Journal of Algebra and Related Topics
Vol. 9, No 1, (2021), pp 131-141

ON PRIME AND SEMIPRIME IDEALS OF Γ-SEMIHYPERRINGS

J.J. PATIL * AND K.F. PAWAR

ABSTRACT. The Γ-semihyperring is a generalization of the concepts of a semiring, a semihyperring and a Γ-semiring. In this paper, the notions of completely prime ideals and prime radicals for Γ-semihyperring are introduced and studied some important properties accordingly. We also introduced the notions of m-system and complete m-system. Then characterizations of prime ideals and completely prime ideals of Γ-semihyperring with the help of m-system and complete m-system has been taken into account. It is our attempt to find a bridge between semiprime (completely semiprime) ideals and prime (completely prime) ideals of a Γ-semihyperring.

1. Introduction and preliminaries

The notion of hypergroup was introduced by Marty [7] in 1934. Later many authors studied algebraic hyperstructure which are generalization of classical algebraic structure. In classical algebraic structure the composition of two elements is an element while in an algebraic hyperstructure composition of two elements is a set. Let H be a non-empty set then, the map $\circ : H \times H \to \wp^*(H)$ is called a hyperopertion where $\wp^*(H)$ is the family of all non-empty subsets of H and the couple $(H, \circ)$ is called a hypergroupoid. Moreover, the couple $(H, \circ)$ is called a semihypergroup if for every $a, b, c \in H$ we have $(a \circ b) \circ c = a \circ (b \circ c)$. The notion of Γ-semihyperring as a generalization of semiring, semihyperring and Γ-semiring was introduced by Dehkordi and Davvaz [2]. In

MSC(2010): Primary: 16Y99; Secondary: 20N20
Keywords: m-system, complete m-system, additively idempotent Γ-semihyperring.
Received: 3 September 2020, Accepted: 30 May 2021.
$*$Corresponding author .

[12], Pawar et al. introduced the notion of regular (strongly regular) Γ-semihyperrings and gave it's characterization with the help of ideals of Γ-semihyperrings. In [6], Kim et al. introduced the notion of hyper R-subgroup of a hypernear-ring and investigate some properties of hypernear-ring with respect to the hyper R-subgroup.

The concept of prime ideal is originated from prime numbers of the integers and it plays very important role in different algebraic structures like ring theory, semiring theory, semigroup theory etc. The relation between semiprime ideals and prime ideals in a (non-commutative) semigroups is established by Park and Kim [8]. In [4], Iseki provided number of results characterizing prime ideals of semiring. The notion of a Γ-semiring was introduced by Rao, [14]. In [3], Dutta and Sardar introduced the notion of prime ideals and prime radicals of Γ-semiring and study via it's operator semirings. In [5], Kim introduced the notion of prime and semiprime ideal and made their characterization in Γ-seminear-ring. In Γ-semihypergroup c-system, n-system, m-system and complete prime Γ-radical are introduced and studied by Pawar and Safoora [13].

The hyperstructure theory has vast applications in various streams of sciences. Our main aim to introduce and to study briefly the concepts of classical algebraic structure in hyperstructure theory. In this paper, we introduced the notions of prime (completely prime) ideals, m-system (complete m-system) for Γ-semihyperring. Also, prime (completely prime) ideals of Γ-semihyperring has characterized with the help of m-system (complete m-system). We tried to connect semiprime (completely semiprime) ideals and prime (completely prime) ideals of a Γ-semihyperring.

Here are some useful definitions which are taken from [2].

Definition 1.1. Let R be a commutative semihypergroup and Γ be a commutative group. Then R is called a Γ-semihyperring if there is a map $R \times \Gamma \times R \to \wp^*(R)$ (images to be denoted by $a\alpha b$, for all $a, b \in R$ and $\alpha \in \Gamma$) and $\wp^*(R)$ is the set of all non-empty subsets of R satisfying the following conditions:

(1) $a\alpha(b + c) = a\alpha b + a\alpha c$
(2) $(a + b)\alpha c = a\alpha c + b\alpha c$
(3) $a(\alpha + \beta)c = a\alpha c + a\beta c$
(4) $a\alpha(b\beta c) = (a\alpha b)\beta c$, for all $a, b, c \in R$ and for all $\alpha, \beta \in \Gamma$.

Example 1.2. [9] Let $R = \mathbb{Q}, \Gamma = \{\gamma_\alpha | \alpha \in \mathbb{N}\}$ and $A_\alpha = \alpha\mathbb{Z}^+$, we define $x\alpha y \to xA_\alpha y, \alpha \in \Gamma$ and $x, y \in R$. Then R is a Γ-semihyperring under ordinary addition and multiplication.

Definition 1.3. A Γ-semihyperring R is said to be commutative if $a\alpha b = b\alpha a$, for all $a, b \in R$ and $\alpha \in \Gamma$.

In the Example 1.2, R is a commutative Γ-semihyperring.

Definition 1.4. A Γ-semihyperring R is said to be with zero if there exists $0 \in R$ such that $a \in a + 0$ and $0 \in 0\alpha a, 0 \in a\alpha 0$, for all $a \in R$ and $\alpha \in \Gamma$.

In the Example 1.2, R is a Γ-semihyperring with zero.

Let A and B be two non-empty subsets of a Γ-semihyperring R and $x \in R$, then

$$A + B = \{x \mid x \in a + b, a \in A, b \in B\}$$
$$A\Gamma B = \{x \mid x \in a\alpha b, a \in A, b \in B, \alpha \in \Gamma\}.$$

Definition 1.5. A non empty subset R_1 of Γ-semihyperring R is called a Γ-subsemihyperring if it is closed with respect to the multiplication and addition, that is $R_1 + R_1 \subseteq R_1$ and $R_1\Gamma R_1 \subseteq R_1$.

Example 1.6. [12]

Let $R = \{a, b, c, d\}$ then R is commutative semihypergroup with following hyperoperations

$+$	a	b	c	d
a	$\{a\}$	$\{a,b\}$	$\{a,c\}$	$\{a,d\}$
b	$\{a,b\}$	$\{b\}$	$\{b,c\}$	$\{b,d\}$
c	$\{a,c\}$	$\{b,c\}$	$\{c\}$	$\{c,d\}$
d	$\{a,d\}$	$\{b,d\}$	$\{c,d\}$	$\{d\}$

$\cdot$	a	b	c	d
a	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c,d\}$
b	$\{a,b\}$	$\{b\}$	$\{b,c\}$	$\{b,c,d\}$
c	$\{a,b,c\}$	$\{b,c\}$	$\{c\}$	$\{c,d\}$
d	$\{a,b,c,d\}$	$\{b,c,d\}$	$\{c,d\}$	$\{d\}$

Here $\{a, b\}$ is a Γ-subsemihyperring of R.

Definition 1.7. A right (left) ideal I of a Γ-semihyperring R is an additive subsemihypergroup of $(R, +)$ such that $I\Gamma R \subseteq I (R\Gamma I \subseteq I)$. If I is both right and left ideal of R then we say that I is a two sided ideal or simply an ideal of R.

Example 1.8. [1] Let $R = \{a, b, c, d\}, \Gamma = \mathbb{Z}_2$ and $\alpha = \bar{0}, \beta = \bar{1}$. Then R is Γ-semihyperring with the following hyperoperations

$+$	a	b	c	d
a	$\{a, b\}$	$\{a, b\}$	$\{c, d\}$	$\{c, d\}$
b	$\{a, b\}$	$\{a, b\}$	$\{c, d\}$	$\{c, d\}$
c	$\{c, d\}$	$\{c, d\}$	$\{a, b\}$	$\{a, b\}$
d	$\{c, d\}$	$\{c, d\}$	$\{c, d\}$	$\{a, b\}$

β	a	b	c	d
a	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$
b	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$
c	$\{a, b\}$	$\{a, b\}$	$\{c, d\}$	$\{c, d\}$
d	$\{a, b\}$	$\{a, b\}$	$\{c, d\}$	$\{c, d\}$

For any $x, y \in R$ we define $x\alpha y = \{a, b\}$. Here $\{a, b\}$ is a ideal of a Γ-semihyperring.

2. Prime ideals and semiprime ideals in Γ-semihyperrings

In this section, we proved some basic properties on prime ideals and completely prime ideals of Γ-semihyperring R. We also established relationship between (completely) semiprime and (completely) prime ideals of R.

Definition 2.1. A non-empty ideal P of a Γ-semihyperring R is said to be prime if $A\Gamma B \subseteq P$, then $A \subseteq P$ or $B \subseteq P$, for any ideal A, B of R.

Definition 2.2. A non-empty ideal P of a Γ-semihyperring R is said to be semiprime if $A\Gamma A \subseteq P$, then $A \subseteq P$, for any ideal A of R.

Definition 2.3. An ideal P of a Γ-semihyperring R is said to be completely prime if $a\Gamma b \subseteq P$, then $a \in P$ or $b \in P$, for any $a, b \in R$.

Definition 2.4. An ideal P of a Γ-semihyperring R is said to be completely semiprime if $a\Gamma a \subseteq P$, then $a \in P$, for any $a \in R$.

Definition 2.5. A subset E of a Γ-semihyperring R is said to be an additively idempotent for any $a, b \in E$ implies that $a + b \subseteq E$.

Definition 2.6. A Γ-semihyperring R is said to be a additively idempotent if all subsets of R are additively idempotent.

In the Example 1.6, Γ-semihyperring R is an additively idempotent.

It is easy to check that every completely prime ideal of a Γ-semihyperring R is a prime ideal of R and prime (completely prime) ideal of Γ-semihyperring R is a semiprime (completely semiprime) ideal of R. Also, throughout this paper we consider Γ-semihyperring R is an additively idempotent.

Definition 2.7. An ideal generated by subset A of a Γ-semihyperring R is the smallest ideal of R containing A. It is denoted by $< A >$.

Lemma 2.8. *Let R be a Γ-semihyperring. Then for any subset A of R, $< A >= A \cup A\Gamma R \cup R\Gamma A \cup R\Gamma A\Gamma R$.*

For any element $a \in R$, $< a >= a \cup a\Gamma R \cup R\Gamma a \cup R\Gamma a\Gamma R$ is the smallest ideal of R containing a. Also, we can observe $< a >_r= a\cup a\Gamma R$ ($< a >_l= a\cup R\Gamma a$) is the smallest right (left) ideal of R that containing a.

Lemma 2.9. *Let R be a Γ-semihyperring. Then the following conditions are equivalent:*

(1) *P is a prime ideal of a Γ-semihyperring R.*
(2) *$a\Gamma R\Gamma b \subseteq P$ implies that $a \in P$ or $b \in P$, for any $a, b \in R$.*
(3) *If $< a > \Gamma < b >\subseteq P$, then $a \in P$ or $b \in P$.*
(4) *If A and B are right ideals in R such that $A\Gamma B \subseteq P$, then either $A \subseteq P$ or $B \subseteq P$.*
(5) *If A and B are left ideals in R such that $A\Gamma B \subseteq P$, then either $A \subseteq P$ or $B \subseteq P$.*

Proof. (1) $\Rightarrow$ (2)
Let $a\Gamma R\Gamma b \subseteq P$. Then $(R\Gamma a\Gamma R)\Gamma(R\Gamma b\Gamma R) \subseteq P$ and since $R\Gamma a\Gamma R$ and $R\Gamma a\Gamma R$ are ideals of R, by (1) we have $R\Gamma a\Gamma R \subseteq P$ or $R\Gamma b\Gamma R \subseteq P$. Assume that $R\Gamma a\Gamma R \subseteq P$. Consider $< a >= a \cup a\Gamma R \cup R\Gamma a \cup R\Gamma a\Gamma R$ which is the smallest ideal of a Γ-semihyperring R containing an element a. Now,

$$\begin{aligned} < a >^3 &=< a > \Gamma < a > \Gamma < a > \\ &\subseteq R\Gamma a\Gamma R \\ &\subseteq P. \end{aligned}$$

So by (1) we get, $< a >\subseteq P$ and so $a \in P$. Similarly if $R\Gamma b\Gamma R \subseteq P$, we have $b \in P$.
(2) $\Rightarrow$ (3)
Let $a, b \in R$ such that $< a > \Gamma < b >\subseteq P$. Then $a\Gamma R\Gamma b \subseteq< a > \Gamma < b >\subseteq P$. Therefore, by (2) either $a \in P$ or $b \in P$.
(3) $\Rightarrow$ (4)
Suppose that A and B are right ideals of R and $A\Gamma B \subseteq P$. Let $A \nsubseteq P$.

Then there exists $a \in A$ but $a \notin P$. If $b \in B$, then $< a > \Gamma < b > \subseteq A\Gamma B \cup R\Gamma A\Gamma B \subseteq P$. So by (3), $b \in P$. Thus we get, $B \subseteq P$.
(4) $\Rightarrow$ (5)
Suppose that A and B are left ideals in R and $A\Gamma B \subseteq P$. Let $A \nsubseteq P$. Then there exists an element $a \in A \setminus P$. If $b \in B$, then $< a > \Gamma < b > \subseteq A\Gamma B \cup A\Gamma B\Gamma R \subseteq P$. But $< a >$ and $< b >$ are right ideals also. So by (4), $b \in P$. Therefore we get, $B \subseteq P$.
(5) $\Rightarrow$ (1)
Let A, B be ideals of R and $A\Gamma B \subseteq P$. Since A, B are left ideal also. So by (5), either $A \subseteq P$ or $B \subseteq P$. Thus we get, P is a prime ideal of R. □

Lemma 2.10. *Let R be a Γ-semihyperring. Then the following conditions are equivalent:*

(1) *P is a semiprime ideal of a Γ-semihyperring R.*
(2) *$a\Gamma R\Gamma a \subseteq P$ implies that $a \in P$, for any $a \in R$.*

Proof. The proof is on the line of lemma 2.9. □

Lemma 2.11. *Let P_i be any sets of a prime ideals of a Γ-semihyperring R $(i \in I)$ and $P = \cap\{P_i \mid i \in I\}$ is non-empty. Then P is a semiprime ideal of R.*

Proof. Let A be an ideal of a Γ-semihyperring R and $A\Gamma A \subseteq P$. Then for any $i \in I$, $A\Gamma A \subseteq P_i$. Since every prime ideal is semiprime, $A \subseteq P_i$, for any $i \in I$. Hence $A \subseteq P = \cap\{P_i \mid i \in I\}$. So P is a semiprime ideal of R. □

Definition 2.12. A subset M of a Γ-semihyperring R is said to be an m-system if $a, b \in M$, then $a\alpha x\beta b \cap M \neq \phi$, for some $\alpha, \beta \in \Gamma$ and some $x \in R$.

Example 2.13. [10] Let X be a non-empty set and τ is a topology on X. We define the hyperoperation of the addition and the multiplication on τ as $A, B \in \tau, A + B = A \cup B, A \cdot B = A \cap B$. Then τ is a Γ-semihyperring, where Γ is a commutative group, if we define $x\alpha y \to x \cdot y$ for every $x, y \in \tau, \alpha \in \Gamma$.
Consider τ as usual topology on set of real $\mathbb{R}$. Then $M = \{(-a, a) \mid a \in \mathbb{R}$ and $a > 0\}$ is the m-system of a Γ-semihyperring τ.

Definition 2.14. A subset M of a Γ-semihyperring R is said to be a complete m-system if $a, b \in M$, then $a\alpha b \cap M \neq \phi$, for some $\alpha \in \Gamma$.

Lemma 2.15. *An ideal P of a Γ-semihyperring R is a prime ideal if and only if the compliment of P in R is an m-system.*

Proof. The lemma 2.9 is a generalization of given lemma. □

Lemma 2.16. *An ideal P of a Γ-semihyperring R is a completely prime ideal if and only if the compliment of P in R is a complete m-system.*

Proof. Let P be a completely prime ideal of a Γ-semihyperring R and $a, b \in R \setminus P$. Suppose that $a\Gamma b \subseteq P$ then by hypothesis either $a \in P$ or $b \in P$. A contradiction to our supposition, so there exists $\alpha \in \Gamma$ such that $a\alpha b \cap (R\setminus P) \neq \phi$ i.e. $R\setminus P$ is a complete m-system. Conversely, let $R\setminus P$ be a complete m-system where P is an ideal of a Γ-semihyperring R and $a\Gamma b \subseteq P$. Suppose that $a, b \in R \setminus P$. Since $R \setminus P$ is a complete m-system so there exists $\alpha \in \Gamma$ such that $a\alpha b \cap (R \setminus P) \neq \phi$. A contradiction to our supposition, thus either $a \in P$ or $b \in P$. □

The following two lemmas are very important showing exitance of an m-system (a complete m-system) M for any $a \in R \setminus P$, where P is a semiprime (completely semiprime) Γ-semihyperring such that a belongs to M but does not contain any element of P.

Lemma 2.17. *Let P be a semiprime ideal of a Γ-semihyperring R. Then for any $a \in R \setminus P$ there exists an m-system of R, say M such that $a \in M$ and $M \cap P = \phi$.*

Proof. Let $a \notin P$. Choose sets $M_1, M_2, M_3, ...$ inductively as follows: $M_1 = \{a\}$. Since $a \notin P$ and P is a semiprime ideal of R then by the Lemma 2.10, $a\alpha r\beta a \cap P^c \neq \phi$, for some $\alpha, \beta \in \Gamma$ and for some $r \in R$. Let $M_2 = a\alpha r\beta a \cap P^c$. For any $a_{2_i} \in M_2 \subseteq P^c$ then again by the Lemma 2.10, $a_{2_i}\alpha_{2_i}r_{2_i}\beta_{2_i}a_{2_i} \cap P^c \neq \phi$, for some $\alpha_{2_i}, \beta_{2_i} \in \Gamma$ and some $r_{2_i} \in R$. Now, let $M_{2_i} = a_{2_i}\alpha_{2_i}r_{2_i}\beta_{2_i}a_{2_i} \cap P^c$ and $M_3 = \cup_{i\in I} M_{2_i}$. In similar manner, we can define sets $M_4, M_5, ...$ Let $M = \cup_{i\in i} M_i$. Then clearly $M \cap P = \phi$. Suppose that $a, b \in M$, then $a \in M_i, b \in M_j$ for some $i, j \in I$. For the convenience, let us assume that i is less than or equals to j. Then $a_i\Gamma R\Gamma aj \cap M_{j+1} \neq \phi$. So we get, M is an m-system and $M \cap P = \phi$ and $a \in M$. □

Lemma 2.18. *Let P be a completely semiprime ideal of a Γ-semihyperring R. Then for any $a \in R \setminus P$ there exists a complete m-system of R, say M such that $a \in M$ and $M \cap P = \phi$.*

Proof. Let $a \notin P$. Choose sets $M_1, M_2, M_3, ...$ inductively as follows: $M_1 = \{a\}$. Since $a \notin P$ and P is a completely semiprime ideal of R then $a\alpha a \cap P^c \neq \phi$, for some $\alpha \in \Gamma$. Let $M_2 = a\alpha a \cap P^c$. For any $a_{2_i} \in M_2 \subseteq P^c$ then $a_{2_i}\alpha_{2_i}a_{2_i} \cap P^c \neq \phi$, for some $\alpha_{2_i} \in \Gamma$. Now, let $M_{2_i} = a_{2_i}\alpha_{2_i}a_{2_i} \cap P^c$ and $M_3 = \cup_{i\in I} M_{2_i}$. In similar manner we can, define sets $M_4, M_5, ...$ Let $M = \cup_{i\in i} M_i$. Then clearly $M \cap P = \phi$. Suppose that $a, b \in M$, then $a \in M_i, b \in M_j$ for some $i, j \in I$. For the convenience, let us assume that i is less than or equals to j. Then $a_i\Gamma a_j \cap M_{j+1} \neq \phi$. So we get, M is a complete m-system and $M \cap P = \phi$ and $a \in M$. $\square$

Definition 2.19. [11] A Γ-semihyperring R is said to be a duo if every one sided ideal of R is a two sided ideal.

In the Example 1.8, Γ-semihyperring R is a duo.

The relationship between semiprime (completely semiprime) ideals is established with prime ideals (completely prime ideals) in Γ-semihyperring R with the help of a following theorems.

Theorem 2.20. *Every semiprime ideal of a additively idempotent Γ-semihyperring R is an intersection of some prime ideals of R.*

Proof. Let P be a semiprime ideal of a additively idempotent Γ-semihyperring R and $\{Q_i \mid i \in I\}$ be the set of all prime ideals of R containing P. Then this set is non-empty because R itself is a prime ideal of R. Let $a \notin P$. Then by the Lemma 2.17, there exists an m-system M such that $M \cap P = \phi$ and $a \in M$. Now, consider the set of all m-systems M of R such that $a \in M$ and $M \cap P = \phi$. Let $T = \{M \mid M \text{ is an } m\text{-system of } R \text{ and } a \in M, M \cap P = \phi\}$. Then T is a non-empty. By the Zorn's Lemma, there exists a maximal element in T, say M'. Also, let $X = \{J \mid J \text{ is an ideal of } R \text{ and } J \cap M' = \phi, J \subseteq P\}$. Then X is non-empty since P is in X. If we apply the Zorn's Lemma on set X, then there exists a maximal element in X, say Q. If $x, y \in R \setminus Q$, then $(< x > \cup Q) \cap M' \neq \phi$ and $(< y > \cup Q) \cap M' \neq \phi$ since $(< x > \cup Q)$ and $(< y > \cup Q)$ are ideals of R properly containing Q. Where $< x >= x \cup x\Gamma R \cup R\Gamma x \cup x\Gamma R\Gamma x$ and $< y >= y \cup y\Gamma R \cup R\Gamma y \cup y\Gamma R\Gamma y$. Hence different cases arises out of which one case is there are some elements $s, t, u, v \in R, \alpha_1, \alpha_2, \beta_1, \beta_2 \in \Gamma$ such that $s\alpha_1 x\alpha_2 t \cap M' \neq \phi$ and $u\beta_1 y\beta_2 v \cap M' \neq \phi$. That is there are $m_1 \in s\alpha_1 x\alpha_2 t$ and $m_2 \in u\beta_1 y\beta_2 v$ such that $m_1, m_2 \in M'$. But M' is an m-system so there is some $r \in R, \alpha, \beta \in \Gamma$ such that $m_1\alpha r\beta m_2 \cap M' \neq \phi$. So we get, $(s\alpha_1 x\alpha_2 t)\alpha r\beta(u\beta_1 y\beta_2 v) \cap M' \neq \phi$. From the fact $(s\alpha_1 x\alpha_2 t)\alpha r\beta(u\beta_1 y\beta_2 v) \cap R \setminus Q \neq \phi$ that is $x\alpha_2 t\alpha r\beta u\beta_1 y \subseteq x\Gamma R\Gamma y \cap R \setminus Q \neq \phi$. Similarly, for the other cases we can easily show $x\Gamma R\Gamma y \cap R \setminus Q \neq \phi$. So we have, $R \setminus Q$ is an m-system i.e Q is a prime ideal of R containing P. From the maximality of M', $M' = R \setminus Q$. Since $a \notin P$, this means $\cap\{Q_i \mid i \in I\} \subseteq P$. Since the converse inclusion is obvious, we have that $P = \cap_{i\in I}\{Q_i \mid Q_i \text{ is an prime ideal of } R \text{ containing } P\}$. Thus completes the proof. $\square$

Corollary 2.21. *Any completely semiprime ideal of a additively idempotent* Γ*-semihyperring* R *is an intersections of prime ideals of* R.

Theorem 2.22. *Every completely semiprime ideal of a additively idempotent and duo* Γ*-semihyperring* R *is an intersection of some completely prime ideals of* R.

Proof. Let P be a completely semiprime ideal of a additively idempotent Γ-semihyperring R and $\{Q_i \mid i \in I\}$ be the set of all completely prime ideals of R containing P. Then this set is non-empty because R itself is a completely prime ideal of R. Let $a \notin P$. Then by the Lemma 2.18, there exists a complete m-system M such that $M \cap P = \phi$ and $a \in M$. Now, consider the set of all complete m-systems M of R such that $a \in M$ and $M \cap P = \phi$. Let $T = \{M \mid M \text{ is a complete } m\text{-system of } R \text{ and } a \in M, M \cap P = \phi\}$. Then T is a non-empty. By the Zorn's Lemma, there exists a maximal element in T, say $M^{'}$. Also, let $X = \{J \mid J \text{ is an ideal of } R \text{ and } J \cap M^{'} = \phi, J \subseteq P\}$. Then X is non-empty since P is in X. If we apply the Zorn's Lemma on set X, there exists a maximal element in X, say Q. If $x, y \in R \setminus Q$, then $(< x >_l \cup Q) \cap M^{'} \neq \phi$ and $(< y >_r \cup Q) \cap M^{'} \neq \phi$. Since $(< x >_l \cup Q)$ and $(< y >_r \cup Q)$ are ideals of R properly containing Q. Where $< x >_l = R\Gamma x \cup x$ and $< y >_r = y\Gamma R \cup y$. Hence some cases arises out of which one is there are some cases elements $s, t \in R, \alpha_1, \alpha_2 \in \Gamma$ such that $s\alpha_1 x \cap M^{'} \neq \phi$ and $y\alpha_2 t \cap M^{'} \neq \phi$. That is there are $m_1 \in s\alpha_1 x$ and $m_2 \in y\alpha_2 t$ such that $m_1, m_2 \in M^{'}$. But $M^{'}$ is a complete m-system so there is some $\alpha \in \Gamma$ such that $m_1 \alpha m_2 \cap M^{'} \neq \phi$. So we get, $(s\alpha_1 x)\alpha(y\alpha_2 t) \cap M^{'} \neq \phi$. From the fact $(s\alpha_1 x)\alpha(y\alpha_2 t) \cap R \setminus Q \neq \phi$ that is $x\alpha y \subseteq x\Gamma y \cap R \setminus Q \neq \phi$. Similarly, for the other cases we can easily show $x\Gamma y \cap R \setminus Q \neq \phi$. So we have, $R \setminus Q$ is a complete m-system i.e. Q is a completely prime ideal of R containing P. From the maximality of $M^{'}$, $M^{'} = R \setminus Q$. Since $a \notin P$, this means $\cap\{Q_i \mid i \in I\} \subseteq P$. Since the converse inclusion is obvious, we have that $P = \cap_{i \in I}\{Q_i \mid Q_i \text{ is a completely prime ideal of } R \text{ containing } P\}$. Hence completes the proof. □

Theorem 2.23. *If* I *is an ideal of a* Γ*-semihyperring* R *and* P *is prime ideal of* R*, then* $I \cap P$ *is a prime ideal of* I *consider* I *as a* Γ*-semihyperring.*

Definition 2.24. The prime radical of a Γ-semihyperring R defined as the intersection of all prime ideal of R and it is denoted by $P(R)$.

Theorem 2.25. *For a* Γ*-semihyperring* R, $P(R) = \{r \in R \mid \textit{every } m\textit{-system of } R \textit{ which contains } r \textit{ contains } 0\}$.

Proof. Let $P^{'} = \{r \in R \mid$ every m-system of R which contains r contains $0\}$. Let $y \notin P(R)$. Then $y \notin P$ for some prime ideal P of R. By the Lemma 2.15, P^c is an m-system of R. Since $0 \in P, 0 \notin P^c$. Then P^c is an m-system of R contains y but not contain 0. So $y \notin P^{'}$. Thus we get, $P^{'} \subseteq P(R)$. Now, let $y \notin P^{'}$. Then there is an m-system M of R such that $P^{'} \cap M = \phi$. So $y \notin P(R)$. Thus we get, $P(R) \subseteq P'$. Hence completes the proof. □

Theorem 2.26. *Let R be a Γ-semihyperring. If I is an ideal of R, then $P(I) = I \cap P(R)$, where $P(I)$ denotes the prime radical of I consider I as a Γ-semihyperring.*

Proof. Let Ω be the collection of all prime ideals of Γ-semihyperring R and Λ be the collection of all prime ideal of I. By the Theorem 2.23, $P \in \Omega$ implies that $P \cap I \in \Lambda$. So $P(I) = \cap_{P \in \Lambda} P \subseteq \cap_{Q \in \Omega}(I \cap Q) = I \cap (\cap_{Q \in \Omega} Q) = I \cap P(R)$. Let $a \notin P(I)$. Then by the Theorem 2.25, $0 \notin M$ for some m-system M of I containing a. Since M is also an m-system of R by the Theorem 2.25, $a \notin P(R)$. Therefore we get, $P(R) \subseteq P(I)$. So $I \cap P(R) \subseteq P(I)$. Thus we get, $P(I) = I \cap P(R)$. □

3. Acknowledgements

The authors express their gratitude to anonymous referees for useful suggestions that improved the present paper.

References

1. S. O. Dehkordi and B. Davvaz, *Ideal theory in* Γ*- semihyperrings*, Iranian Journal of Science and Technology A, (3) **37** (2013), 251-263.
2. S. O. Dehkordi and B. Davvaz, Γ*- semihyperrings: ideals, homomorphism and relation*, Afrika Mathematica, **26** (2015), 849-861.
3. T. Dutta and S. Sardar, *On prime ideals and prime radicals of a* Γ*-semiring* , An. Stiint. Univ. Al. I. Cuza Iasi, Mat.(NS), **46** (2000), 319-329.
4. K. Iseki, *Ideal theory of semiring*, Proc. Japan Acad., (8) **32** (1956), 554-559.
5. K. Kim, *On Prime and Semiprime Ideals In Gamma-Seminear-Rings*, Scientiae Mathematicae Japonicae, **4** (2001), 885-889.
6. K. Kim, B. Davvaz and E. Roh, *On Hyper R-subgroups of Hypernear-rings*, Scientiae Mathematicae Japonicae, (e-2007), 649-656.
7. F Marty, *Sur une generalization de la notion de groupe*, 8th congres Math. Scandinaves, Stockholm, (1934), 45-49.
8. Y. K. Park and J. P. Kim, *Prime and semiprime ideals in semigroups*, Kyungpook Math. J., **32** (1992), 629-633
9. J. Patil and K. Pawar, *On quasi-ideals of* Γ*-semihyperrings*, Journal of hyperstructure, **8(2)** (2019), 123-134.
10. J. Patil and K. Pawar, *On Uniformly Strongly Prime* Γ*-semihyperring*, Kragujevac Journal of Mathematics, **48(1)** (2024), 79-87.
11. J. Patil and K. Pawar, *On Duo* Γ*-semihyperrings*, Communicated.

12. K. Pawar,J. Patil and B. Davvaz, *On a regular* Γ- *semihyperrings and idempotent* Γ- *semihyperrings*, Kyungpook Math. J., **59** (2019), 35-45.
13. K. Pawar and S. Ansari, *On Systems, Maximal* Γ*-hyperideals and Complete Prime* Γ*-radical In* Γ*-semihypergroups*, Journal of hyperstructure, **8(2)** (2019), 135-149.
14. M. Rao, Γ*-semiring. I*, Sotheaseast Asian Bull. Math., (1) **19** (1995), 49-54.

J.J. Patil
Department of Mathematics, Indraraj Arts, Commerce and Science college, Sillod, Dist: Aurangabad,
Dr. Babashasheb Ambedkar Marathwada University,
Dist:Aurangabad- 431 112, India.
Email: jjpatil.777@rediffmail.com

K.F. Pawar
Department of Mathematics, School of Mathematical Sciences,
Kavayitri Bahinabai Chaudhari North Maharashtra University,
Jalgaon - 425 001, India.
Email: kfpawar@nmu.ac.in