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THE ANNIHILATOR GRAPH OF MODULES OVER COMMUTATIVE RINGS

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ABSTRACT. Let M be a module over a commutative ring R, $Z_*(M)$ be its set of weak zero-divisor elements, and if $m \in M$, then let $I_m = (Rm :_R M) = \{r \in R : rM \subseteq Rm\}$. The annihilator graph of M is the (undirected) graph $AG(M)$ with vertices $\tilde{Z}_*(M) = Z_*(M) \setminus \{0\}$, and two distinct vertices m and n are adjacent if and only if $(0 :_R I_m I_n M) \neq (0 :_R m) \cup (0 :_R n)$. We show that $AG(M)$ is connected with diameter at most two and girth at most four. Also, we study some properties of the zero-divisor graph of reduced multiplication-like R-modules.

1. Introduction

Let R be a commutative ring with identity and $Z(R)$ its set of zero-divisors. The concept of the graph of zero-divisors of a ring was first introduced by Beck in [7]. The zero-divisor graph of R (denoted by $\Gamma(R)$) was introduced by Anderson and Levingston in [2], with vertices $Z(R)^* = Z(R) \setminus \{0\}$ and two distinct vertices x and y adjacent if $xy = 0$. In [5], Badawi introduces, for a commutative ring R with nonzero identity, its annihilator graph $AG(R)$. The set of vertices of this graph is $Z(R)^* = Z(R) \setminus \{0\}$ and two distinct vertices x and y adjacent if and only if $ann_R(xy) \neq ann_R(x) \cup ann_R(y)$ where $ann_R(a) = \{r \in R : ra = 0\}$ for every $a \in R$. It follows that each edge (path) of $\Gamma(R)$ is an edge (path) of $AG(R)$, so $\Gamma(R)$ is an induced subgraph of $AG(R)$.

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In [8], Behboodi gives several generalizations of the concept of zero-divisor elements in a commutative ring to modules and he introduces zero-divisor graphs for modules over commutative rings. The reader is referred to [4], [6], [10], [11], [12] and [13] for a detailed discussion of associated graphs to algebraic structures.

All rings in this paper are commutative with nonzero identity. Let R be a commutative ring and M be an R-module. For $x \in M$, we denote the annihilator of the factor module M/Rx by I_x (i.e., $I_x := \{r \in R : rM \subseteq Rx\} = (Rx :_R M)$). It is clear that for each $x \in M$, $(0 :_R M) \subseteq I_x$ and $Rx = M$ if and only if $I_x = R$. In particular, if $M = R$ then for each $x \in R$, $I_x = Rx$. This means that $I_xI_y = 0$ if and only if $xy = 0$. (see [8])

An element m of M is called a weak zero-divisor if either $m = 0$ or $I_mI_nM = (Rm :_R M)(Rn :_R M)M = 0$ for some nonzero element $n \in M$ with $I_n \subset R$. The set of all weak zero-divisor elements of M is denoted by $Z_*(M)$. It is clear that when $M = R$, then $Z_*(R) = Z(R)$. For an R-module M, we let $\tilde{Z}_*(M) = Z_*(M) \setminus \{0\}$. Then we associate the zero-divisor graph $\Gamma_*(M)$ with vertices $\tilde{Z}_*(M)$ and the vertices m and n are adjacent if and only if $I_mI_nM = 0$. (see [8, Definition 1.1]) In this article, we introduce the annihilator graph $AG(M)$ for an R-module M, with vertices $\tilde{Z}_*(M) = Z_*(M) \setminus \{0\}$, and two distinct vertices m and n are adjacent if and only if $(0 :_R I_mI_nM) \neq (0 :_R m) \cup (0 :_R n)$. One can show that $(0 :_R m) \cup (0 :_R n) \subseteq (0 :_R I_mI_nM)$ since $I_mI_nM \subseteq Rm \cap Rn$. So the reverse of the inclusion is important. Also, it follows that each edge (path) of $\Gamma_*(M)$ is an edge (path) of $AG(M)$, so $\Gamma_*(M)$ is an induced subgraph of $AG(M)$. In the second section, we consider several definitions and results which we use throughout this paper. We devote Section 3 to the reduced multiplication-like R-modules (defined later) and their properties. In Section 4, we show that $AG(M)$ is connected with diameter at most two (Theorem 4.5). If $AG(M)$ is not identical to $\Gamma_*(M)$, then we show that $gr(AG(M))$ (i.e., the length of a smallest cycle) is at most four (Theorem 4.8). We show that for a reduced multiplication-like R-module M, if $Z_*(M)$ is a submodule, then it is a prime submodule. In the final section, we determine when $AG(M)$ is identical to $\Gamma_*(M)$. Also we consider some properties of the zero-divisor graph of reduced multiplication-like R-modules.

We begin with some notation and definitions. Let Γ be a simple graph. The vertex set of Γ is denoted by $V(\Gamma)$. We recall that a graph is connected if there exists a path connecting any two distinct vertices. The distance $\mathrm{d}(a, b)$ is the length of the shortest path from a to b, if such a path does not exist, then $d(a,b)=\infty$. The diameter of a graph Γ, denoted by $\mathrm{diam}(\Gamma)$, is equal to $\sup\{d(a,b) : a,b \in V(\Gamma)\}$. A graph is complete, if it is connected with diameter less than or equal to one. The girth of a graph Γ, denoted by $\mathrm{gr}(\Gamma)$, is the length of a shortest cycle in Γ, provided Γ contains a cycle; otherwise; $\mathrm{gr}(\Gamma)=\infty$, in this case Γ is called an acyclic graph. We say that two (induced) subgraphs Γ_1 and Γ_2 of Γ are disjoint if Γ_1 and Γ_2 have no common vertices and no vertex of Γ_1 (respectively, Γ_2) is adjacent (in Γ) to any vertex not in Γ_1 (respectively, Γ_2). We denote the complete bipartite graph on m and n vertices by $K^{m,n}$.

2. Preliminaries

We devote this section to the several definitions and results which we use throughout this paper.

Definition 2.1. Let R be a commutative ring and let M be an R-module.
(1) For every submodule N of an R-module M, the ideal $\{r \in R : rM \subseteq N\}$ is denoted by $(N :_R M)$. So $(0 :_R M)$ is the annihilator of M.
(2) For any submodules N and K of M, we define the product of N and K by $NK = (N :_R M)(K :_R M)M$.(see [3, Definition 3.1]).
(3) A submodule N of M is said to be nilpotent if there exists a positive integer k such that $N^k = (N :_R M)^k M = 0$.(see [3, Definition 3.4])
(4) An R-module M is defined to be a multiplication module if for each submodule N of M, $N = IM$ for some ideal I of R. In this case we can take $I = (N :_R M)$.
(5) Let M be an R-module. We say that M is a multiplication-like module, if for each nonzero submodule N of M, $(0 :_R M) \subset (N :_R M)$.(see [8, Definition 2.1])
(6) A ring R is reduced, if it has no nonzero nilpotent elements and an R-module M is called reduced, if for any $m \in M$ and any $a \in R$, $ma = 0$ implies $Rm \cap aM = 0$. (see [1])
(7) A proper submodule N of an R-module M is prime if whenever $rm \in N$ for some $r \in R$ and $m \in M$, then $m \in N$ or $r \in (N :_R M)$.
(8) A proper submodule N of an R-module M is semiprime if for every ideal I of R and every submodule K of M, $I^k K \subseteq N$ for some positive integer k implies that $IK \subseteq N$. An R-module M is called a semiprime module if $\langle 0 \rangle \subset M$ is a semiprime submodule. (see [9])
(9) A submodule K of an R-module M is called essential, if for each submodule L of M, $K \cap L = 0$ implies that $L = 0$. An R-module M is called a uniform module if the intersection of any two nonzero

submodules is nonzero. So every submodule of an uniform module is an essential submodule.

We have the following result that is proved in ([3, Lemma 3.7]).

Lemma 2.2. *Let M be an R-module. Then the following statements are equivalent.*
(a) 0 is the only nilpotent submodule of M.
(b) For all submodules N and K of M with $NK = 0$, we have $N \cap K = 0$.

3. Reduced and multiplication-like modules

The aim of this section is to study the properties of reduced and multiplication-like modules. We first consider the following lemma.

Lemma 3.1. *Let M be a reduced R-module. Then the following hold.*
(1) If $a^k m = 0$ for some $a \in R$, $m \in M$ and positive integer k, then $am = 0$.
(2) If $I^k M = 0$ for some ideal I of R and positive integer k, then $IM = 0$.
(3) M is a semiprime R-module.
(4) For any elements $m, n \in M$ and $r \in R$, $Rrm \cap Rn = 0$ if and only if $Rm \cap Rrn = 0$.
(5) Every submodule of M is reduced.

Proof. (1) Let $a^k m = 0$ for some $a \in R$ and $m \in M$. We can assume that $k \neq 1$. Then $a(a^{k-1}m) = 0$. Since M is a reduced module, we have $Ra^{k-1}m \cap aM = 0$. So $a^{k-1}m = 0$, since $a^{k-1}m \in Ra^{k-1}m \cap aM$. By a similar way, we get $am = 0$.
(2) Suppose that $I^k M = 0$ for some ideal I of R and positive integer k. Let $a \in I$ and $m \in M$. Then $a^k m \in I^k M = 0$, so $am = 0$ by (1). Thus $IM = 0$.
(3) Let K be a submodule of M and $I^k K = 0$ for some positive integer k. Let $a \in I$ and $m \in K$. Then $a^k m \in I^k K = 0$, so $am = 0$ by (1). Therefore $IK = 0$. Hence 0 is a semiprime submodule of M and then M is a semiprime R-module.
(4) Let $Rrm \cap Rn = 0$ and $y \in Rm \cap Rrn$. Then $y = sm = s'rn$ for some $s, s' \in R$. So $r^2 s'n = rsm \in Rrm \cap Rn = 0$. Since M is a reduced module, it follows that $y = rs'n = 0$ by (1). Similarly, if $Rm \cap Rrn = 0$, then we have $Rrm \cap Rn = 0$.
(5) Let N be a submodule of M, $m \in N$ and $a \in R$ such that $ma = 0$. Then $Rm \cap aN \subseteq Rm \cap aM = 0$, since M is a reduced module. □

The next proposition has a crucial role in this paper.

Proposition 3.2. *Let M be a reduced multiplication-like module and $r \in R$, $n, m \in M$ such that $rm \neq 0$ and $rn \neq 0$. Then the following hold.*
(1) 0 is the only nilpotent submodule of M.
(2) $I_n I_{rm} M \neq 0$ if and only if $I_m I_{rn} M \neq 0$.

Proof. (1) Let N be a nonzero nilpotent submodule of M. Then $(N : M)^k M = N^k = 0$ for some positive integer k. Since M is a reduced module, we have $(N : M)M = 0$ by Lemma 3.1 (2). Thus $(N :_R M) \subseteq (0 :_R M)$. It is clear that $(0 :_R M) \subseteq (N :_R M)$. Hence $(0 :_R M) = (N :_R M)$, a contradiction since M is a multiplication-like module. Then $N = 0$.
(2) Assume that $I_n I_{rm} M \neq 0$. Then $Rn \cap Rrm \neq 0$, since $I_n I_{rm} M \subseteq Rn \cap Rrm$. So $Rm \cap Rrn \neq 0$ by Lemma 3.1 (4). Thus $I_m I_{rn} M = (Rm)(Rrn) \neq 0$ by (1) and Lemma 2.2. The converse is similar. □

In view of Proposition 3.2, we have the following lemma.

Lemma 3.3. *Let M be an R-module and m, n two nonzero elements of M. Then the followig hold.*
(1) If $Rm \cap Rn = 0$, then m and n are adjacent in $\Gamma_*(M)$.
(2) If M is a reduced multiplication-like module, then $Rm \cap Rn = 0$ if and only if m and n are adjacent in $\Gamma_*(M)$.

Proof. (1) It is clear since $I_m I_n M \subseteq Rm \cap Rn = 0$.
(2) Assume that m and n are adjacent in $\Gamma_*(M)$. Then $(Rm)(Rn) = I_m I_n M = 0$. So $Rm \cap Rn = 0$ by Lemma 2.2 and Proposition 3.2. The coverse is clear by (1). □

4. Basic properties of $AG(M)$

In this section, we compute the diameter of graph $AG(M)$. We begin with a lemma containing several useful properties of $AG(M)$.

Lemma 4.1. *Let R be a commutative ring and M be an R-module.*
(1) Let m, n be distinct elements of $\tilde{Z}_*(M)$. Then $m - n$ is not an edge of $AG(M)$ if and only if $(0 :_R I_m I_n M) = (0 :_R n)$ or $(0 :_R I_m I_n M) = (0 :_R m)$.
(2) If $m - n$ is an edge of $\Gamma_*(M)$ for some distinct $m, n \in \tilde{Z}_*(M)$, then $m - n$ is an edge of $AG(M)$. In particular, if P is a path in $\Gamma_*(M)$, then P is a path in $AG(M)$.
(3) If $m - n$ is not an edge of $AG(M)$ for some distinct $m, n \in \tilde{Z}_*(M)$, then $(0 :_R m) \subseteq (0 :_R n)$ or $(0 :_R n) \subseteq (0 :_R m)$.
(4) If $d_{\Gamma_*(M)}(m, n) = 3$ for some distinct $m, n \in \tilde{Z}_*(M)$, then $m - n$ is

an edge of $AG(M)$.

Proof. (1) Assume that $m - n$ is not an edge of $AG(M)$. Then $(0:_R I_mI_nM) = (0:_R m) \cup (0:_R n)$ by definition. Since $(0:_R I_mI_nM)$ is a union of two ideals, we have $(0:_R I_mI_nM) = (0:_R m)$ or $(0:_R I_mI_nM) = (0:_R n)$. Conversely, suppose that $(0:_R I_mI_nM) = (0:_R m)$ or $(0:_R I_mI_nM) = (0:_R n)$. Then $(0:_R I_mI_nM) = (0:_R m) \cup (0:_R n)$ and so $m - n$ is not an edge of $AG(M)$.
(2) Suppose that $m - n$ is an edge of $\Gamma_*(M)$ for some distinct $m, n \in \tilde{Z}_*(M)$. Then $I_mI_nM = 0$ and $(0:_R I_mI_nM) = R$. Since $m \neq 0$ and $n \neq 0$, it follows that $(0:_R m) \neq R$ and $(0:_R n) \neq R$. Thus $m - n$ is an edge of $AG(M)$. The particular statement is clear.
(3) Suppose that $m - n$ is not an edge of $AG(M)$ for some distinct $m, n \in \tilde{Z}_*(M)$. Then $(0:_R I_mI_nM) = (0:_R m) \cup (0:_R n)$. Since $(0:_R I_mI_nM)$ is a union of two ideals, we have $(0:_R I_mI_nM) = (0:_R m)$ or $(0:_R I_mI_nM) = (0:_R n)$. So $(0:_R m) \subseteq (0:_R n)$ or $(0:_R n) \subseteq (0:_R m)$.
(4) Suppose that $d_{\Gamma_*(M)}(m, n) = 3$ for some distinct $m, n \in \tilde{Z}_*(M)$. Then there is a path $m - a - b - n$ of length 3 between m and n. So $I_mI_aM = I_aI_bM = I_bI_nM = 0$, $I_mI_bM \neq 0$ and $I_aI_nM \neq 0$. Then $I_a(I_mI_nM) = I_b(I_nI_mM) = 0$, so $I_a, I_b \subseteq (0:_R I_mI_nM)$. Since $I_mM \subseteq Rm$, we have $(0:_R m) \subseteq (0:_R I_mM)$. So $I_bm \neq 0$, since $I_mI_bM \neq 0$. Similarly $I_an \neq 0$. But we have $I_mI_aM = I_bI_nM = 0$. The proof will now break into two cases:
Case 1. If $I_am = 0$ and $I_bn = 0$, then $I_a \in (0:_R m) \setminus (0:_R n)$ and $I_b \in (0:_R n) \setminus (0:_R m)$, since $I_bm \neq 0$ and $I_an \neq 0$. Hence $m - n$ is an edge of $AG(M)$ by Part (3).
Case 2. If either $I_am \neq 0$ or $I_bn \neq 0$, then $I_a \subseteq (0:_R I_mI_nM) \setminus ((0:_R m) \cup (0:_R n))$ or $I_b \subseteq (0:_R I_mI_nM) \setminus ((0:_R m) \cup (0:_R n))$, so $(0:_R I_mI_nM) \neq (0:_R m) \cup (0:_R n)$ and the proof is complete. □

In [8, Proposition 1.4], it is shown that, for an R-module M with $I = (0:_R M)$, $\Gamma_*({}_RM) = \Gamma_*({}_{R/I}M)$. Now, we have the following lemma.

Lemma 4.2. *Let M be an R-module with $I = (0:_R M)$. Then $AG_R(M) = AG_{R/I}(M)$.*

Proof. Let $x \in \tilde{Z}_*(M)$. Then there exists $0 \neq y \in M$ such that $I_xI_yM = 0$. It is clear that $I \subseteq (0:_R x) \cap (0:_R y)$, $(0:_{R/I} I_xI_yM) = (0:_R I_xI_yM)/I$, $(0:_{R/I} x) = (0:_R x)/I$ and $(0:_{R/I} y) = (0:_R y)/I$. By ([8, Proposition 1.4]), $x \in Z_*({}_RM)$ if and only if $x \in Z_*({}_{R/I}M)$.

It is clear that the vertices x and y are adjacent in $AG_R(M)$ if and only if x and y are adjacent in $AG_{R/I}(M)$. Therefore $AG_R(M) = AG_{R/I}(M)$. □

We need the following lemma that is a generalization of [8, Lemma 1.7].

Lemma 4.3. *Let R be a commutative ring and M be an R-module and $x, y \in \tilde{Z}_*(M)$. Then the following hold.*
(1) If $r \in (0 :_R m)$ for some element $m \in M$, then $I_m I_{rx} M = 0$ for every element $x \in M$. In particular, if $rx \neq 0$, then $m - rx$ is an edge in $\Gamma_*(M)$ for every element $x \in M$.
(2) If $x - y$ is an edge in $\Gamma_*(M)$, then for each $r \notin (0 :_R x) \cap (0 :_R y)$, either $x - ry$ or $y - rx$ is also an edge in $\Gamma_*(M)$.
(3) If $x - y$ is an edge in $\Gamma_*(M)$, then $rx - sy$ is also an edge in $\Gamma_*(M)$ for every $r, s \in R$ such that $rx \neq 0$ and $sy \neq 0$.

Proof. (1) Let $\lambda \in I_m I_{rx} M$. Then $\lambda \in I_m Rrx$, since $I_{rx} M \subseteq Rrx$. So $\lambda = (r_1 s_1 + ... + r_n s_n) rx$ for some $r_i \in I_m$ and $s_i \in R$ with $i = 1, 2, ..., n$. Then $r_i M \subseteq Rm$ for each i. Thus $r_i s_i x \in Rm$ and hence $r_i s_i rx \in Rrm = 0$ for each i. So $\lambda = 0$. The in particular statement is clear.
(2) Let $0 \neq r \in R$ and $x - y$ is an edge in $\Gamma_*(M)$. Then $I_x I_y M = 0$. If $r \in (0 :_R x)$, then $r \notin (0 :_R y)$ and $x - ry$ is an edge in $\Gamma_*(M)$ by (1). So suppose that $r \notin (0 :_R x)$. Then $rx \neq 0$. If $r \in (0 :_R y)$, then $y - rx$ is an edge in $\Gamma_*(M)$ by (1). If $r \notin (0 :_R y)$, then $ry \neq 0$ and $x - ry$ is an edge in $\Gamma_*(M)$ by [8, Lemma 1.7].
(3) Since $I_{rx} \subseteq I_x$ and $I_{sy} \subseteq I_y$, it follows that $I_{rx} I_{sy} M \subseteq I_x I_y M = 0$ and hence $rx - sy$ is also an edge in $\Gamma_*(M)$.

□

The next proposition gives a partial answer to the question " Is $AG(M)$ a connected graph or not?". Here we answer this question by [8, Theorem 1.8].

Proposition 4.4. *Let R be a commutative ring and M be an R-module. If $m - n$ is not an edge of $AG(M)$ for some distinct $m, n \in \tilde{Z}_*(M)$, then there is a $k \in \tilde{Z}_*(M) \setminus \{m, n\}$ such that $m - k - n$ is a path in $\Gamma_*(M)$ and hence $m - k - n$ is a path in $AG(M)$.*

Proof. By [8, Theorem 1.8], $diam(\Gamma_*(M)) \leq 3$. So for every element $m, n \in \tilde{Z}_*(M)$, if either $d_{\Gamma_*(M)}(m, n) = 1$ or 3, then $m - n$ is an edge of

$AG(M)$ and m and n are adjacent in $AG(M)$ by Lemma 4.1. Hence if m and n are not adjacent in $AG(M)$, then $d_{\Gamma_*(M)}(m,n) = 2$. So there exists $k \in \tilde{Z}_*(M) \setminus \{m,n\}$ such that $m-k-n$ is a path in $\Gamma_*(M)$ and hence $m-k-n$ is a path in $AG(M)$. □

In view of Proposition 4.4, we have the following theorem.

Theorem 4.5. *Let R be a commutative ring and M be an R-module with $|\tilde{Z}_*(M)| \geq 2$. Then $AG(M)$ is connected and $diam(AG(M)) \leq 2$.*

Lemma 4.6. *Let R be a commutative ring and M be an R-module. Suppose that $m-n$ is an edge of $AG(M)$ that is not an edge of $\Gamma_*(M)$ for some distinct $m,n \in \tilde{Z}_*(M)$. If there is a $x \in (0 :_M I_mI_n) \setminus \{m,n\}$ such that $I_xI_mM \neq 0$ and $I_xI_nM \neq 0$, then $m-x-n$ is a path in $AG(M)$ that is not a path in $\Gamma_*(M)$, and hence $C: m-x-n-m$ is a cycle in $AG(M)$ of length three and each edge of C is not an edge of $\Gamma_*(M)$.*

Proof. Suppose that $m-n$ is an edge of $AG(M)$ that is not an edge of $\Gamma_*(M)$. Then $I_mI_nM \neq 0$. Assume that there is a $x \in (0 :_M I_mI_n) \setminus \{m,n\}$ such that $I_xI_mM \neq 0$ and $I_xI_nM \neq 0$. Since $I_n(I_mI_xM) \subseteq I_mI_nRx = 0$, so $I_n \subseteq (0 :_R I_mI_nM)$. If $I_nx = 0$, then $I_nI_xM \subseteq I_nRx = 0$, a contradiction. So $I_nx \neq 0$. Similarly $I_nm \neq 0$. Hence $I_n \subseteq (0 :_R I_mI_xM) \setminus ((0 :_R m) \cup (0 :_R x))$, so $(0 :_R I_mI_xM) \neq (0 :_R m) \cup (0 :_R x)$. We conclude that $m-x$ is an edge of $AG(M)$. Similarly $I_m \subseteq (0 :_R I_nI_xM) \setminus ((0 :_R n) \cup (0 :_R x))$ and so $x-n$ is an edge of $AG(M)$. Hence $m-x-n$ is a path in $AG(M)$. Since $I_xI_mM \neq 0$ and $I_xI_nM \neq 0$, thus $m-x-n$ is not a path in $\Gamma_*(M)$. It is clear that $m-x-n-m$ is a cycle in $AG(M)$ of length three and each edge of C is not an edge of $\Gamma_*(M)$. □

By [8, Theorem 1.8] for every R-module M, $gr(\Gamma_*(M)) \leq 4$, So $gr(AG(M)) \in \{3,4\}$ by Lemma 4.1 (2). But the following result shows that for a reduced multiplication-like R-module M, there is a cycle C of length 3 or 4 in $AG(M)$ such that C is not a cycle in $\Gamma_*(M)$.

Theorem 4.7. *Let M be a reduced multiplication-like R-module. Suppose that $m-n$ is an edge of $AG(M)$ that is not an edge of $\Gamma_*(M)$ for some distinct $m,n \in \tilde{Z}_*(M)$. Then there exist $x,y \in M \setminus \{m,n\}$ such that one of the following statement holds.*
(1) $x-n-m-x$ and $y-m-n-y$ are two cycles in $AG(M)$ of length three such that each edge of C is not an edge of $\Gamma_*(M)$.
(2) $n-x-y-m-n$ is a cycle in $AG(M)$ of length 4 that is not a cycle in $\Gamma_*(M)$.

Proof. Let $m - n$ be an edge of $AG(M)$ that is not an edge of $\Gamma_*(M)$. First suppose that $(0 :_M I_m I_n) = 0$. Since $n \in Z_*(M)$, there exists $x \in M$ such that $I_x \subset R$ and $I_n I_x M = 0$, so $I_x I_n I_m M = 0$. Thus $I_x M = 0$, since $(0 :_M I_m I_n) = 0$. Therefore $I_x I_n M = I_x I_m M = 0$. Hence $m - x - n - m$ is a cycle of length three in $AG(M)$.
Now, assume that $(0 :_M I_m I_n) \neq 0$. Since $m - n$ is an edge of $AG(M)$ that is not an edge of $\Gamma_*(M)$, it follows that $(0 :_R I_m I_n M) \neq (0 :_R m) \cup (0 :_R n)$ and $I_m I_n M \neq 0$. Since M is a reduced module, we have $I_m^2 I_n M \neq 0$ and $I_m I_n^2 M \neq 0$ by Lemma 3.1. Now let $r \in (0 :_R I_m I_n M) \setminus ((0 :_R m) \cup (0 :_R n))$. Then $r I_m I_n M = 0$, $rm \neq 0$ and $rn \neq 0$. It is clear that $rm, rn \in (0 :_R I_m I_n)$. Now, we show that $rm, rn \notin \{m, n\}$. If $rm = m$, then $I_m^2 I_n M = I_m I_n (I_m M) \subseteq I_m I_n Rm = I_m I_n Rrm = 0$, a contradiction. Similarly $rm \neq n$, since $I_m I_n^2 M \neq 0$. Hence $rm \notin \{m, n\}$. By a similar way, we get $rn \notin \{m, n\}$. Set $x = rm$ and $y = rn$. Now, we show that $rm - m$ is an edge of $AG(M)$ that is not an edge of $\Gamma_*(M)$. If $rm - m$ is an edge of $\Gamma_*(M)$, then $I_{rm} I_m M = 0$. It means that $(Rrm)(Rm) = (Rrm :_R M)(Rm :_R M)M = I_{rm} I_m M = 0$, so $Rrm \cap Rm = 0$ by Proposition 3.2 and Lemma 2.2, which is a contradiction. So $rm - m$ is not an edge of $\Gamma_*(M)$. Similarly, $rn - n$ is not an edge of $\Gamma_*(M)$. The proof will now break into two cases:
Case 1. If $I_{rm} I_n M \neq 0$, then $I_m I_{rn} M \neq 0$ by Proposition 3.2. We show that $I_n \subseteq (0 :_R I_m I_{rm} M) \setminus ((0 :_R m) \cup (0 :_R rm))$. Since $I_n (I_m I_{rm} M) \subseteq I_n I_m Rrm = 0$, we have $I_n \subseteq (0 :_R I_m I_{rm} M)$. Since $(0 :_R m) \subseteq (0 :_R rm)$, it suffices to show that $I_n(rm) \neq 0$. If $I_n(rm) = 0$, then $I_n I_{rm} M \subseteq I_n Rrm = 0$ which is a contradiction. So $(0 :_R I_m I_{rm} M) \neq (0 :_R m) \cup (0 :_R rm)$. Hence $m - rm$ is an edge of $AG(M)$ that is not an edge of $\Gamma_*(M)$. Similarly, $rn - n$ is an edge of $AG(M)$ that is not an edge of $\Gamma_*(M)$. Now we show that $I_m \subseteq (0 :_R I_n I_{rm} M) \setminus ((0 :_R n) \cup (0 :_R rm))$. Since $I_m (I_{rm} I_n M) \subseteq I_m I_n Rrm = 0$, it follows that $I_m \subseteq (0 :_R I_n I_{rm} M)$. Since $I_{rm} I_m M \neq 0$ and $I_{rm} M \subseteq Rrm$, we have $I_m(rm) \neq 0$. Also, if $I_m n = 0$, then $I_m I_n M \subseteq I_m Rn = 0$ which is a contradiction. So $rm - n$ is an edge of $AG(M)$ that is not an edge in $\Gamma_*(M)$. Similarly $m - rn$ is an edge of $AG(M)$ that is not an edge in $\Gamma_*(M)$. Then in this case, $x - n - m - x$ and $y - m - n - y$ are two cycles in $AG(M)$ of length three such that each edge of C is not an edge of $\Gamma_*(M)$.
Case 2. If $I_{rm} I_n M = 0$, then $I_m I_{rn} M = 0$ by Proposition 3.2. So $I_{rm} I_{rn} M \subseteq I_{rm} I_n M = 0$. Hence $n - x - y - m - n$ is a cycle in $AG(M)$ of length 4 that is not a cycle in $\Gamma_*(M)$. □

In view of Theorem 4.7, we have the following result.

Theorem 4.8. *Let M be a reduced multiplication-like R-module and suppose that $AG(M) \neq \Gamma_*(M)$. Then $gr(AG(M)) \in \{3, 4\}$. Furthermore, there is a cycle C in $AG(M)$ that C is not a cycle in $\Gamma_*(M)$.*

Proposition 4.9. *Let M be a reduced multiplication-like R-module and suppose that $AG(M) \neq \Gamma_*(M)$. Suppose that $m - n$ is not an edge of $AG(M)$ for some distinct $m, n \in \tilde{Z}_*(M)$. If $(0 :_R m) = (0 :_R n) = 0$, then $Z_*(M) = M$ and $gr(AG(M)) = 3$.*

Proof. Suppose that $m - n$ is not an edge of $AG(M)$. Then $(0 :_R I_m I_n M) = (0 :_R m) \cup (0 :_R n) = 0$ and $I_m I_n M \neq 0$ by Lemma 4.1 (2). Since $m \in \tilde{Z}_*(M)$, we have $I_x I_m M = 0$ for some nonzero element $x \in M$. Thus $I_x(I_m I_n M) = 0$ and then $I_x \subseteq (0 :_R I_m I_n M) = 0$. Hence $I_y I_x M = 0$ for every element $y \in M$. Thus $Z_*(M) = M$ and x is adjacent to every element of M. If $gr(AG(M)) = 4$, then there exists a cycle $a - b - c - d - a$, since x adjacent to every element, it follows that $a - x - b - a$, $a - x - d - a$, $b - x - c - b$ and $c - x - d - c$ are four cycles of length 3 in $AG(M)$, which is a contradiction. So $gr(AG(M)) = 3$ by Theorem 4.8. □

Proposition 4.10. *Let M be an R-module. Then the following hold.*
(1) If M has no essential submodule, then $Z_*(M) = M$.
(2) If M is a reduced multiplication-like R-module and $\Gamma_*(M)$ is a complete graph with $Z_*(M) = M$, then M has no essential submodule.
(3) If M is a reduced multiplication R-module and $AG(M)$ is a complete graph with $Z_*(M) = M$, then M has no essential submodule.

Proof. (1) Let $0 \neq x \in M$. Since M has no essential submodule, Rx can not be an essential submodule. So there exists $0 \neq y \in M$ such that $Rx \cap Ry = 0$. Then $I_x I_y M \subseteq Rx \cap Ry = 0$ and so $x \in Z_*(M)$.
(2) Let M have an essential submodule N and $0 \neq x \in M$. Then $N \cap Rx \neq 0$ and there is a nonzero element $y \in N \cap Rx$. So $0 \neq y \in Ry \cap Rx$; thus $Rx \cap Ry \neq 0$. Since $\Gamma_*(M)$ is a complete graph, it follows that $I_x I_y M = 0$. Then $Rx \cap Ry = 0$ by Lemma 2.2 and Proposition 3.2 which is a contradiction.
(3) Let M have an essential submodule N and $0 \neq x \in M$. Then $N \cap Rx \neq 0$ and there is a nonzero element $y \in N \cap Rx$. So $0 \neq y \in Ry \cap Rx$; thus $Rx \cap Ry \neq 0$ and $y \in Rx$. So $y = ax$ for some $0 \neq a \in R$. Since $AG(M)$ is a complete graph, we have $(0 :_R I_x I_y M) \neq (0 :_R x) \cup (0 :_R y)$ and since $(0 :_R x) \subseteq (0 :_R y)$, we have $(0 :_R I_x I_y M) \neq (0 :_R y)$. Hence there exists $0 \neq s \in (0 :_R I_x I_y M) \setminus (0 :_R y)$. So $sy \neq 0$ and $sI_x I_y M = 0$. Since $y \in Rx$, $I_y \subseteq I_x$. Therefore $sI_y^2 M \subseteq sI_x I_y M = 0$ and thus $sI_y M = 0$, since M is a reduced module. By assumption M

is a multiplication module, then $sRy = sI_yM = 0$. Hence $s \in (0 :_R y)$ which is a contradiction. □

We end this section with the following proposition.

Proposition 4.11. *Let M be a reduced multiplication-like R-module. If $Z_*(M)$ is a submodule, then $Z_*(M)$ is a prime submodule of M.*

Proof. Let $rx \in Z_*(M)$ for some $r \in R$ and $x \in M$. First suppose that $rx = 0$. So $r \in (0 :_R x)$. If $rM = 0$, then $rM \subseteq Z_*(M)$. So let $rM \neq 0$. Then $rn \neq 0$ for some nonzero element $n \in M$. Then $I_xI_{rn}M = 0$ by Lemma 4.3. Thus $x \in Z_*(M)$. Now, let $rx \neq 0$ and $x \notin Z_*(M)$. It suffices to show that $rM \subseteq Z_*(M)$. There exists a nonzero element $y \in M$ such that $I_{rx}I_yM = 0$. If $r \notin (0 :_R y)$, then $ry \neq 0$ and $I_{ry}I_xM = 0$ by Proposition 3.2. Then $x \in Z_*(M)$ which is a contradiction. So we can assume that $ry = 0$. Let $0 \neq \lambda \in rM$. Then $\lambda = rm$ for some nonzero element $m \in M$. So $I_yI_{rm}M = I_yI_\lambda M = 0$ by Lemma 4.3. Therefore $\lambda \in Z_*(M)$ and we have $rM \subseteq Z_*(M)$. □

5. When is $AG(M)$ identical to $\Gamma_*(M)$?

In this section, we determine when $AG(M)$ is identical to $\Gamma_*(M)$. By Lemma 4.1, each edge (path) of $\Gamma_*(M)$ is an edge (path) of $AG(M)$. So $\Gamma_*(M)$ is an induced subgraph of $AG(M)$ and if $\Gamma_*(M)$ is a complete graph, then $AG(M) = \Gamma_*(M)$ is a complete graph. So in [8], whenever we have "$\Gamma_*(M)$ is a complete graph", we conclude that $AG(M) = \Gamma_*(M)$.

First, we consider several examples of the annihilator and the zero-divisor graphs of R-modules.
The following is an example of a nonreduced R-module M, where $AG(M) \neq \Gamma_*(M)$ and these graphs are not complete.

Example 5.1. Let $M = \mathbb{Z}_{12}$ and $R = \mathbb{Z}$. Since $(\bar{2}^2)(\bar{3}) = 0$ but $(\bar{2})(\bar{3}) = \bar{6} \neq 0$, it follows that M is nonreduced. Also, M has seven weak zero-divisors. It is easy to check that $I_{\bar{2}} = I_{\bar{10}} = 2\mathbb{Z}$, $I_{\bar{3}} = I_{\bar{9}} = 3\mathbb{Z}$, $I_{\bar{4}} = I_{\bar{8}} = 4\mathbb{Z}$ and $I_{\bar{6}} = 6\mathbb{Z}$. So $I_{\bar{2}}I_{\bar{6}}M = I_{\bar{10}}I_{\bar{6}}M = 0$, $I_{\bar{3}}I_{\bar{4}}M = I_{\bar{3}}I_{\bar{8}}M = I_{\bar{9}}I_{\bar{4}}M = I_{\bar{9}}I_{\bar{8}}M = 0$ and $I_{\bar{4}}I_{\bar{6}}M = I_{\bar{8}}I_{\bar{6}}M = 0$. It is clear that $0 \neq \bar{4} \in I_{\bar{2}}I_{\bar{10}}M = (2\mathbb{Z})(2\mathbb{Z})\mathbb{Z}_{12}$, then $\bar{2}$ is not adjacent to $\bar{10}$ in $\Gamma_*(M)$. One can see $(0 :_R I_{\bar{2}}I_{\bar{10}}M) = 3\mathbb{Z}$ and $(0 :_R \bar{2}) \cup (0 :_R \bar{10}) = 6\mathbb{Z}$. Therefore $(0 :_R I_{\bar{2}}I_{\bar{10}}M) \neq (0 :_R \bar{2}) \cup (0 :_R \bar{10})$. So $\bar{2} - \bar{10}$ is an edge in $AG(M)$ that is not an edge of $\Gamma_*(M)$. Thus $AG(M) \neq \Gamma_*(M)$.

The following is an example of an R-module M, where $AG(M) = \Gamma_*(M)$ is not a complete graph.

Example 5.2. Let $M = \mathbb{Z} \oplus \mathbb{Z}_2$ and $R = \mathbb{Z}$. It is easy to see that for every $0 \neq n \in \mathbb{Z}$, $I_{(n,\bar{0})} = r\mathbb{Z}$ such that $r = lcm(2, n)$, $I_{(n,\bar{1})} = n\mathbb{Z}$ and $I_{(0,\bar{1})} = 0$. So $I_{(n,\bar{1})}I_{(0,\bar{1})}M = I_{(n,\bar{0})}I_{(0,\bar{1})}M = 0$ and every element is adjacent to $(0, \bar{1})$ and we have $I_{(n,\bar{1})}I_{(n,\bar{0})}M \neq 0$, $I_{(n,\bar{1})}I_{(m,\bar{1})}M \neq 0$ and $I_{(n,\bar{0})}I_{(m,\bar{0})}M \neq 0$ for every nonzero elements $n, m \in M$. Similarly for every element α and β of M with $\alpha, \beta \in M \setminus \{(0, \bar{1})\}$, we have $(0 :_R \alpha) = (0 :_R \beta) = (0 :_R I_\alpha I_\beta M) = 0$. Thus $\Gamma_*(M)$ and $AG(M)$ are not complete graphs and $AG(M) = \Gamma_*(M)$.

The following is an example of a nonreduced R-module M, where $AG(M) \neq \Gamma_*(M)$ and $\Gamma_*(M)$ is not a complete graph, but $AG(M)$ is the complete graph K_3.

Example 5.3. Let $M = \mathbb{Z}_8$ and $R = \mathbb{Z}$. Since $(\bar{2}^2)(\bar{2}) = 0$ but $(\bar{2})(\bar{2}) = \bar{4} \neq 0$, so M is nonreduced. Also, M has three weak zero-divisors. It is easy to check that $I_{\bar{2}} = I_{\bar{6}} = 2\mathbb{Z}$, $I_{\bar{4}} = 4\mathbb{Z}$ and $I_{\bar{3}} = I_{\bar{5}} = I_{\bar{7}} = \mathbb{Z}$. So $I_{\bar{2}}I_{\bar{4}}M = I_{\bar{4}}I_{\bar{6}}M = 0$ and $I_{\bar{2}}I_{\bar{6}}M \neq 0$. Then $\bar{2}$ is not adjacent to $\bar{6}$ in $\Gamma_*(M)$. One can see $(0 :_R I_{\bar{2}}I_{\bar{6}}M) = 2\mathbb{Z}$ and $(0 :_R \bar{2}) \cup (0 :_R \bar{6}) = 4\mathbb{Z}$. Therefore $(0 :_R I_{\bar{2}}I_{\bar{6}}M) \neq (0 :_R \bar{2}) \cup (0 :_R \bar{6})$. So $\bar{2} - \bar{6}$ is an edge in $AG(M)$ that is not an edge of $\Gamma_*(M)$. Thus $AG(M) \neq \Gamma_*(M)$ and $AG(M)$ is the complete graph K_3.

The following is an example of a reduced R-module M, where $AG(M) = \Gamma_*(M)$ is not a complete graph.

Example 5.4. Let $M = \mathbb{Z}_6$ and $R = \mathbb{Z}$. Then M has three weak zero-divisors. It is easy to check that M is a reduced module and $I_{\bar{2}} = I_{\bar{4}} = 2\mathbb{Z}$, $I_{\bar{3}} = 3\mathbb{Z}$ and $I_{\bar{1}} = I_{\bar{5}} = \mathbb{Z}$. So $I_{\bar{2}}I_{\bar{3}}M = I_{\bar{4}}I_{\bar{3}}M = 0$ and $I_{\bar{2}}I_{\bar{4}}M \neq 0$. Then $\bar{2}$ is not adjacent to $\bar{4}$ in $\Gamma_*(M)$. One can see $(0 :_R I_{\bar{2}}I_{\bar{4}}M) = 3\mathbb{Z}$ and $(0 :_R \bar{2}) \cup (0 :_R \bar{4}) = 3\mathbb{Z}$. Therefore $(0 :_R I_{\bar{2}}I_{\bar{4}}M) = (0 :_R \bar{2}) \cup (0 :_R \bar{4})$. Then $\bar{2}$ is not adjacent to $\bar{4}$ in $AG(M)$. Thus $AG(M) = \Gamma_*(M)$ and these graphs are not complete.

Now, we study the case when $(0 :_R M)$ is a prime ideal.

Theorem 5.5. *Let R be a commutative ring and m, n be distinct nonzero elements of an R-module M for which $(0 :_R M)$ is a prime ideal. Suppose that $m, n \in \tilde{Z}_*(M)$. Then $m - n$ is an edge of $AG(M)$ if and only if $m - n$ is an edge of $\Gamma_*(M)$.*

Proof. Suppose that $m - n$ is an edge of $AG(M)$ such that it is not an edge of $\Gamma_*(M)$. It is clear that $I_mI_nM \neq 0$. We show that $m - n$ is not an edge of $AG(M)$. If $0 \neq r \in (0 :_R I_mI_nM) \setminus ((0 :_R m) \cup (0 :_R n))$, then $rI_mI_nM = 0$. So $rI_mI_n \in (0 :_R M)$. Since $(0 :_R M)$ is a prime ideal and $I_mI_nM \neq 0$, we have $r \in (0 :_R M)$. Therefore $rM = 0$

and $r \in (0 :_R m) \cap (0 :_R n)$ which is a contradiction. Therefore $(0 :_R I_m I_n M) = (0 :_R m) \cup (0 :_R n)$ and then $m - n$ is not an edge of $AG(M)$. The converse is clear by Lemma 4.1 (2). □

In view of Theorem 5.5 we have the following result.

Corollary 5.6. *Let R be a commutative ring and M be an R-module M for which $(0 :_R M)$ is a prime ideal.Then $AG(M) = \Gamma_*(M)$.*

Proposition 5.7. *Let M be an R-module. Then the following hold.*
(1) If M is a reduced multiplication-like uniform module, then $AG(M) = \Gamma_*(M) = \emptyset$.
(2) $AG(M) = \Gamma_*(M) = \emptyset$ if and only if M is a multiplication-like uniform module and $(0 :_R M)$ is a prime ideal.

Proof. (1) Let $m \in \tilde{Z}_*(M)$. Then there exists $n \in \tilde{Z}_*(M)$ such that $I_m I_n M = 0$. Since M is a uniform module, we have $Rm \cap Rn \neq 0$. Let $0 \neq x \in Rm \cap Rn$. Then $x = rm = sn$ for some nonzero elements $r, s \in R$. So $I_x^2 M = I_{rm} I_{sn} M \subseteq I_m I_n M = 0$. Therefore $I_x M = 0$ since M is a reduced module. Hence $I_x = (0 :_R M)$ which is a contradiction. So $\tilde{Z}_*(M) = \emptyset$
(2) Let $AG(M) = \emptyset$. Then for every nonzero elements $m, n \in M$, m and n are not adjacent. If $Rm \cap Rn = 0$, then $I_m I_n M \subseteq Rm \cap Rn = 0$ which is a contradiction. So $Rm \cap Rn \neq 0$ for every nonzero elements $m, n \in M$. This implies that M is a uniform module. Now, we show that M is a multiplication-like module. Let $0 \neq x \in M$. If $(0 :_R M) = I_x$, then $I_x M = 0$. Therefore $I_x I_y M = 0$ for every $0 \neq y \in M$ which is a contradiction. So $(0 :_R M) \subset I_x$ for every $0 \neq x \in M$ and then M is multiplication-like by [8, Lemma 2.3]. Now, suppose that $rs \in (0 :_R M)$ and $r, s \notin (0 :_R M)$. So there exist $x, y \in M$ such that $rx \neq 0$ and $sy \neq 0$. Since $rsM = 0$, it follows that $s \in (0 :_R rx)$ and $r \in (0 :_R sy)$. Then $I_{rx} I_{sy} M = 0$ by Lemma 4.3. Hence $rx, sy \in \tilde{Z}_*(M)$ which is a contradiction. Thus $(0 :_R M)$ is a prime ideal.
Conversely, let M is a multiplication-like uniform module and $(0 :_R M)$ is a prime ideal. Then it is clear that M is a reduced R-module. So the result is clear by (1). □

Lemma 5.8. *Let M be a simple R-module. Then $AG(M) = \Gamma_*(M) = \emptyset$.*

Proof. Since M is a simple module, $Rm = M$ for every nonzero element $m \in M$. Then $I_m = R$. So for every elements $m, n \in M$, we have $I_m I_n M = M \neq 0$ and thus $\tilde{Z}_*(M) = \emptyset$. □

Theorem 5.9. *Let M_1 and M_2 be two isomorphic simple R-modules and $M = M_1 \oplus M_2$. Then $Z_*(M) = M$ and $AG(M) = \Gamma_*(M)$ is a complete graph.*

Proof. By [14, Theorem 2.1], we can assume that $M = M_1 \oplus M_1$ and we have $(R(x, y) :_R M) = (0 :_R M_1)$ for every elements $x, y \in M_1$. So $I_{(x,y)} = (0 :_R M_1)$ for every elements $x, y \in M_1$. Now, let $(0, 0) \neq (a, b) \in M$. Then $I_{(x,y)}I_{(a,b)}M = (0 :_R M_1)^2M = 0$. So $\tilde{Z}_*(M) = M \setminus \{0\}$ and $\Gamma_*(M)$ is a complete graph. Thus $AG(M)$ is a complete graph. □

We end this paper with the following proposition that shows for a reduced multiplication-like R-module M, if $Z_*(M) = M$, then $\Gamma_*(M)$ is not a bipartite graph.

Proposition 5.10. *Let M be a reduced multiplication-like R-module. If $\Gamma_*(M)$ is a bipartite graph with parts V_1 and V_2 then $Z_*(M) \neq M$.*

Proof. Suppose that $Z_*(M) = M$. Then $\tilde{Z}_*(M) = V_1 \cup V_2$. Let $\bar{V_i} = V_i \cup \{0\}$ for $i = 1, 2$. We show that $\bar{V_1}$ is a submodule of M. Let $x_1, x_2 \in \bar{V_1}$ and $r \in R$. We have to show that $x_1 + x_2, rx_1 \in \bar{V_1}$. If $rx_1 = 0$, then $rx_1 \in \bar{V_1}$, so suppose that $rx_1 \neq 0$. Since $\Gamma_*(M)$ is a bipartite graph, x_1 is adjacent to an element of V_2, say y_1. Then $I_{x_1}I_{y_1}M = 0$, so $I_{rx_1}I_{y_1}M = I_{x_1}I_{y_1}M = 0$. Since $y_1 \in \bar{V_2}$, we have $rx_1 \in \bar{V_1}$.
If $x_1 + x_2 = 0$, then it is clear that $x_1 + x_2 \in \bar{V_1}$. Now suppose that $0 \neq x_1 + x_2 \notin V_1$, then $x_1 + x_2 \in V_2$ since $Z_*(M) = M = V_1 \cup V_2$. So $I_{x_1}I_{x_1+x_2}M = I_{x_2}I_{x_1+x_2}M = 0$. This implies that $Rx_1 \cap R(x_1 + x_2)M = Rx_2 \cap R(x_1 + x_2)M = 0$ by Lemma 2.2 and Proposition 3.2. If $I_{x_1+x_2} = 0$, then $x_1 + x_2$ is adjacent to every vertices which is a contradiction. Hence $I_{x_1+x_2} \neq 0$. Let $(0 :_R x_1) = 0$ and $0 \neq r \in I_{x_1+x_2}$. Then $rM \subseteq R(x_1+x_2)$. So $0 \neq rx_1 \in rM \subseteq R(x_1+x_2)$; thus $0 \neq rx_1 \in Rx_1 \cap R(x_1 + x_2)$ which is a contradiction. Therefore $(0 :_R x_1) \neq 0$, Similarly $(0 :_R x_2) \neq 0$. If $r \in (0 :_R x_1) \setminus (0 :_R x_2)$, then $rx_1 = 0$ and $rx_2 \neq 0$. So $0 \neq rx_2 = r(x_1 + x_2) \in Rx_2 \cap R(x_1 + x_2)$ which is a contradiction. The result is similar when $r \in (0 :_R x_2) \setminus (0 :_R x_1)$. Now, assume that $(0 :_R x_1) = (0 :_R x_2) \neq 0$. We may assume that $(0 :_R x_1 + x_2) \neq 0$ since $(0 :_R x_1) \subseteq (0 :_R x_1 + x_2)$. If $(0 :_R x_1 + x_2) = I_{x_1+x_2}$, then $I^2_{x_1+x_2}M = I_{x_1+x_2}(I_{x_1+x_2}M) \subseteq I_{x_1+x_2}R(x_1 + x_2) = 0$, then $I_{x_1+x_2}M = 0$ since M is reduced. So $x_1 + x_2$ is adjacent to all vertics which is a contradiction. Therefore $(0 :_R x_1 + x_2) \neq I_{x_1+x_2}$. Let $s \in I_{x_1+x_2} \setminus (0 :_R x_1 + x_2)$. If $sx_1 = 0$, then $sx_2 = 0$ since $(0 :_R x_1) = (0 :_R x_2)$ which is a contradiction. Hence $sx_1 \neq 0$. So $sx_1 \in sM \subseteq R(x_1 + x_2)$, thus $0 \neq sx_1 \in Rx_1 \cap R(x_1 + x_2)$ which is

a contradiction. Therefore we have $x_1 + x_2 \in V_1$. Similarly, V_2 is a submodule of M.
Now suppose that $m \in \tilde{Z}_*(M)$. We can assume that $m \in V_1$. So there exists $n \in \tilde{Z}_*(M)$ such that $I_m I_n M = 0$. Thus $n \in \bar{V_2}$. If $m + n = 0$, then $m = -n \in V_1 \cap V_2$ which is a contradiction. So, let $m + n \neq 0$. Then $m + n \in \tilde{Z}_*(M)$, since $Z_*(M) = M$. Let $m + n \in V_1$. Then $m + n \in \bar{V_1}$. Therefore $n \in \bar{V_1} \cap \bar{V_2} = 0$ which is a contradiction. The proof is similar if $m + n \in V_2$. □

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