(I_{l+1}) : I_1$, then $ax \in \sigma(I_{l+1})$. Hence $a \in \sigma(I_{l+1}) : x = \sigma(I_l)$. The reverse is easy.
ii) Let $l \geq \rho_\sigma^f(A)$ and P be a prime ideal of A such that $P \in Ass_A\big(A/\sigma(I_l)\big)$.
(a) Suppose that A is a local ring with maximal ideal P. There exists $a \in A$ such that $a \notin \sigma(I_l)$ and $P = \sigma(I_l) : a$. Then $aP \subseteq \sigma(I_l)$ and

$aI_1P \subseteq \sigma(I_{l+1})$. Hence $P \subseteq \sigma(I_{l+1}) : aI_1$. We have $aI_1 \nsubseteq \sigma(I_{l+1})$, otherwise $a \in \sigma(I_{l+1}) : I_1 = \sigma(I_l)$ (See (i)) and $a \in \sigma(I_l)$ which is absurd. Thus, since P is maximal, we have $P = \sigma(I_{l+1}) : aI_1$. Let $(b_1, ..., b_r)$ be a finite system of generators of I_1. Then $aI_1 = (b_1a, ..., b_ra)$ and $P = \sigma(I_{l+1}) : (b_1a, ..., b_ra) = \bigcap_{i=1}^{r} \big(\sigma(I_{l+1}) : b_ia\big)$. That is, $P \subseteq \sigma(I_{l+1}) : b_ia$ for every $i = 1, ..., r$. There exists $k \in \{1, ..., r\}$ such that $a \notin \sigma(I_{l+1}) : b_k$. Otherwise, if for every $i = 1, ..., r$ $a \in \sigma(I_{l+1}) : b_i$, then $a \in \bigcap_{i=1}^{r} \big(\sigma(I_{l+1}) : b_i\big) = \sigma(I_{l+1}) : (b_1, ..., b_r) = \sigma(I_{l+1}) : I_1 = \sigma(I_l)$ (See (i)) and $a \in \sigma(I_l)$ which is absurd. Therefore $P = \sigma(I_{l+1}) : b_ka$ and $P \in Ass_A\big(A/\sigma(I_{l+1})\big)$. Then $Ass_A\big(A/\sigma(I_l)\big) \subseteq Ass_A\big(A/\sigma(I_{l+1})\big)$ for all $l \geq \rho_\sigma^f(A)$.

(b) Suppose that A is not a local ring with maximal ideal P. We know that A_P is a local ring with maximal ideal PA_P. Let $\widehat{\sigma}_P$ be the semi-prime operation in the set of all ideals of A_P such that $\widehat{\sigma}_P(JA_P) = \sigma(J)A_P$, for all ideal J of A. By i) we have $\sigma(I_{l+1}) : I_1 = \sigma(I_l)$, then $\sigma(I_{l+1})A_P : I_1A_P = \sigma(I_l)A_P$ and $\widehat{\sigma}_P(I_{l+1}A_P) : I_1A = \widehat{\sigma}_P(I_lA_P)$, for every $l \geq \rho_\sigma^f(A)$. By (a) we have

$$Ass_{A_P}\big(A_P/\widehat{\sigma}_P(I_lA_P)\big) \subseteq Ass_{A_P}\big(A_P/\widehat{\sigma}_P(I_{l+1}A_P)\big).$$

If $P \in Ass_A\big(A/\sigma(I_l)\big)$, then $PA_P \in Ass_{A_P}\big(A_P/\widehat{\sigma}_P(I_lA_P)\big)$. It follows that $PA_P \in Ass_{A_P}\big(A_P/\widehat{\sigma}_P(I_{l+1}A_P)\big) = Ass_{A_P}\big(A_P/\sigma(I_{l+1})A_P\big)$. Hence $P \in Ass_A\big(A/\sigma(I_{l+1})\big)$ and $Ass_A\big(A/\sigma(I_l)\big) \subseteq Ass_A\big(A/\sigma(I_{l+1})\big)$ for all $l \geq \rho_\sigma^f(A)$.

$\square$

4. Δ-closure and $(I^k)_\sigma$

Let A be a Noetherian ring, I be a nonzero ideal of A and σ be a semi-prime operation in the set of ideals of A. Let $\big(\sigma(I^n)\big)_{n\in\mathbb{N}}$ be the σ-closure of the filtration $\big(I^n\big)_{n\in\mathbb{N}}$. Let Δ be a nonempty, multiplicatively closed set of nonzero ideals in A and $J_\Delta = \bigcup_{K\in\Delta}(JK : K)$ be the Δ-closure of J, for all ideal J of A. We consider that $(I^k)_\sigma = \sigma(I^{n+k}) : \sigma(I^n)$, with $k \geq 1$ and $n \gg 0$ be two integers. The following proposition shows the conditions under which equality $(I^k)_\sigma = \big(\sigma(I^k)\big)_\Delta$ holds.

Proposition 4.1. *If the filtration $\big(\sigma(I^n)\big)_{n\in\mathbb{N}}$ is I-good and $\Delta = \big\{I^n \mid n \in \mathbb{N}-\{0\}\big\}$, then there exists an integer $n_0 \geq 1$ such that $(I^k)_\sigma = \big(\sigma(I^k)\big)_\Delta$ for all $k \geq n_0$.*

Proof. Let $k \geq 1$ be an integer and $x \in \big(\sigma(I^k)\big)_\Delta = \bigcup_{K\in\Delta}\big(\sigma(I^k)K : K\big)$. Since $\Delta = \big\{I^n, \quad n \in \mathbb{N}-\{0\}\big\}$, there exists an integer $l \geq 1$ such that $xI^l \subseteq \sigma(I^k)I^l$. Since σ is a semi-prime operation, $x\sigma(I^l) \subseteq$

$\sigma(I^{k+l})$. Thus $x \in (I^k)_\sigma$, since $\{\sigma(I^{n+k}) : \sigma(I^n)\}_{n\in\mathbb{N}}$ is an increasing sequence (see[3]). Conversely, let $x \in (I^k)_\sigma$ and $x \notin \big(\sigma(I^k)\big)_\Delta$, it follows that $xI^n \subseteq x\sigma(I^n) \subseteq \sigma(I^{k+n})$ for $n \gg 0$ and $xI^m \nsubseteq I^m\sigma(I^k)$ for all integer $m \geq 1$. Since the filtration $\big(\sigma(I^n)\big)_{n\in\mathbb{N}}$ is I-good, there exists an integer $n_0 \geq 1$ such that $I^n\sigma(I^s) = \sigma(I^{s+n})$ for all $s \geq n_0$ and $n \geq 1$. Therefore, for $k \geq n_0$ and $n \gg 0$ we have $xI^n \nsubseteq \sigma(I^{k+n})$ and $xI^n \subseteq \sigma(I^{k+n})$, which is absurd. Hence the equality.

□

Example 4.2. In [9] the authors show that, if I is an ideal of an analytically unramified semi-local ring, then there exists a positive integer k such that $(I^{n+k})' = I^n(I^k)'$ for all integers $n \geq 0$, with $(I^n)'$ is the integral closure of the ideal I^n. In others words, in such ring the filtration $\big((I^n)'\big)_{n\in\mathbb{N}}$ is I-good.

Let σ and δ be two closure operations such that $\sigma(J) \subseteq \delta(J)$, for all ideals J. We notice that $\delta(I) \subseteq \delta[\sigma(I)] \subseteq \delta[\delta(I)] = \delta(I)$ and $\delta(I) \subseteq \sigma[\delta(I)] \subseteq \delta[\delta(I)] = \delta(I)$ for all ideals I. Therefore $\delta(I) = \delta[\sigma(I)] = \sigma[\delta(I)]$, for all ideals I. Hence the following corollary is a consequence of the Proposition 4.1.

Corollary 4.3. *Let σ be a semi-prime operation comparable to the Δ-closure operation such that the filtration $\big(\sigma(I^n)\big)_{n\in\mathbb{N}}$ is I-good. Then $(I^k)_\sigma = \sigma\big((I^k)_\Delta\big)$.*

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