Journal of Algebra and Related Topics
Vol. 9, No 1, (2021), pp 143-158

ON SECONDARY SUBHYPERMODULES

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Abstract. Let R be a Krasner hyperring and M be an R- hypermodule. Let $\psi : S^h(M) \to S^h(M) \cup \{\emptyset\}$ be a function, where $S^h(M)$ denote the set of all subhypermodules of M. In the first part of this paper, we introduce the concept of a secondary hypermodule over a Krasner hyperring. A non-zero hypermodule M over a Krasner hyperring R is called secondary if for every $r \in R$, $rM = M$ or $r^n M = 0$ for some positive integer n. Then we investigate some basic properties of secondary hypermodules. Second, we introduce the notion of ψ-secondary subhypermodules of an R-hypermodule and we obtain some properties of such subhypermodules.

1. Introduction

Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Hyperstructures have many applications to several sectors of both pure and applied mathematics, for instance in geometry, lattices, cryptography, automata, graphs and hypergraphs, fuzzy set, probability and rough set theory and so on (see [4] and [5]). The hypergroup notion was introduced in 1934 by a French mathematician F. Marty [8], at the 8th Congress of Scandinavian Mathematicians. The notion of hyperrings was introduced by M. Krasner in 1983, where the addition is a hyperoperation, while the

MSC(2010): Primary: 20N20; Secondary: 13E05
Keywords: Krasner hyperring, hypermodule, secondary hypermodule, ψ-secondary subhypermodule.
Received: 21 February 2021, Accepted: 31 May 2021.
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multiplication is an operation [7]. Also, hypermodules over a hyperring is a generalization of the classical modules over a ring. Several kinds of hyperrings and hypermodules were introduced and studied by many authors. Prime, primary, and maximal subhypermodules of a hypermodule were discussed by M. M. Zahedi and R. Ameri in [10]. Also, R. Ameri et al in [1] studied prime and primary subhypermodules of (m, n)-hypermodules. The principal notions of algebraic hyperstructure theory can be found in [5] and [10].

In this paper, we define the concept of secondary hypermodules and get some basic properties of such hypermodules. Also, we introduce and study a generalization of secondary subhypermodules which is called ψ-secondary subhypermodules and we give a number of results of these subhypermodules.

2. Basic definitions and results

Definition 2.1. [7] Let H be a nonempty set and $P^*(H)$ denotes the set of all nonempty subsets of H. If $+ : H \times H \longrightarrow P^*(H)$ is a map such that the following conditions hold, then we say that $(H, +)$ is a canonical hypergroup.

(i) for every $x, y, z \in H$, $x + (y + z) = (x + y) + z$;
(ii) for every $x, y \in H$, $x + y = y + x$;
(iii) there exists $0 \in H$ such that $0 + x = \{x\}$ for every $x \in H$;
(iv) for every $x \in H$ there exists a unique element $x' \in R$ such that $0 \in x + x'$, it is denoted by $-x$;
(v) for every $x, y, z \in H$, $z \in x+y$ implies $y \in -x+z$ and $x \in z-y$.

Let $A \subset H$. Then A is called a *subhypergroup* of H if $0 \in H$ and $(A, +)$ is itself a hypergroup.

Definition 2.2. A *Krasner hyperring* is an algebraic hyperstructure $(R, +, \cdot)$ which satisfies the following axioms:

(1) $(R, +)$ is a canonical hypergroup;
(2) $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element, i.e., $x \cdot 0 = 0 \cdot x = 0$;
(3) the operation " $\cdot$" is distributive over the hyperoperation "+", which means that for all x, y, z of R we have:

$$x \cdot (y + z) = x \cdot y + x \cdot z \text{ and } (x + y) \cdot z = x \cdot z + y \cdot z.$$

A Krasner hyperring $(R, +, \cdot)$ is called *commutative with unit element* $1 \in R$; if we have

(a) $xy = yx$ for all $x, y \in R$,
(b) $1x = x1$ for all $x \in R$.

A subset S of a hyperring R is said to be a *subhyperring* of R if $(S, +, \cdot)$ is itself a hyperring.

A subhyperring I of a hyperring R is a *left (right) hyperideal* of R if $rx \in I (xr \in I)$ for all $r \in R$, $x \in I$. I is called a *hyperideal* if I is both a left and a right hyperideal.

Definition 2.3. [10] Let I be a hyperideal of a Krasner hyperring R. The radical of I (in abbreviation $rad(I)$) is the set of all $x \in R$ such that $x^n \in I$ for some $n \in \mathbb{N}$. It is clear that $rad(I)$ is a hyperideal of R.

Definition 2.4. [10] Let R be a Krasner hyperring and P be a proper hyperideal of R.

(1) P is called a prime hyperideal of R if $ab \in P$ for some $a, b \in P$, then $a \in P$ or $b \in P$.
(2) P is called a primary hyperideal of R if $ab \in P$ for some $a, b \in P$, then $a \in P$ or $b \in rad(P)$.

Definition 2.5. [10] Let R and S be hyperrings. A mapping ϕ from R into S is said to be a hyperring homomorphism, if for all $a, b \in R$;

(1) $\phi(a + b) = \phi(a) + \phi(b)$, $\phi(0) = 0$.
(2) $\phi(ab) = \phi(a)\phi(b)$.

Definition 2.6. [10] Let $(R, +, \cdot)$ be a hyperring with unit element 1. An R-(left) hypermodule M is a commutative hypergroup $(M, +)$ together with a map $\cdot : R \times M \longrightarrow M$ defined by

$$(a, m) \mapsto a \cdot m = am \in M$$

such that for all $r_1, r_2 \in R$ and $m_1, m_2, m \in M$ we have

(1) $r_1 \cdot (m_1 + m_2) = r_1 \cdot m_1 + r_2 \cdot m_2$;
(2) $(r_1 + r_2) \cdot m = (r_1 \cdot m) + (r_2 \cdot m)$;
(3) $(r_1 \cdot r_2) \cdot m = r_1 \cdot (r_2 \cdot m)$;
(4) $1m = m$;
(5) $r0_M = 0_R m = 0_M$.

A nonempty subset N of an R-hypermodule M is called a *subhypermodule* if N is an R-hypermodule with the operations of M.

Proposition 2.7. *Let $N \neq \emptyset$ be a subset of an R-hypermodule M. Then N is a subhypermodule of M, if and only if, for every $x, y \in N$ and $r \in R$, we have $x - y \subseteq N$ and $rx \in N$.*

Definition 2.8. (a) A proper subhypermodule P of a R-hypermodule M is called prime (primary) whenever $rm \in P$ with $r \in h(R)$ and

$m \in h(M)$, implies that $m \in N$ or $rM \subseteq N$ ($m \in N$ or $r^nM \subseteq N$ for some positive integer n).

(b) A proper subhypermodule N of M is said to be maximal, provided that for subhypermodule K of M with $N \subseteq K \subseteq M$, then $N = K$ or $K = M$.

Example 2.9. [7] If $(R, +, \cdot)$ is a ring and G a subset of R such that $(G, \cdot)$ is a group, then we can define an equivalence relation $\cong$ on R as follows:

$$(\forall x, y \in R)\ (x \cong y \Leftrightarrow xG = yG)$$

The equivalence class represented by x is $P(x) = \{y \in R \mid yG = xG\} = xG$. Let $\frac{R}{G}$ be the set of all equivalence classes. Define a hyperoperation $\oplus$ on $\frac{R}{G}$ as follows:

$$\begin{aligned} P(x) \oplus P(y) &= \{P(t) \mid P(t) \cap (P(x) + P(y)) \neq \emptyset\} \\ &= \{tG \mid \exists g_1, g_2 \in G \text{ such that } t = xg_1 + yg_2\} \end{aligned}$$

and define a binary operation $\cdot$ on $\frac{R}{G}$ by $xG \cdot yG = xyG$ (or $P(x) \cdot P(y) = P(xy)$). Then $(\frac{R}{G}, \oplus, \cdot)$ forms a hyperring. Moreover, if we choose R to be a field, then we get that $(\frac{R}{G}, \oplus, \cdot)$ is a hyperfield.

Lemma 2.10. [3] *Let R be a hyperring with unit element. Then R is an R-hypermodule.*

Definition 2.11. Let I be a hyperideal of a hyperring R and let $R/I = \{r + I | r \in R\}$. Define the hyperoperations $\oplus$ and $\otimes$ on R/I by $(a + I) \oplus (b + I) = a + b + I$ and $(a + I) \otimes (b + I) = ab + I$. Then $(R/I, \oplus, \otimes)$ is called a quotient hyperring.

Proposition 2.12. [3] *Let M be an R-hypermodule and N be a subhypermodule of M. The quotient hypergroup $M/N = \{m + N \mid m \in M\}$ is an R-hypermodule under the multiplication defined by $r(m + N) = rm + N$. This is called the quotient hypermodule.*

In the following, we recall the construction of the hyperrings of fractions [6]. Let R be any hyperring and let S be any multiplicatively closed subset of R with $1 \in S$. Define a relation "$\sim$" on $R \times S$ by $(a, s) \sim (b, t)$,if and only if $0 \in (at - bs)u$, for some $u \in S$. Denote the equivalence class of (a, s) with $\frac{a}{s}$ and let $S^{-1}R$ denote the set of all equivalence classes. We endow the set $S^{-1}R$ with a hyperring structure, by defining the addition and the multiplication between fractions as follows:

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st} \quad \textit{and} \quad \frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}$$

We know that $S^{-1}R$ forms a hyperring under these operations.

Similarly, one constructs hypermodule of fractions. Let M be an R-hypermodule and S be a multiplicatively closed subset of R. Define a relation "$\sim$" on $M \times S$ by

$$(m, s) \sim (m', s') \Leftrightarrow (\exists t \in S)(0 \in t(ms' - m's)),$$

i. e. that there exists $t \in S$ such that $tms' = tm's$. Such a relation is obviously an equivalence relation. Let $\frac{m}{s}$ denote the equivalence class of the pair (m, s), and let, for a given M and S, the symbol $S^{-1}M$ denote the set of all such fractions. Then $S^{-1}M$ is an $S^{-1}R$-hypermodule.

3. SECONDARY HYPERMODULES

In this section we introduce and study the concept of secondary hypermodules over a Krasner hyperring.

Definition 3.1. A non-zero hypermodule M over a Krasner hyperring R is called secondary if for every $r \in R$, $rM = M$ or $r^n M = 0$ for some positive integer n. In which case, $rad((0 :_R M)) = P$ is a prime hyperideal of R, M is said to be P-secondary.

An R- hypermodule M is called simple, if it is non-zero and has no non-zero proper subhypermodule.

Lemma 3.2. *Let M be an R-hypermodule and N a subhypermodule of M. Then N is a maximal subhypermodule of M if and only if M/N is a simple R-hypermodule.*

Proof. It is straightforward. □

Let M be an R-hypermodule. An element $r \in R$ is called a zero-divisor on M, if there exists $0 \neq m \in M$ such that $rm = 0$.

Lemma 3.3. *Let M be a simple hypermodule over hyperring R. Then every zero divisor on M is an annihilator of M.*

Proof. Let r be a zero-divisor on M. Then there exists $0 \neq a \in M$ such that $ra = 0$. Since M is a simple R-hypermodule, we get $Ra = M$. Hence, $rM = r(Ra) = (Rr)a = R(ra) = 0$. Thus, r is an annihilator of M. □

Proposition 3.4. *Let M be an R-hypermodule. Then every maximal subhypermodule is a prime subhypermodule.*

Proof. Let N be a maximal subhypermodule of M. Let $rm \in N$ where $r \in R$ and $m \in M \setminus N$. Since $0 \neq (m + N) \in M/N$ and $r(m + N) = 0$, we get r is a zero-divisor on hypermodule M/N; hence by Lemma 3.2 and Lemma 3.3, $r \in (N :_R M)$, as required. □

Proposition 3.5. *Let R be a Krasner hyperring and M be a free R-hypermodule. Then the following hold:*

(a) *If I is a primary hyperideal of R, then IM is a primary subhypermodule of M.*

(b) *If I is a prime hyperideal of R, then IM is a prime subhypermodule of M.*

Proof. (a) We have $IM \neq M$ since $I \neq R$ and M is free hypermodule. Let $\{m_i\}_{i \in I}$ be a basis of M and let $rm \in IM$ with $m \notin IM$ where $r \in R$ and $m \in M$. Hence $m \in \sum_{i=1}^{n} r_i m_i$ with $r_i \in R$. Since $m \notin IM$, there exists, $1 \leq j \leq n$, such that $r_j \notin I$. There are elements $b_1, b_2, \cdots, b_n \in I$ such that $\sum_{i=1}^{n}(rr_i)m_i = \sum_{i=1}^{n} b_i m_i$, and so $0 \in \sum_{i=1}^{n}(rr_i - b_i)m_i$, so $rr_i = b_i$ for every $i = 1, \cdots, n$. Since $rr_j \in I$ and $r_j \notin I$, then $r^m \in I$ for some $m \in \mathbb{N}$; thus $r^m M \subseteq IM$, as required.
(b) The proof is similar to that of (a). □

Definition 3.6. A subhypermodule N of M is said to be pure subhypermodule if $aN = N \cap aM$ for every $a \in R$.

Proposition 3.7. *Let R be a Krasner hyperring and M be an R-hypermodule, and N be a non-zero pure subhypermodule of M. Then M is a P-secondary hypermodule if and only if both N and M/N are P-secondary R-hypermodules.*

Proof. Assume that M is P-secondary and let $a \in R$. If $a \in P$, then $a^n N \subseteq a^n M = 0$ and $a^n(M/N) = 0$ for some $n \in \mathbb{N}$. If $a \notin P$, then $aN = N \cap aM = N$ and $a(M/N) = M/N$, hence N and M/N are P-secondary R-hypermodules. Conversely, assume that N and M/N are P secondary R-hypermodules and let $b \in R$. If $b \in P$, then $b^t M \subseteq N$ and $0 = b^t N = N \cap b^m M = b^t M$ for some $t \in \mathbb{N}$, so b is nilpotent on M. If $b \notin P$, then $N = bN = N \cap bM$ and $b(M/N) = M/N$, hence $bM = M$, as needed. □

Theorem 3.8. *Let M be a secondary hypermodule and N be a P-prime subhypermodule of M. Then N is a P-secondary hypermodule.*

Proof. Assume that M is a Q-secondary hypermodule and $r \in R$. If $r \in Q$, then $r^s N \subseteq r^s M = 0$ for some $s \in \mathbb{N}$, so r is a nilpotent on N. Suppose that $r \notin Q$; we show that $rN = N$. So assume that $a \in N$. Then there exists $b \in M$ such that $a = rb$. As N is a prime subhypermodule and $rb \in N$, then $b \in N$. It follows that $rN = N$, so N is a Q-secondary R-hypermodule. Now we need to show that $P = Q$. Since the inclusion $P \subseteq Q$ is trivial, we will prove the reverse inclusion. Suppose $c \in Q$. Then $c^m M = 0$ for some $m \in \mathbb{N}$ since M is a Q-secondary hypermodule. As $M \neq N$, there is an element $x \in M$

such that $x \notin N$. Therefore, $c^m x = 0 \in N$, so N prime gives $c \in P$; hence $Q \subseteq P$, as required. $\square$

Corollary 3.9. *Let R be a Krasner hyperring, M an R-hypermodule and N a P-secondary subhypermodule of M. Then the following hold:*

(a) *If K is a primary subhypermodule of M, then $N \cap K$ is P-secondary.*

(b) *If K is a prime subhypermodule of M, then $N \cap K$ is P-secondary.*

Proof. (a) Assume that $a \in R$ and let $a \in P$. Then $a^m(N \cap K) \subseteq a^m N = 0$ for some m, so a is nilpotent on $N \cap K$. Suppose that $a \notin P$; we show that a divides $N \cap K$. It suffices to show that $N \cap K \subseteq a(N \cap K)$. If $b \in N \cap K$, then $b = am$ for some $m \in N$. Then $am \in K$. It follows that $m \in K$, otherwise, if $m \notin K$ and $a^s \in (K :_R M)$ for some s, then $m \in N = a^s N \subseteq a^s M \subseteq K$ which is a contradiction, so $m \in K$; hence $b \in a(N \cap K)$ and the proof is complete.
(b) The proof is similar to that (a). $\square$

Definition 3.10. A hypermodule M is said to be secondary representable, if it can be written as a sum $M = M_1 + M_2 + \cdots + M_k$ with each M_i secondary, and if such representation exists then the attached primes of M are $Att(M) = \{(0 :_R M_1), \cdots , (0 :_R M_k)\}$.

Theorem 3.11. (a) *Every primary subhypermodule of a representable R-hypermodule is representable.*

(b) *Every prime subhypermodule of a representable R-hypermodule is representable.*

Proof. (a) Assume that $M = \sum_{i=1}^{k} S_i$ is a minimal secondary representation of M with $Att(M) = \{P_1, \cdots , P_k\}$ and let N be a P-primary subhypermodule of M. There exists a subhypermodule S_i, say S_1, such that $S_1 \nsubseteq N$ since $N \neq M$. First, we show that $P = P_1$. Let $a \in P_1$. So there exist $n \in \mathbb{N}$ and $y \in S_1 - N$ such that $a^n y = 0$. Hence $a \in P$ since N is P-primary. Therefore $P_1 \subseteq P$. For the other containment, suppose that there exists an element $c \in P$ with $c \notin P_1$. Then $S_1 = c^s S_1 \subseteq c^s M \subseteq N$ for some s, which is a contradiction. Thus $P = P_1$. Likewise, if $S_j \nsubseteq N$ for $j \neq 1$, then $P = P_1 = P_j$ which is a contradiction. We will show that $S_i \subseteq N$ for $i = 2, \cdots , k$. As $P \neq P_i$, we divide the proof into two cases:

Case 1: $P \nsubseteq P_i$.
There exists an element $p \in P$ with $p \notin P$. Let $b \in S_i$. Then $S_i = p^t S_i \subseteq p^t M \subseteq N$ for some t.

Case 2: $P_i \nsubseteq P$.
There exists an element $p \in P$ with $p \notin P$. Then there exists an integer

n such that $a^n b = 0 \in N$, so $b \in N$ since N is primary; hence $b \in N$. Thus $S_i \subseteq N$. It follows that $N = N \cap M = N \cap S_1 + \sum_{i=1}^{k} S_i$. Now the assertion follows from Corollary 3.9.
(b) The proof is similar to that (a). $\square$

Corollary 3.12. *Let R be a hyperring, M a representable R-hypermodule and N a primary (resp. prime) R-subhypermodule of M. Then $Att(N) \subseteq Att(M)$.*

4. ψ-SECONDARY SUBHYPERMODULES

In this section, we define and study ψ-secondary subhypermodules of a hypermodule over a Krasner hyperring.

Definition 4.1. Let M be an R-hypermodule. We say that a non-zero subhypermodule N of an R-hypermodule M is a secondary (weak secondary) subhypermodule, if $r \in R$, K a subhypermodule of M, $rN \subseteq K$ ($rN \subseteq K$ and $rM \nsubseteq K$), then $N \subseteq K$ or $r^n N = 0$ for some $n \in \mathbb{N}$.

Clearly, every secondary subhypermodule of an R-hypermodule M is a weak secondary subhypermodule of M. But the converse is not true in general, as we see in the following example.

Example 4.2. Let $\mathbb{Z}$ be the ring of integers and $G = \{-1, 1\}$ is the multiplicative subgroup $\mathbb{Z}$. Then by using Example 2.9, $R = \frac{\mathbb{Z}}{G}$ is a hyperring. By using Lemma 2.10, $M = \frac{\mathbb{Z}}{G}$ is an $R = \frac{\mathbb{Z}}{G}$ hypermodule. Then R-hypermodule M is weak secondary which is not secondary.

Definition 4.3. Let M be an R-hypermodule, $S^h(M)$ be the set of all subhypermodules of M, and let $\psi : S^h(M) \to S^h(M) \cup \{\emptyset\}$ be a function. We say that a non-zero subhypermodule N of M is a ψ-secondary subhypermodule of M if $r \in R$, K a subhypermodule of M, $rN \subseteq K$, and $r\psi(N) \nsubseteq K$, then $N \subseteq K$ or $r^n N = 0$ for some $n \in \mathbb{N}$.

In Definition 4.3, since $r\psi(N) \nsubseteq K$ implies that $r(\psi(N) + N) \nsubseteq K$, there is no loss of generality in assuming that $N \subseteq \psi(N)$ in the rest of this paper. Let M be an R-hypermodule. We use the following functions $\psi : S^h(M) \to S^h(M) \cup \{\emptyset\}$.

$$\psi_M(N) = M, \ \forall N \in S^h(M),$$

$$\psi_i(N) = (N :_M Ann_R^i(N)), \ \forall N \in S^h(M), \ \forall i \in \mathbb{N},$$

$$\psi_\sigma(N) = \sum_{i=1}^{\infty} \psi_i(N), \ \forall N \in S^h(M).$$

Then it is clear that for any subhypermodule and every positive integer n, we have the following implications:

$$\textit{secondary} \Rightarrow \psi_{n-1} - \textit{secondary} \Rightarrow \psi_n - \textit{secondary} \Rightarrow \psi_\sigma - \textit{secondary}$$

For functions $\psi, \theta : S^h(M) \to S^h(M) \cup \{\emptyset\}$, we write $\psi \leq \theta$ if $\psi(N) \subseteq \theta(N)$ for each $N \in S^h(M)$. So whenever $\psi \leq \theta$, any ψ-secondary subhypermodule is θ-secondary.

Theorem 4.4. *Let M be an R-hypermodule and N be a subhypermodule of R. Then the following statements are equivalent:*

(i) *N is a secondary subhypermodule of M.*
(ii) *$N \neq 0$ and $rN \subseteq K$, where $r \in R$ and K is a subhypermodule of M, implies either $r^n N = 0$ for some $n \in \mathbb{N}$ or $N \subseteq K$.*

Proof. $(i) \Rightarrow (ii)$ is obvious.
$(ii) \Rightarrow (i)$ Let $r \in R$ and $r^n N \neq 0$ for any $n \in \mathbb{N}$. Since $rN \subseteq rN$, so $N \subseteq rN$ by assumption. Therefore $rN = N$, as needed. □

Theorem 4.5. *Let M be an R-hypermodule and N be a subhypermodule of M. Let $\psi : S^h(M) \to S^h(M) \cup \{\emptyset\}$ be a function and N be a ψ-secondary subhypermodule of R-hypermodule M such that $Ann_R(N)\psi(N) \nsubseteq N$. Then N is a secondary subhypermodule of M.*

Proof. Let $r \in R$ and K be a subhypermodule of M such that $rN \subseteq K$. If $r\psi(N) \nsubseteq K$, then we are done because N is a ψ-secondary subhypermodule of R-hypermodule M. Thus suppose that $r\psi(N) \subseteq K$. If $r\psi(N) \nsubseteq N$, then $r\psi(N) \nsubseteq N \cap K$. Since $rN \subseteq N \cap K$, then $N \subseteq N \cap K \subseteq K$ or $r^n N = 0$ for some $n \in \mathbb{N}$, as required. So let $r\psi(N) \subseteq N$. If $Ann_R(N)\psi(N) \nsubseteq K$, then $(r + Ann_R(N))\psi(N) \nsubseteq K$. Thus $t \in r + s$ such that $t\psi(N) \nsubseteq K$ for some $s \in Ann_R(N)$. As $tN \subseteq K$ implies that $N \subseteq K$ or $t^n N = 0$ for some $n \in \mathbb{N}$. We have $r \in t-s$, so $r^n N \subseteq (t-s)^n N = 0$. Hence let $Ann_R(N)\psi(N) \subseteq K$. Since $Ann_R(N)\psi(N) \nsubseteq N$, there exists $s \in Ann_R(N)$ such that $s\psi(N) \nsubseteq N$. Thus $s\psi(N) \nsubseteq N \cap K$. Hence we have $(r + s)\psi(N) \nsubseteq N \cap K$. So $t \in r + s$ such that $t\psi(N) \nsubseteq N \cap K$. Therefore, $(r + s)N \subseteq N \cap K$ implies that $tN \subseteq N \cap K$, hence $N \subseteq N \cap K \subseteq K$ or $t^n N = 0$ for some $n \in \mathbb{N}$ since N is a ψ-secondary subhypermodule of M. Hence $r^n N \subseteq (t - s)^n N = 0$, as needed. □

Corollary 4.6. *Let M be an R-hypermodule, N a subhypermodule of M. Let $\psi : S^h(M) \to S^h(M) \cup \{\emptyset\}$ be a function and N be a ψ-secondary subhypermodule of M such that $(N :_M Ann^2_R(N)\psi(N) \subseteq \psi(N)$. Then N is a ψ_σ-secondary subhypermodule of M.*

Proof. If N is a secondary subhypermodule of M, then the result is clear. So assume that N is not a secondary subhypermodule of M. Then by Theorem 4.5, we have $Ann_R(N)\psi(N) \subseteq N$. Therefore, by assumption,

$$(N :_M Ann_R^2(N)) \subseteq \psi(N) \subseteq (N :_M Ann_R(N)).$$

We conclude that $\psi(N) = (N :_M Ann_R^2(N)) = (N :_M Ann_R(N))$, because $(N :_M Ann_R(N)) \subseteq (N :_M Ann_R^2(N))$. So we get

$$(N :_M Ann_R^3(N)) = (((N :_M Ann_R^2(N)) :_M Ann_R(\psi(N))) =$$

$$((N :_M Ann_R(N)) :_M Ann_R(N)) = (N :_M Ann_R^2(N)) = \psi(N).$$

By continuing, we get that $\psi(N) = (N :_M Ann_R^i(N))$ for all $i \geq 1$. Hence $\psi(N) = \psi_\sigma(N)$, as needed. □

Theorem 4.7. *Let M be an R-hypermodule and $\psi : S^h(M) \to S^h(M) \cup \{\emptyset\}$ be a function. Let N be a subhypermodule of M such that for all hyperideals I and J of R, $(N :_M I) \subseteq (N :_M J)$ implies that $J \subseteq I$. If N is not a secondary subhypermodule of M, then N is not a ψ_1-secondary subhypermodule of M.*

Proof. Since N is not a secondary subhypermodule of M, there exists $r \in R$ and a subhypermodule K of M such that $r^n N \neq 0$ for each $n \in \mathbb{N}$ and $N \nsubseteq K$, but $rN \subseteq K$ by Theorem 4.4. We have $N \nsubseteq N \cap K$ and $rN \subseteq N \cap K$. If $r(N :_M Ann_R(N)) \nsubseteq N \cap K$, then N is not a ψ_1-secondary subhypermodule of M. Hence let $r(N :_M Ann_R(N)) \subseteq N \cap K$. Thus $r(N :_M Ann_R(N)) \subseteq N \cap K \subseteq N$. Therefore, $(N :_M Ann_R(N)) \subseteq (N :_M r)$ and so by assumption, $r \in Ann_R(N)$, which is a contradiction. □

Corollary 4.8. *Let M be an R-hypermodule and $\psi : S^h(M) \to S^h(M) \cup \{\emptyset\}$ be a function. Let N be a subhypermodule of M such that for all hyperideals I and J of R, $(N :_M I) \subseteq (N :_M J)$ implies that $J \subseteq I$. Then N is a secondary subhypermodule of M if and only if N is a ψ_1-secondary subhypermodule of M.*

An R-hypermodule M is said to be a comultiplication hypermodule if for every subhypermodule N of M, there exists a hyperideal I of R such that $N = (0 :_M I)$. It is easy to see that M is a comultiplication module if and only if $N = (0 :_M Ann_R(N))$ for each subhypermodule N of M.

Definition 4.9. Let R be a hyperring and $\varphi : S^h(R) \to S^h(R) \cup \{\emptyset\}$ be a function. A proper hyperideal P of R is called φ-primary, if for $a, b \in R$, $ab \in P - \varphi(P)$, then $a \in P$ or $b \in rad(P)$.

Definition 4.10. Let M be an R-hypermodule and $\varphi : S^h(M) \to S^h(M) \cup \{\emptyset\}$ be a function. A proper subhypermodule N of M is said to be φ-primary, if for each $r \in R$ and $m \in M$, $rm \in N \setminus \varphi(N)$, then $m \in N$ or $r \in rad((N :_R M))$.

Theorem 4.11. *Let M be an R-hypermodule, $\varphi : S^h(R) \to S^h(R) \cup \{\emptyset\}$, and $\psi : S^h(M) \to S^h(M) \cup \{\emptyset\}$ be functions. Then the following hold:*

(i) *If S is a ψ-secondary subhypermodule of M with $Ann_R(\psi(S)) \subseteq \varphi(Ann_R(S))$, then $Ann_R(S)$ is a φ-primary hyperideal of R.*

(ii) *If M is a comultiplication R-hypermodule, S is a subhypermodule of M such that $\psi(S) = (0 :_M \varphi(Ann_R(S)))$, and $Ann_R(S)$ is a φ-primary hyperideal of R, then S is a ψ-secondary subhypermodule of M.*

Proof. (i) Let $ab \in Ann_R(S) \setminus \varphi(Ann_R(S))$ for some $a, b \in R$. Then $ab\psi(S) \neq 0$ by assumption. If $a\psi(S) \subseteq (0 :_M b)$, then $ab\psi(S) = 0$, a contradiction. Thus $a\psi(S) \nsubseteq (0 :_M b)$. Therefore, $S \subseteq (0 :_M b)$ or $a^n S = 0$ for some $n \in \mathbb{N}$ because S is a ψ-secondary subhypermodule of M. Hence $a \in Ann_R(S)$ or $b \in Ann_R(S)$, as required.

(ii) Let $a \in R$ and K be a subhypermodule of M such that $aS \subseteq K$ and $a\psi(S) \nsubseteq K$. As $aS \subseteq K$, we have $S \subseteq (K :_M a)$. It follows that

$$S \subseteq ((0 :_M Ann_R(K)) :_M a) = (0 :_M aAnn_R(K)).$$

This implies that $aAnn_R(K) \subseteq Ann_R((0 :_M aAnn_R(K))) \subseteq Ann_R(S)$. Hence $aAnn_R(K) \subseteq Ann_R(S)$. If $aAnn_R(K) \subseteq \varphi(Ann_R(S))$, then

$$\psi(S) = ((0 :_M \varphi(Ann_R(S)) = ((0 :_M Ann_R(K)) :_M a).$$

As M is a comultiplication R-hypermodule, we have $a\psi(S) \subseteq K$, a contradiction. Thus $aAnn_R(K) \nsubseteq \varphi(Ann_R(S))$ and so as $Ann_R(S)$ is a φ-primary hyperideal of R, we conclude that $a^n S = 0$ for some $n \in \mathbb{N}$ or

$$S = (0 :_M Ann_R(S)) \subseteq (0 :_M Ann_R(K)) = K,$$

as needed. □

Example 4.12. Let $\mathbb{Z}$ be the ring of integers and $G = \{-1, 1\}$ is the multiplicative subgroup $\mathbb{Z}$. Then by using Example 2.9, $R = \frac{\mathbb{Z}}{G}$ is a hyperring. The hyperideals of R are of the form $\langle nG \rangle$, where $n \in \mathbb{Z}$. Also, by using Lemma 2.10, $M = \frac{\mathbb{Z}}{G}$ is an $R = \frac{\mathbb{Z}}{G}$ hypermodule. Consider the subhypermodule $S = \langle 2G \rangle$. Clearly, M is not a comultiplication R-hypermodule. Suppose that $\varphi : S^h(R) \to S^h(R) \cup \{\emptyset\}$ and $\psi : S^h(M) \to S^h(M) \cup \{\emptyset\}$ be functions such that $\varphi(I) = I$ for each hyperideal I of R and $\psi(S) = M$. Then $Ann_R(S) = 0$ is a φ-primary hyperideal of R and $\psi(S) = M = (0 :_M \varphi(Ann_R(S)))$. But as $(3G)S \subseteq \langle 6G\rangle$, $S \nsubseteq \langle 6G\rangle$ and $(3G)^n S \neq 0$ for each $n \in \mathbb{N}$, we have that S is not a ψ-secondary subhypermodule of M.

Proposition 4.13. *Let M be an R-hypermodule, $\psi : S^h(M) \to S^h(M) \cup \{\emptyset\}$ be a function and N be a ψ-secondary subhypermodule of M. Then we have the following statements.*

(i) *If K is a subhypermodule of M with $K \subset N$ and $\psi_K : S(M/K) \to S(M/K) \cup \{\emptyset\}$ be a function such that $\psi_K(N/K) = \psi(N)/K$, then N/K is a ψ_K-secondary subhypermodule of M/K.*

(ii) *Let N be a finitely generated subhypermodule of M, S be a multiplicatively closed subset of R with $Ann_R(N) \cap S = \emptyset$, and $S^{-1}\psi : S^h(S^{-1}M) \to S^h(S^{-1}M) \cup \{\emptyset\}$ be a function such that $(S^{-1}\psi)(S^{-1}N) = S^{-1}\psi(N)$. Then $S^{-1}N$ is a $S^{-1}\psi$-secondary subhypermodule of $S^{-1}M$.*

Proof. (i) Since $K \subset N$, then $N/K \neq 0$. Let $r \in R$, L/K be a subhypermodule of M/K, $r(N/K) \subseteq L/K$ and $r\psi(N/K) \nsubseteq L/K$. We get $rN \subseteq L$ and $r\psi(N) \nsubseteq L$. Therefore, $r^n N = 0$ for some $n \in \mathbb{N}$ or $N \subseteq L$ since N is a ψ-secondary subhypermodule of M. Hence $r^n(N/K) = 0$ for some $n \in \mathbb{N}$ or $N/K \subseteq L/K$, as needed.
(ii) Since N is finitely generated and $Ann_R(N) \cap S = \emptyset$, we get $S^{-1}(N) \neq 0$. Let $\frac{r}{s} \in h(S^{-1}R)$, $S^{-1}(K)$ be a subhypermodule of $S^{-1}M$ and $\frac{r}{s}(S^{-1}\psi)(S^{-1}N) \nsubseteq S^{-1}K$. Thus we get $rN \subseteq K$ and $r\psi(N) \nsubseteq K$ $((S^{-1}\psi)(S^{-1}N) = S^{-1}\psi(N))$. Hence $N \subseteq K$ or $r^n N = 0$ for some $n \in \mathbb{N}$ since N is a ψ-secondary subhypermodule of M. Therefore, $S^{-1}N \subseteq S^{-1}K$ or $(\frac{r}{s})^n \psi(S^{-1}N) = 0$ for some $n \in \mathbb{N}$, and so $S^{-1}N$ is a $S^{-1}\psi$-secondary subhypermodule of $S^{-1}M$. □

Definition 4.14. Let M and M' be R-hypermodules. A mapping f from M into M' is said to be a homomorphism, if

(1) for any $m, n \in M$, $f(m+n) = f(m) + f(n)$,
(2) for any $r \in R$ and $m \in M$, $f(rm) = rf(m)$.

Proposition 4.15. *Let M and M' be R-hypermodules and $f : M \to M'$ be a monomorphism. Let $\psi : S^h(M) \to S^h(M) \cup \{\emptyset\}$ and $\psi' : S(M') \to S(M') \cup \{\emptyset\}$ be functions such that $\psi(f^{-1}(N')) = f^{-1}(\psi'(N'))$, for each subhypermodule N' of M'. If N' is a ψ-secondary subhypermodule of M' such that $N' \subseteq Im(f)$, then $f^{-1}(N')$ is a ψ-secondary subhypermodule of M.*

Proof. Since $N' \neq 0$ and $N' \subseteq Im(f)$, we have $f^{-1}(N') \neq 0$. Let $a \in R$ and K be a subhypermodule of M such that $af^{-1}(N') \subseteq K$ and $a\psi(f^{-1}(N')) \nsubseteq K$. Then by assumptions, $aN' \subseteq f(K)$ and $a\psi'(N') \nsubseteq f(K)$. Thus $a^n N' = 0$ for some $n \in \mathbb{N}$ or $N' \subseteq f(K)$ since N' is a

ψ'-secondary subhypermodule of M'. Therefore, $a^n f^{-1}(N') = 0$ for some $n \in \mathbb{N}$ or $f^{-1}(N') \subseteq K$, as required. $\square$

A proper subhypermodule N of an R-hypermodule M is said to be *completely irreducible* if $N = \bigcap_{i\in I} N_i$, where $\{N_i\}_{i\in I}$ is a family of subhypermodules of M, implies that $N = N_i$ for some $i \in I$. It is easy to see that every subhypermodule of M is an intersection of completely irreducible subhypermodules of M.

Remark 4.16. Let N, K be subhypermodules of an R-hypermodule M. To prove $N \subseteq K$, it is enough to show that if L is a completely irreducible subhypermodule of M such that $K \subseteq L$, then $N \subseteq L$.

Proposition 4.17. *Let M be an R-hypermodule and $\psi : S^h(M) \to S^h(M) \cup \{\emptyset\}$ be a function and let N be a ψ_1-secondary subhypermodule of M. Then we have the following statements:*

(i) *If for $a \in R$, $aN \neq N$, then $(N :_M rad(Ann_R(N))) \subseteq (N :_M a)$.*
(ii) *If J is a hyperideal of R such that $rad(Ann_R(N)) \subseteq J$ and $JN \neq N$, then $(N :_M rad(Ann_R(N))) = (N :_M J)$.*

Proof. (i) Let $a \in R$ such that $aN \neq N$. If $a^n N = 0$ for some $n \in \mathbb{N}$, then clearly $(N :_M rad(Ann_R(N))) \subseteq (N :_M a)$. Hence let $a^n N \neq 0$ for each $n \in \mathbb{N}$. Now let H be a completely irreducible subhypermodule of M such that $N \subseteq H$. Then $N \nsubseteq aN \cap H$ and $aN \subseteq aN \cap H$. Thus as N is a ψ_1-secondary subhypermodule of M, we have $a(N :_M Ann_R(N)) \subseteq aN \cap H \subseteq H$. Hence $a(N :_M Ann_R(N)) \subseteq N$ by Remark 4.16. Therefore, $a(N :_M rad(Ann_R(N))) \subseteq a(N :_M Ann_R(N))$ implies that $a(N :_M rad(Ann_R(N))) \subseteq N$. Hence $(N :_M rad(Ann_R(N))) \subseteq (N :_M a)$.
(ii) As $JN \neq N$, we have $aN \neq N$ for each $a \in J$. Thus by part (i), for each $a \in J$, $(N :_M rad(Ann_R(N))) \subseteq (N :_M a)$. This implies that

$$(N :_M J) = \bigcap_{a\in J}(N :_M a) \supseteq (N :_M rad(Ann_R(N))).$$

The inverse inclusion follows from the fact that $rad(Ann_R(N)) \subseteq J$. $\square$

Theorem 4.18. *Let M be an R-hypermodule, $\psi : S^h(M) \to S^h(M) \cup \{\emptyset\}$ be a function. If $(0 :_M a)$ is a ψ_1-secondary of subhypermodule of M such that $(0 :_M a) \subseteq a(0 :_M aAnn_R(0 :_M a))$, then $(0 :_M a)$ is a secondary subhypermodule of M.*

Proof. Let $N := (0 :_M a)$ be a ψ_1-secondary subhypermodule of M. Then $(0 :_M a) \neq 0$. Let $b \in R$ and K be a subhypermodule of M such that $b(0 :_M a) \subseteq K$. If $b(N :_M Ann_R(N)) \nsubseteq K$, then $b^n(0 :_M a) = 0$

for some $n \in \mathbb{N}$ or $(0 :_M a) \subseteq K$ since $(0 :_M a)$ is a ψ_1-secondary subhypermodule of M. So let $b(N :_M Ann_R(N)) \subseteq K$. Now we have $(a+b)(0 :_M a) \subseteq K$. If $(a+b)(N :_M Ann_R(N)) \nsubseteq K$, then there exists $t \in a+b$ such that $t(N :_M Ann_R(N)) \nsubseteq K$ and since $t(0 :_M a) \subseteq K$, then as $(0 :_M a)$ is a ψ_1-secondary subhypermodule of M, then $t^n(0 :_M a) = 0$ for some $n \in \mathbb{N}$ or $(0 :_M a) \subseteq K$ and so $a \in t-b$, hence $a^n(0 :_M a) \subseteq (t-b)^n(0 :_M a) = 0$, we are done. Hence assume that $(a+b)(N :_M Ann_R(N)) \subseteq K$. Then $b(N :_M Ann_R(N)) \subseteq K$ gives that $a(N :_M Ann_R(N)) \subseteq K$. Therefore by assumption, $(0 :_M a) \subseteq K$ and the result follows from Theorem 4.4. □

Theorem 4.19. *Let M be an R-hypermodule, $\psi : S^h(M) \to S^h(M) \cup \{\emptyset\}$ be a functions, and N be a non-zero subhypermodule of M. Then the following are equivalent:*

(i) *N is a ψ-secondary subhypermodule of M;*

(ii) *For a subhypermodule L of M with $N \nsubseteq L$, we have*

$$rad((L :_R N)) = rad(Ann_R(N)) \cup rad((L :_R \psi(N)));$$

(iii) *For a subhypermodule L of M with $N \nsubseteq L$, we have $rad((L :_R N)) = rad(Ann_R(N))$ or $rad((L :_R N)) = rad((L :_R \psi(N)))$;*

(iv) *For any hyperideal I of R and any subhypermodule K of M, if $IN \subseteq K$ and $I \nsubseteq rad((K :_R \psi(N)))$, then $IN = 0$ or $N \subseteq K$;*

(v) *For each $a \in R$ with $a\psi(N) \nsubseteq aN$, we have $aN = N$ or $a^nN = 0$ for some $n \in \mathbb{N}$.*

Proof. $(i) \Rightarrow (ii)$ Let for a subhypermodule L of M with $N \nsubseteq L$, we have $a \in rad((L :_R N)) \setminus rad((L :_R \psi(N)))$. Then $a^n\psi(N) \subseteq L$ for some $n \in \mathbb{N}$ and $a^n\psi(N) \nsubseteq L$. Since N is a ψ-secondary subhypermodule of M, we have $a \in rad(Ann_R(N))$. As we may assume that $N \subseteq \psi(N)$, the other inclusion always holds.

$(ii) \Rightarrow (iii)$ This follows from the fact that if a hyperideal is a union of two hyperideals, it is equal to one of them.

$(iii) \Rightarrow (iv)$ Let I be a hyperideal of R and K be a subhypermodule of M such that $IN \subseteq K$ and $I \nsubseteq rad((K :_R \psi(N)))$. Suppose $I \nsubseteq rad(Ann_R(N))$ and $N \nsubseteq K$. We show that $I \subseteq rad((K :_R \psi(N)))$. Let $a \in I$ and first let $a \notin rad(Ann_R(N))$. Then since $aN \subseteq K$, we have $rad((K :_R N)) \neq rad(Ann_R(N))$. Hence by assumption $rad((K :_R N)) = rad((K :_R \psi(N)))$. So $a \in rad((K :_R \psi(N)))$. Now let $a \in I \cap rad(Ann_R(N))$. Let $b \in I \setminus rad(Ann_R(N))$. Then $a+b \subseteq I \setminus rad(Ann_R(N))$. Hence by the first case, we have $b \in rad((K :_R \psi(N)))$ and $(b+a) \in rad((K :_R \psi(N)))$. This gives that $a \in rad((K :_R \psi(N)))$. Thus in any case $a \in rad((K :_R \psi(N)))$. Thus $I \subseteq rad((K :_R \psi(N)))$, as desired.

$(iv) \Rightarrow (i)$ The proof is straightforward.

$(i) \Rightarrow (v)$ Let $a \in R$ such that $a\psi(N) \nsubseteq aN$. Then $aN \subseteq aN$ implies that $N \subseteq aN$ or $a^n N = 0$ for some $n \in \mathbb{N}$ by part (i). Thus $N = aN$ or $a^n N = 0$ for some $n \in \mathbb{N}$, as required.

$(v) \Rightarrow (i)$ Let $a \in R$ and K be a subhypermodule of M such that $aN \subseteq K$ and $a\psi(N) \nsubseteq K$. If $a\psi(N) \subseteq aN$, then $aN \subseteq K$ implies that $a\psi(N) \subseteq K$, a contradiction. Thus by part (v), $a^n N = 0$ for some $n \in \mathbb{N}$ or $aN = N$. Therefore, $N \subseteq K$ or $a^n N = 0$ for some $n \in \mathbb{N}$, as needed. □

Example 4.20. Let N be a non-zero subhypermodule of M and let $\psi : S^h(M) \to S^h(M) \cup \{\emptyset\}$ be a function. If $\psi(N) = N$, then N is a ψ-secondary subhypermodule of M by Theorem 4.19 $(v) \Rightarrow (i)$.

Proposition 4.21. *Let M be an R-hypermodule and let N and K be weak secondary subhypermodules of M such that $N \cap K \neq 0$ and $r(N \cap K) = rN \cap rK$ for each $r \in R$, then $N \cap K$ is a weak secondary subhypermodule of M.*

Proof. Let $a \in R$ with $aM \nsubseteq a(N \cap K)$. If $aM \subseteq aN$ and $aM \subseteq aK$, then $aM \subseteq a(N \cap K)$, a contradiction. If $aM \nsubseteq aN$ and $aM \nsubseteq aK$, then by Theorem 4.19 $(i) \Rightarrow (v)$, $aN = N$ or $a^n N = 0$ for some $n \in \mathbb{N}$ and $aK = K$ or $a^m K = 0$ for some $m \in \mathbb{N}$. If $a^n N = 0$ or $a^m K = 0$, then $a^t(N \cap K) = 0$ for some $t \in \mathbb{N}$ and we are done. So let $aN = N$ and $aK = K$. Then $a(N \cap K) = N \cap K$. Finally, if $aM \nsubseteq aN$, $aM \subseteq aK$, and $aN = N$, then $aN \subseteq aM \subseteq aK$. Hence $N \cap K \subseteq N = aN = aN \cap aK = a(N \cap K)$. It follows that $a(N \cap K) = N \cap K$, as needed. □

Acknowledgments

The authors would like to thank the referee for careful reading.

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