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ON CP-FRAMES

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Abstract. Let $\mathcal{R}_c(L)$ be the pointfree version of $C_c(X)$, the subring of $C(X)$ whose elements have countable image. We shall call a frame L a CP-frame if the ring $\mathcal{R}_c(L)$ is regular. We give some characterizations of CP-frames and we show that L is a CP-frame if and only if each prime ideal of $\mathcal{R}_c(L)$ is an intersection of maximal ideals if and only if every ideal of $\mathcal{R}_c(L)$ is a z_c-ideal. In particular, we prove that any P-frame is a CP-frame but not conversely, in general. In addition, we study some results about CP-frames like the relation between a CP-frame L and ideals of closed quotients of L. Next, we characterize CP-frames as precisely those L for which every prime ideal in the ring $\mathcal{R}_c(L)$ is a z_c-ideal. Finally, we show that this characterization still holds if prime ideals are replaced by essential ideals, radical ideals, convex ideals, or absolutely convex ideals.

1. Introduction

Throughout this paper, all spaces are assumed to be completely regular and the term "ring" means commutative ring with identity. First, recall whenever any cozero set of a topological space X is closed then we call it a *P-space*. These spaces are studied and variously characterized in [18]; see also [21]. In 2013, $C_c(X)$, the largest subalgebra of $C(X)$ whose elements have countable image, is studied in [16]. This subalgebra has received some more attention, recently, for instance, you can see [3, 8, 12, 17, 23, 24, 27, 28]. In [16], the authors introduced the concept of CP-space as below: a topological space X is a CP-space if $C_c(X)$ is a regular ring. Also, the

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relation between P-spaces and CP-spaces is studied. In [25], the ring $\mathcal{R}_c(L)$ is introduced as the pointfree version of the ring $C_c(X)$.

A principal intention of this paper is to introduce countably P-frame (briefly CP-frame) L mentioned in the title in terms of some algebraic properties of the ring $\mathcal{R}_c L$. We introduce CP-frame as follows: a completely regular frame L is called CP-frame whenever $\mathcal{R}_c(L)$ is regular (Definition 3.1). We show that every P-frame is a CP-frame (Proposition 3.5), but not conversely, in general. In the final section, we provide alternative characterizations of CP-frames (Theorem 4.3). Moreover, we describe CP-frames L via ideals of closed quotients of L (see Proposition 4.7). In addition, we characterize CP-frames as precisely those L for which every prime ideal in the ring $\mathcal{R}_c L$ is a z_c-ideal. Finally, we show that this characterization still holds if prime ideals are replaced by essential ideals, radical ideals, convex ideals, or absolutely convex ideals (see Proposition 4.10 and Corollary 4.11).

2. Preliminaries

Our references for frames are [22] and [29]. All our frames are completely regular.

For a frame L, a pair (n, N) is said to be a *quotient* of L if $n : L \to N$ a surjective frame map where N is a frame. We often say that N is a quotient frame of L, for simplicity (see [29] for details). For any element $x \in L$, $\uparrow x = \{y \in L : x \leq y\}$ is a frame in the induced order of L. It is evident that the largest element of $\uparrow x$ is the same as that of L while the smallest element of $\uparrow x$ is x, and $\overline{x} : L \to \uparrow x$, given by $a \longmapsto x \vee a$, is a surjective frame map making $\uparrow x$ (or the pair $(\overline{x}, \uparrow x)$ a quotient of L and is called a *closed quotient* of L. We denote the family of all closed quotients of L by $CQ(L)$ and a non-empty subfamily $\mathcal{P}$ of $CQ(L)$ is called an *ideal of closed quotients of L* whenever it satisfies the next two properties:

- $(\overline{x}, \uparrow x) \leq (\overline{y}, \uparrow y)$ and $(\overline{y}, \uparrow y) \in \mathcal{P}$ imply $(\overline{x}, \uparrow x) \in \mathcal{P}$.
- if $(\overline{x}, \uparrow x)$ and $(\overline{y}, \uparrow y) \in \mathcal{P}$, then $(\overline{x}, \uparrow x) \vee (\overline{y}, \uparrow y) = (\overline{(x \wedge y)}, \uparrow (x \wedge y)) \in \mathcal{P}$.

You can check that the family of all closed quotients of L is an example of the ideals of closed quotients of L (for more details see [2, 5]).

We regard the *Stone-Čech compactification* of the frame L, which is denoted by βL, as the frame of completely regular ideals of L (for more details see [6, 7, 29]). We will use the notations of [6] for regarding the *frame of reals* $\mathcal{L}(\mathbb{R})$ and also the f-ring $\mathcal{R}(L)$ of all *continuous real-valued functions* on a frame L. For every $r \in \mathbb{R}$, define the constant frame map $\mathbf{r} \in \mathcal{R}(L)$ by $\mathbf{r}(p, q) = \top$, whenever $p < r < q$, and otherwise $\mathbf{r}(p, q) = \bot$. An element α of $\mathcal{R}(L)$ is said to be bounded if there exists a $n \in \mathbb{N}$ such that $\alpha(-n, n) = \top$.

The set of all bounded elements of $\mathcal{R}(L)$ is denoted by $\mathcal{R}^*(L)$ which is a sub-f-ring of $\mathcal{R}(L)$. It is also well known that $\mathcal{R}^*(L) \cong \mathcal{R}(\beta L)$, in connection with the Stone-Čech compactification of a frame L. For a beautiful account of the ring $\mathcal{R}(L)$ and the frame $\mathcal{L}(\mathbb{R})$, the reader is referred to [6] (see also [5]).

A *cozero element* of L is an element of the form $\mathrm{coz}(\alpha)$ for some $\alpha \in \mathcal{R}(L)$ (see [6]). We denote the cozero part of L by $\mathrm{Coz}[L]$.

For a frame L and $x \in L$, $x^* = \bigvee\{y \in L\colon y \wedge x = 0\}$ is the pseudocomplement of x. A frame L in which $c \vee c^* = \top$ for any $c \in \mathrm{Coz}[L]$ is called a P-frame. For more details about P-frames, we refer to [5] and [10].

For any pairs $(p, q) \in \mathbb{Q}^2$, put:

$$\langle p, q\rangle := \{\, x \in \mathbb{Q} : p < x < q \,\} \text{ and } \rrbracket p, q\llbracket := \{\, x \in \mathbb{R} : p < x < q \,\}.$$

For any topological space X, put $\mathfrak{O}X := \{O \subseteq X :\ O \text{ is an open subset of } X\}$. The homomorphism $\tau : \mathcal{L}(\mathbb{R}) \to \mathfrak{O}\mathbb{R}$ given by $(p, q) \mapsto \rrbracket p, q\llbracket$ is an isomorphism (see [6, Proposition 2]).

Recall from [25] that an element α of $\mathcal{R}(L)$ is said to be an *overlap* of a subset S of $\mathbb{R}$, denoted by $\alpha \blacktriangleleft S$, if $\tau(u) \cap S \subseteq \tau(v) \cap S$ implies $\alpha(u) \le \alpha(v)$, for every $u, v \in \mathcal{L}(\mathbb{R})$. Also, we recall from [25, Lemma 3.5] that for any $\alpha \in \mathcal{R}(L)$ and any $S \subseteq \mathbb{R}$, the following statements are equivalent:

(1) $\alpha \blacktriangleleft S$.
(2) $\tau(u) \cap S = \tau(v) \cap S$ implies $\alpha(u) = \alpha(v)$, for any $u, v \in \mathcal{L}(\mathbb{R})$.
(3) $\tau(p, q) \cap S = \tau(v) \cap S$ implies $\alpha(p, q) = \alpha(v)$, for any $v \in \mathcal{L}(\mathbb{R})$ and any $p, q \in \mathbb{Q}$.
(4) $\tau(p, q) \cap S \subseteq \tau(v) \cap S$ implies $\alpha(p, q) \le \alpha(v)$, for any $v \in \mathcal{L}(\mathbb{R})$ and any $p, q \in \mathbb{Q}$.

We say that $\alpha \in \mathcal{R}(L)$ has *pointfree countable image* if there exists a countable subset S of $\mathbb{R}$ such that $\alpha \blacktriangleleft S$. We write $\mathcal{R}_c(L)$ for the set of all $\alpha \in \mathcal{R}(L)$ such that α has the pointfree countable image (also, see [15]). It is shown in [25]) that for each completely regular frame L, the set $\mathcal{R}_c(L)$ is a sub-f-ring of $\mathcal{R}(L)$. An ideal I of $\mathcal{R}_c(L)$ is called a *z_c-ideal* if for every $\alpha, \beta \in \mathcal{R}_c(L)$, $\mathrm{coz}(\alpha) = \mathrm{coz}(\beta)$ with $\beta \in I$ imply $\alpha \in I$ (for more details, see [13]).

3. On regularity of $\mathcal{R}_c L$

In this section, we introduce CP-frames. First, we recall the following contents. Let A be a ring. A *Von Neumann inverse* (abbreviated VN-inverse) of $x \in A$ is an element y such that $x = xyx$. We call a ring A a *regular ring* whenever each its element has a VN-inverse. As we saw in [18, 4J], a topological X is a P-space if and only if $C(X)$ is a regular ring. Also, recall from [16] that a topological space X is said to be a CP-space if $C_c(X)$ is regular. Adapting this to frames, we offer the following definition.

Definition 3.1. A frame L is called a CP-frame if $\mathcal{R}_c(L)$ is regular.

We are going to show that every P-frame is a CP-frame. To do this, we use the next results. In [5], the authors have introduced an interesting function which is important for us. Now, let us talk about this function. Let a be a complemented element of L. Define the frame map $e_a : \mathcal{L}(\mathbb{R}) \to L$ by

$$e_a(p,q) = \begin{cases} \top & \text{if } p < 0 < 1 < q \\ a' & \text{if } p < 0 < q \leq 1 \\ a & \text{if } 0 \leq p < 1 < q \\ \bot & \text{otherwise,} \end{cases}$$

for each $p, q \in \mathbb{Q}$. Then $e_a \in \mathcal{R}(L)$, $\operatorname{coz}(e_a) = a$, and $\operatorname{coz}(1 - e_a) = a'$. Also, for each $\alpha \in \mathcal{R}(L)$, $\alpha e_a \in \mathcal{R}(L)$ and it is easy to check that

$$(\alpha e_a)(p,q) = \begin{cases} \alpha(p,q) \vee a' & \text{if } p < 0 < q \\ \alpha(p,q) \wedge a & \text{otherwise,} \end{cases}$$

for each $p, q \in \mathbb{Q}$.

We recall from [20, 29] that $\mathcal{L}(\mathbb{R})$ may be equivalently defined as the frame generated by the $(p, -)$ and $(-, q)$, where $p, q \in \mathbb{Q}$, subject to the relations

$(R'1)$ $(p, -) \wedge (-, q) = 0$ whenever $p \geq q$,
$(R'2)$ $(p, -) \vee (-, q) = \top$ whenever $p < q$,
$(R'3)$ $(p, -) = \bigvee_{r \in \mathbb{Q}, r > p} p(r, -)$,
$(R'4)$ $(-, q) = \bigvee_{s \in \mathbb{Q}, s < q} q(-, s)$,
$(R'5)$ $\bigvee_{p \in \mathbb{Q}} (p, -) = \top$ and
$(R'6)$ $\bigvee_{q \in \mathbb{Q}} (-, q) = \top$.

It is well known that a map α from the subbase of $\mathcal{L}(\mathbb{R})$ into a frame L defines a frame homomorphism $\mathcal{L}(\mathbb{R}) \to L$ if and only if it sends the relations $(R'1)$-$(R'6)$ above to identities in L.

Proposition 3.2. *Let $\alpha \in \mathcal{R}_c(L)$ such that $\operatorname{coz}(\alpha) \vee (\operatorname{coz}(\alpha))^* = \top$. Then $\iota_\alpha \colon \mathcal{L}(\mathbb{R}) \to L$ given by*

$$\iota_\alpha(p,q) = \begin{cases} (\operatorname{coz}(\alpha))^* \vee \alpha((-, \frac{1}{p}) \vee (\frac{1}{q}, -)) & if\, p < 0 < q \\ \alpha(\frac{1}{q}, \frac{1}{p}) & if\, p < q \leq 0 \text{ or } 0 \leq p < q \end{cases}$$

for each $p, q \in \mathbb{Q}$, determines a real-valued continuous functions in $\mathcal{R}_c(L)$.

Proof. First, it should be pointed out that when $p < q \leq 0$ if $q = 0$, then we have $\iota_\alpha(p,q) = \alpha(-,\frac{1}{p})$ and also when $0 \leq p < q$ if $p = 0$, then we have $\iota_\alpha(p,q) = \alpha(\frac{1}{q},-)$. Since $\mathrm{coz}(\alpha) \vee (\mathrm{coz}(\alpha))^* = \top$, Lemma 4.9 in [1] tells us that $\iota_\alpha \in \mathcal{R}(L)$. So, we need only show that ι_α is an overlap of a countable subset T of $\mathbb{R}$. Since $\alpha \in \mathcal{R}_c(L)$, we can choose a countable subset S of $\mathbb{R}$ such that $\alpha \blacktriangleleft S$. Putting $T = \{\frac{1}{x} : x \in S, x \neq 0\} \cup \{0\}$, then it is routine to show that $\iota_\alpha \blacktriangleleft T$. □

The foregoing proposition tells us that $\iota_\alpha \in \mathcal{R}_c(L)$, and $\mathrm{coz}(\iota_\alpha) = \mathrm{coz}(\alpha)$ when $\alpha \in \mathcal{R}_c(L)$ and $\mathrm{coz}(\alpha) \vee (\mathrm{coz}(\alpha))^* = \top$. This immediately leads to:

Corollary 3.3. *If $a \in L$ such that $a \vee a^* = \top$, then $a, a^* \in \mathrm{Coz}_c[L]$.*

Proposition 3.4. *Let L be a P-frame. For every $\alpha \in \mathcal{R}_c(L)$, $\iota_\alpha\alpha = e_{\mathrm{coz}(\alpha)}$, where ι_α is the real-valued continuous function in the Proposition 3.2.*

Proof. Using the fact that if L is a regular frame and $f, g \colon L \to M$ are frame homomorphisms such that $f(x) \leq g(x)$ for every $x \in L$, then $f = g$, it is enough to show that, for each $p, q \in Q$, $\alpha\iota_\alpha(p,q) \leq e_{\mathrm{coz}(\alpha)}(p,q)$ since $\mathcal{L}(\mathbb{R})$ is a regular frame. Let $p, q \in \mathbb{Q}$. We consider four cases.

Case 1: Assume $0, 1 \notin \tau(p,q)$. If there exist $u, v, w, z \in \mathbb{Q}$ such that $\langle u,v\rangle\langle w,z\rangle \subseteq \langle p,q\rangle$, then $(\frac{1}{v},\frac{1}{u}) \wedge (w,z) = \bot$, which implies that

$$\begin{aligned}\iota_\alpha\alpha(p,q) &= \bigvee\Big\{\iota_\alpha(u,v) \wedge \alpha(w,z) : \langle u,v\rangle\langle w,z\rangle \subseteq \langle p,q\rangle\Big\}\\ &= \alpha\left(\bigvee\Big\{(\frac{1}{v},\frac{1}{u}) \wedge (w,z) : \langle u,v\rangle\langle w,z\rangle \subseteq \langle p,q\rangle\Big\}\right)\\ &= \bot\\ &= e_{\mathrm{coz}(\alpha)}(p,q).\end{aligned}$$

Case 2: Assume $0, 1 \in \tau(p,q)$. Since $e_{\mathrm{coz}(\alpha)}(p,q) = \top$, we have

$$\alpha\iota_\alpha(p,q) \leq e_{\mathrm{coz}(\alpha)}(p,q).$$

Case 3: Assume $0 \notin \tau(p,q)$ and $1 \in \tau(p,q)$. Then

$$\alpha\iota_\alpha(p,q) \leq \mathrm{coz}(\alpha\iota_\alpha) = \mathrm{coz}(\alpha) = e_{\mathrm{coz}(\alpha)}(p,q).$$

Case 4: Assume $0 \in \tau(p,q)$ and $1 \notin \tau(p,q)$. Then

$$\begin{aligned}\alpha\iota_\alpha(p,q) \wedge \mathrm{coz}(\alpha) &= \alpha\iota_\alpha(p,q) \wedge \mathrm{coz}(\alpha\iota_\alpha)\\ &= \alpha\iota_\alpha(p,q) \wedge \alpha\iota_\alpha((-,0) \vee (0,-))\\ &= \alpha\iota_\alpha\Big((p,q) \wedge ((-,0) \vee (0,-))\Big)\\ &= \alpha\iota_\alpha((p,0) \vee (0,q)).\end{aligned}$$

Now, similar to the case 1, we can conclude that $\alpha\iota_\alpha(p,q) \wedge \mathrm{coz}(\alpha) = \bot$, which implies $\alpha\iota_\alpha(p,q) \leq (\mathrm{coz}(\alpha))^* = e_{\mathrm{coz}(\alpha)}(p,q)$. □

Proposition 3.5. *Every P-frame is a CP-frame.*

Proof. Let L be a P-frame. We are going to show that $\mathcal{R}_c(L)$ is regular. Consider $\alpha \in \mathcal{R}_c(L)$. Since $\operatorname{coz}(\alpha) \vee (\operatorname{coz}(\alpha))^* = \top$, Proposition 3.2 says $\iota_\alpha \in \mathcal{R}_c(L)$, and hence, by the foregoing proposition, we have $\alpha\iota_\alpha = e_{\operatorname{coz}(\alpha)}$, which implies $\alpha^2\iota_\alpha = \alpha e_{\operatorname{coz}(\alpha)}$. Since

$$\operatorname{coz}\left(\alpha(1 - e_{\operatorname{coz}(\alpha)})\right) = \operatorname{coz}(\alpha) \wedge \operatorname{coz}(1 - e_{\operatorname{coz}(\alpha)}) = \operatorname{coz}(\alpha) \wedge (\operatorname{coz}(\alpha))^* = \bot,$$

we deduce $\alpha = \alpha e_{\operatorname{coz}(\alpha)}$. Thus, $\alpha^2\iota_\alpha = \alpha$. This shows $\mathcal{R}_c(L)$ is regular. □

We recall from [16] an example of a CP-space which is not a P-space below.

Example 3.6. Let $X = [0, 1] \cup \mathbb{N}$ be the free sum of $[0, 1]$ and the natural numbers $\mathbb{N}$. Clearly, X is a CP-space which is not a P-space.

It is evident that a topological space X is a P-space if and only if the frame $\mathfrak{O}X$ is a P-frame. It is shown in [13] that for every frame L, there is a topological space X such that $\beta L \cong \mathfrak{O}X$ and $C_c(X) \cong \mathcal{R}_c(\mathfrak{O}X) \cong \mathcal{R}_c(\beta L) \cong \mathcal{R}_c^*(L)$. Hence, a topological space X is a CP-space if and only if the frame $\mathfrak{O}X$ is a CP-frame. Thus, by this fact and by Proposition 3.5, if X is a P-space then the frame $\mathfrak{O}X$ is a CP-frame. Therefore, in the above example, $\mathfrak{O}X$ is a CP-frame which is not a P-frame.

4. Some characterizations of CP-frames

In this section, we give some characterizations of CP-frames. We begin with the following lemma.

Lemma 4.1. *Let L be a frame and A be a regular subring of $\mathcal{R}(L)$. Then for every $\alpha \in A$,* $\operatorname{coz}(\alpha)$ *is complemented in* $\operatorname{Coz}[L]$.

Proof. Consider $\alpha \in A$. Since A is regular, we infer that there exists an element β in A such that $\alpha = \alpha^2\beta$. So

$$\operatorname{coz}(\alpha) = \operatorname{coz}(\alpha^2\beta) = \operatorname{coz}(\alpha^2) \wedge \operatorname{coz}(\beta) = \operatorname{coz}(\alpha) \wedge \operatorname{coz}(\beta) = \operatorname{coz}(\alpha\beta).$$

Since $\alpha\beta$ an idempotent element of $\mathcal{R}L$, we conclude that $\operatorname{coz}(\alpha)$ is complemented. □

Recall from [13] that for a frame L, $\operatorname{Coz}_c[L] = \{\operatorname{coz}(\alpha) : \alpha \in \mathcal{R}_c(L)\}$.

Proposition 4.2. *A frame L is a CP-frame if and only if every* $\operatorname{coz}(\alpha) \in \operatorname{Coz}_c[L]$ *is complemented in* $\operatorname{Coz}_c[L]$.

Proof. Let L be a CP-frame. Then $\mathcal{R}_c(L)$ is regular and by Lemma 4.1, we are done.

Conversely, assume the lattice $\operatorname{Coz}_c[L]$ is complemented and let $\alpha \in \mathcal{R}_c(L)$. Since $\operatorname{coz}(\alpha) \vee (\operatorname{coz}(\alpha))^* = \top$, Proposition 3.2 says $\iota_\alpha \in \mathcal{R}_c(L)$.

Now, similar to Proposition 3.5, we would have $\alpha^2\iota_\alpha = \alpha$. Therefore $\mathcal{R}_c(L)$ is regular, that is, L is a CP-frame. □

We recall the notation z-ideal of a ring A as was introduced by Mason in [26]. We refer to z-ideals as defined in [26] as "z-ideals in the sense of Mason". Denoted by $\mathrm{Id}(A)$ and $\mathrm{Max}(A)$ the family of all ideals and the family of all maximal ideals of a ring A, respectively. For $S \subseteq A$, we denote the ideal generated by S with $\langle S \rangle$. For $a \in A$ and $S \subseteq A$, let

$$\mathfrak{M}(a) = \{M \in \mathrm{Max}(A) : a \in M\} \quad \text{and} \quad \mathfrak{M}(S) = \{M \in \mathrm{Max}(A) : M \supseteq S\}.$$

From [26], we recall that an ideal I of a ring A is said to be a *z-ideal in the sense of Mason* if whenever $\mathfrak{M}(x) \supseteq \mathfrak{M}(y)$ and $y \in I$ imply $x \in I$. In [10, Corollary 3.8], Dube shows that an ideal of $\mathcal{R}(L)$ is a z-ideal if and only if it is a z-ideal in the sense of Mason (also see [14]). In [13], the concept of z_c-ideals is introduced. An ideal I in $\mathcal{R}_c(L)$ is a z_c-ideal, if for every $\alpha \in \mathcal{R}_c(L)$, $\mathrm{coz}(\alpha) \in \mathrm{Coz}[I]$ implies $\alpha \in I$. It is shown that every maximal ideal of $\mathcal{R}_c(L)$, is a z_c-ideal and every prime ideal of $\mathcal{R}_c(L)$ is contained in a unique maximal ideal. In particular, it is shown that an ideal I in $\mathcal{R}_c(L)$ is a z_c-ideal if and only if it is a z-ideal in the sense of Mason (see [13]).

We can now characterize CP-frames.

By [26] and [19], if we borrow the proof of [9, Proposition 3.9], word-for-word, we obtain the same result for $\mathcal{R}_c(L)$ in the following theorem.

Theorem 4.3. *The following statements are equivalent for a frame L.*

(1) *L is a CP-frame.*
(2) *Each ideal of $\mathcal{R}_c(L)$ is z_c-ideal.*
(3) *Each ideal of $\mathcal{R}_c(L)$ is an intersection of a set of prime ideals.*
(4) *Each ideal of $\mathcal{R}_c(L)$ is an intersection of a set of maximal ideals.*
(5) *Each prime ideal of $\mathcal{R}_c(L)$ is an intersection of a set of maximal ideals.*
(6) *For each $\alpha, \beta \in \mathcal{R}_c(L)$, $\langle \alpha, \beta \rangle = \langle \alpha^2 + \beta^2 \rangle$.*
(7) *Each principal ideal of $\mathcal{R}_c(L)$ is generated by an idempotent.*
(8) *Each prime ideal of $\mathcal{R}_c(L)$ is maximal.*
(9) *For each $\alpha \in \mathcal{R}_c(L)$, $\mathrm{coz}(\alpha) \vee \mathrm{coz}(\alpha)^* = \top$.*

It is shown in [13] that an ideal J in $\mathcal{R}_c(L)$ is a z_c-ideal if and only if it is a contraction of a z-ideal in $\mathcal{R}(L)$, and also that every maximal ideal of $\mathcal{R}_c(L)$ is a contraction of a maximal ideal in $\mathcal{R}(L)$. Therefore, by these results and Theorem 4.3, we deduce the following:

Corollary 4.4. *For any P-frame L, the following statements hold.*

(1) *For every ideal I of $\mathcal{R}_c(L)$, there exists a z-ideal J of $\mathcal{R}L$ with the property that $I = J \cap \mathcal{R}_c(L)$.*

(2) *For every prime ideal P of $\mathcal{R}_c(L)$, there is a maximal ideal M of $\mathcal{R}(L)$ such that $P = M \cap \mathcal{R}_c(L)$.*
(3) *For every prime ideal P of $\mathcal{R}_c(L)$, there is a z-ideal I of $\mathcal{R}(L)$ with the property that $P = I \cap \mathcal{R}_c(L)$.*
(4) *For every ideal I of $\mathcal{R}(L)$, there is a z-ideal J of $\mathcal{R}(L)$ with $I \cap \mathcal{R}_c(L) = J \cap \mathcal{R}_c(L)$.*
(5) *For every prime ideal P of $\mathcal{R}(L)$, there is a maximal ideal M of $\mathcal{R}(L)$ such that $P \cap \mathcal{R}_c(L) = M \cap \mathcal{R}_c(L)$.*
(6) *For every prime ideal P of $\mathcal{R}(L)$, there exists a z-ideal I of $\mathcal{R}(L)$ with the property that $P \cap \mathcal{R}_c(L) = I \cap \mathcal{R}_c(L)$.*

Now, we introduce an ideal in $\mathcal{R}_c(L)$ and study its relation to CP-frames.

Definition 4.5. For every ideal $\mathcal{P}$ of closed quotients of a frame L, we set $\mathcal{R}_{c\mathcal{P}}(L) = \{\alpha \in \mathcal{R}_c(L) : \ \uparrow(\mathrm{coz}(\alpha))^* \in \mathcal{P}\}$.

$\mathcal{R}_{c\mathcal{P}}(L)$ is the collection of all functions in $\mathcal{R}_c(L)$ whose support belongs to $\mathcal{P}$. For more details about the following results on $\mathcal{R}(L)$, see [2].

Lemma 4.6. *For every ideal $\mathcal{P}$ of closed quotients of L, the following statements hold.*

(1) *$\mathcal{R}_{c\mathcal{P}}(L)$ is a z_c-ideal of $\mathcal{R}_c(L)$.*
(2) *$\mathcal{R}_{c\mathcal{P}}(L)$ is proper if and only if $L \notin \mathcal{P}$.*

Proof. (1). We first show that $\mathcal{R}_{c\mathcal{P}}(L)$ is an ideal of $\mathcal{R}_c(L)$. Consider $\alpha, \beta \in \mathcal{R}_{c\mathcal{P}}(L)$ and $\gamma \in \mathcal{R}_c(L)$. Then $\mathrm{coz}(\alpha - \beta) \leq \mathrm{coz}(\alpha) \vee \mathrm{coz}(\beta)$ implies that $(\mathrm{coz}(\alpha))^* \wedge (\mathrm{coz}(\beta))^* \leq (\mathrm{coz}(\alpha - \beta))^*$. Now, since $\mathcal{P}$ is an ideal of closed quotients of L, it follows that

$$\uparrow(\mathrm{coz}(\alpha - \beta))^* \leq \uparrow((\mathrm{coz}(\alpha))^* \wedge (\mathrm{coz}(\beta))^*) = \uparrow(\mathrm{coz}(\alpha))^* \vee \uparrow(\mathrm{coz}(\beta))^* \in \mathcal{P}.$$

Therefore $\uparrow(\mathrm{coz}(\alpha - \beta))^* \in \mathcal{P}$ and so $\alpha - \beta \in \mathcal{R}_{c\mathcal{P}}(L)$. Similarly, it can be verified that $\alpha\beta \in \mathcal{R}_{c\mathcal{P}}(L)$. Hence, $\mathcal{R}_{c\mathcal{P}}(L)$ is an ideal of $\mathcal{R}_c(L)$.

It is quickly clear that $\mathcal{R}_{c\mathcal{P}}(L)$ is a z_c-ideal.

(2). Since $\uparrow(\mathrm{coz}(\mathbf{1}))^* = L$, we have $L \notin \mathcal{P}$ if and only if $\mathbf{1} \notin \mathcal{R}_{c\mathcal{P}}(L)$. The proof is now complete. □

The next proposition describes CP-frames L via ideals of closed quotients of L.

Proposition 4.7. *The following statements are equivalent for a frame L.*

(1) *L is a CP-frame.*
(2) *For every ideal I of $\mathcal{R}_c(L)$, there exists an ideal $\mathcal{P}$ of closed quotients of L such that $I = \mathcal{R}_{c\mathcal{P}}(L)$.*

Proof. (1) $\Rightarrow$ (2). Let I be an ideal of $\mathcal{R}_c(L)$. Set

$$\mathcal{P} = \Big\{\uparrow a : a \in L \text{ and there exists a } \alpha \text{ in } I \text{ such that } \uparrow a \leq \ \uparrow(\mathrm{coz}(\alpha))^*\Big\}.$$

Clearly, $\mathcal{P}$ is an ideal of closed quotients of L. We show that $I = \mathcal{R}_{c\mathcal{P}}(L)$. Evidently $I \subseteq \mathcal{R}_{c\mathcal{P}}(L)$. To see the reverse inclusion, consider $\alpha \in \mathcal{R}_{c\mathcal{P}}(L)$. Then $\uparrow(\mathrm{coz}(\alpha))^* \in \mathcal{P}$, so, there exists an element β in I such that $\uparrow(\mathrm{coz}(\alpha))^* \leq \uparrow(\mathrm{coz}(\beta))^*$. Therefore, $(\mathrm{coz}(\beta))^* \leq (\mathrm{coz}(\alpha))^*$, it follows that

$$\mathrm{coz}(\alpha) = (\mathrm{coz}(\alpha))^{**} \leq (\mathrm{coz}(\beta))^{**} = \mathrm{coz}(\beta),$$

which implies that $\alpha \in I$ (because I is a z_c-ideal).

(2) ⇒ (1). Let every ideal of $\mathcal{R}_c(L)$ be of the form $\mathcal{R}_{c\mathcal{P}}(L)$ for some ideal $\mathcal{P}$ of closed quotients of L. Then by Lemma 4.6, it is a z_c-ideal, which by Theorem 4.3 implies that L is a CP-frame. □

We close this section with some ring-theoretic characterization of CP-frames (for more details on this characterization of P-frames, see [11]). First, recall an ideal of a ring is said to be an *essential ideal* whenever it intersects any nonzero ideal of the ring nontrivially. The *radical* of an ideal I of a ring A is the ideal $\sqrt{I} = \{x \in A : x^n \in I$ for some $n \in \mathbb{N}\}$. A *radical ideal* is an ideal which equals its radical. One can easily see that any radical ideal is the intersection of a set of prime ideals containing it. We say that a ring A is a *z-good* ring if it has this property: an ideal of A is a z-ideal if and only if its radical ideal is a z-ideal. Equivalently, A is z-good if every ideal of A whose radical is z-ideal is itself a z-ideal. Let A be a lattice-ordered ring (usually abbreviated "l-ring"). An ideal $I \subseteq A$ is said to be *convex* if, for every $x, y \in A$, $0 \leq x \leq y$ and $y \in I$ imply $x \in I$. Also, I is called *absolutely convex* if, for every $x, y \in A$, $|x| \leq |y|$ and $y \in I$ imply $x \in I$. The next lemma shows that the requirement that prime ideals be z-ideals in the sense of Mason characterizes von Neumann regularity among z-good rings.

Lemma 4.8. [10] *A z-good ring is von Neumann regular if and only if every prime ideal in it is a z-ideal.*

The classical function rings $C(X)$ are z-good, see [4, Corollary 2.4]. In [11, Lemma 3.4], the authors showed that $\mathcal{R}(L)$ is a z-good ring. In [13], the authors show that every minimal prime ideal in $\mathcal{R}_c(L)$ is a z_c-ideal. Using this fact and the fact that any minimal prime ideal in a subring of a ring A is the intersection of a set of minimal prime ideals of A, we conclude that an ideal of $\mathcal{R}_c(L)$ is a z_c-ideal if and only if its radical is a z_c-ideal (see [13]). Thus, we have the following result.

Lemma 4.9. *$\mathcal{R}_c(L)$ is a z-good ring.*

These two lemmas are enough to prove the following result that is the counterpart of [11, Proposition 3.1].

Proposition 4.10. *The following statements are equivalent for a frame L.*

(1) *L is a CP-frame.*

(2) *Each essential ideal in $\mathcal{R}_c(L)$ is a z_c-ideal.*
(3) *Each radical ideal in $\mathcal{R}_c(L)$ is a z_c-ideal.*

Proof. It is evident. □

One can see that every radical ideal in $\mathcal{R}_c(L)$ is absolutely convex. With this fact, we arrive the following characterization of CP-frames which is the counterpart of [11, Proposition 3.2].

Corollary 4.11. *The following statements are equivalent for a frame L.*

(1) *L is a CP-frame.*
(2) *Each convex ideal in $\mathcal{R}_c(L)$ is a z_c-ideal.*
(3) *Each absolutely convex ideal in $\mathcal{R}_c(L)$ is a z_c-ideal.*

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