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SOME RESULTS ON THE QUOTIENT OF CO-M MODULES

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Abstract. In this paper, among various results, we prove that if M is a cancellation co-m R-module and L is a non-zero simple submodule of M, then M/L is a co-m R-module. We investigate various conditions under which the quotient module M/N of a co-m module M is also co-m. Moreover, if M is a cancellation Noetherian co-m R-module, then for every second submodule N of M the quotient module M/N is also a co-m R-module. We prove some results concerning socle and radical of co-m modules.

1. Introduction

Throughout this paper all rings are assumed commutative with nonzero identity and all modules are unitary. Multiplication modules play important roles in module theory and have been studied by many authors extensively, see [3], [10], and [22]. We denote the set of all submodules of M by $L(M)$, and also $L^*(M) = L(M) \setminus \{0, M\}$. The notion of $\mathrm{Max}(M)$ denote the set of all maximal submodules of M. A proper submodule N of M is said to be *prime* if from $rm \in N$ for $r \in R$ and $m \in M$ we can deduce that $r \in (N : M)$ or $m \in N$. The set of all prime submodules of M, denoted by $\mathrm{Spec}(M)$. For any submodule N of an R-module M, we define $\mathrm{V}(N)$ to be the set of all prime submodules of M containing N. The singular submodule of a module M will be denoted by $Z(M)$, and is $Z(M) := \{m \in M : \mathrm{Ann}(m) \leq_e R\}$. Then M is called *singular* if $Z(M) = M$ and is *nonsingular* if $Z(M) = \{0\}$

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[21, p.247]. The *second singular submodule* $Z_2(M)$ is the submodule containing $Z(M)$ such that $Z_2(M)/Z(M) = Z(M/Z(M))$ [19, p.37].

An R-module M is called a *multiplication module*, if every submodule N of M has the form $N = IM$ for some ideal I of R. Note that therefore $I \subseteq \mathrm{Ann}_R(M/N)$ and then $N = IM \subseteq \mathrm{Ann}_R(M/N)M \subseteq N$. Therefore $N = \mathrm{Ann}_R(M/N)M$. We note that submodules of a multiplication R-module M need not be multiplication R-modules. It is clear that every cyclic module is multiplication, and that a multiplication module over a local ring is cyclic, see [10, Propostion 4].

According to [5, Definition 3.1], an R-module M is called comultiplication (co-m for short), if for every submodule N of M, there exists an ideal I of R such that $N = \mathrm{Ann}_M(I)$. It is clear that if M is a co-m R-module, then M is a faithful co-m R/I-module such that $I = \mathrm{Ann}_R(M)$, since $\mathrm{Ann}_{R/I}(M) = I = 0_{R/I}$. In [5, Lemma 3.7], it is shown that M is a co-m R-module if and only if for each submodule N of M, we have $N = (0 :_M \mathrm{Ann}_R(N))$. By [5, Example 3.8], every semisimple ring as a module over itself is co-m. Note that every submodule of a co-m module M is also co-m [5, Theorem 3.17 (d)]. But the quotient modules of a co-m module are not necessarily co-m, see [7, Example 2.6].

We investigate some various conditions on module M and some submodules N of M under which the quotient module M/N of a co-m module M is also co-m. We use $N \leq M$ (resp., $N \ll M$, $N \leq_e M$) to indicate that N is a submodule (resp., a small submodule, an essential submodule) of M. For a submodule K of a module M, Zorn's lemma guarantees the existence of a submodule L of M such that L is a maximal essential extension of K in M. A maximal essential extension of N in M is called a *closure* of N in M. A submodule N of an R-module M is said to be *pure* if $IN = IM \cap N$ for every ideal I of R [4, p.232]. Hence, an ideal I of a ring R is *pure*, if $IJ = I \cap J$ for any ideal J of R. A submodule N of an R-module M is said to be *copure* if $(N :_M I) = N + (0 :_M I)$ for every ideal I of R [7, Definition 2.7]. An R-module M is said to be *fully copure* if every submodule of M is copure. It is possible for a co-m R-module M, to have a submodule N for which there exist two ideals $I \neq J$ with the property $(0 :_M I) = N = (0 :_M J)$. For example, if $M = \mathbb{Z}_{2^\infty}$, then $(0 :_M 2\mathbb{Z}) = (0 :_M 6\mathbb{Z})$. It is easy to see that for each submodule N of M there exists a unique ideal I of R such that $N = (0 :_M I)$ if and only if M is co-m and satisfies the double annihilator condition (DAC for short) that is, $\mathrm{Ann}_R(\mathrm{Ann}_M(I)) = I$ for each ideal I of R. Modules with this property are called *strong comultiplication* (s-co-m for short) modules, see [7, Definition 2.1].

An R-module M is said to be *distributive* if the lattice of its submodules is distributive, i.e. $(X+Y)\cap Z=(X\cap Z)+(Y\cap Z)$ for any of its submodules X, Y and Z [12].

2. Preliminaries

In this section, we remind some of the definitions and points that we need in the next sections. We refer the reader to [2], [4], [14], [19], and [21] for all concepts and basic properties of rings and modules not defined here.

Definition 2.1. Let M be an R-module and N be a submodule of M.

(i) N is called *small* (*superfluous*) if for every submodule L of M, $L+N=M$, implies that $L=M$. We denote the set of all small submodules of M by $S(M)$ [4, p.72].

(ii) N is called *essential* (*large*) if for every non-zero submodule L of M, $N\cap L\neq 0$, or equivalently for every submodule L of M, $N\cap L=0$, implies that $L=0$. We denote the set of all essential submodules of M by $ess(M)$ [4, p.72].

(iii) M has the property $(*)$, if for every two ideals I and J of R, $(0:_M I)+(0:_M J)=(0:_M I\cap J)$ $(*)$.

We note that a s-co-m module is a module M with property $(*)$.

Example 2.2. Let $\mathbb{Z}$ be the ring of integer numbers.

(i) Consider $M=\mathbb{Z}_4$ as an $\mathbb{Z}$-module, then the only proper submodule $N=\{\overline{0},\overline{2}\}$ of M is equal to $N=(\overline{0}:_{\mathbb{Z}_4} 2\mathbb{Z})$. Therefore $\mathbb{Z}_4$ is a co-m $\mathbb{Z}$-module.

(ii) Let p be a prime number and consider the $\mathbb{Z}$-module $M=\mathbb{Z}_{p^\infty}$. Choose $N=\mathbb{Z}(1/p+\mathbb{Z})$ and set $I=\mathbb{Z}p^i$, $i\geq 0$. It is clear that $N=\mathrm{Ann}_M(I)$. Therefore $M=\mathbb{Z}_{p^\infty}$ is a co-m $\mathbb{Z}$-module, see [5, Example 3.2].

(iii) Consider $M=\mathbb{Z}$ as a $\mathbb{Z}$-module, then M is not co-m, because $\mathbb{Z}=(0:_{\mathbb{Z}}\mathrm{Ann}_{\mathbb{Z}}(2\mathbb{Z}))\neq 2\mathbb{Z}$, see [5, Example 3.9].

Definition 2.3. Let M be an R-module.

(i) M is *semisimple* if every submodule N of M is a direct summand of M [4, p.116].

(ii) M is *finitely generated* if for every family $\mathfrak{A}$ of submodules of M, $\sum\mathfrak{A}=M$ implies that $\sum\mathfrak{F}=M$ for some finite subset $\mathfrak{F}$ of $\mathfrak{A}$ [4, p.123]. Dually, a module M is *finitely cogenerated* in case for every family $\mathfrak{A}$ of submodules of M, $\bigcap\mathfrak{A}=0$ implies that $\bigcap\mathfrak{F}=0$ for some finite subset $\mathfrak{F}$ in $\mathfrak{A}$ [4, p.124].

(iii) The radical Jacbson of M, denoted by $\mathrm{Rad}(M)$ which is equal to $\mathrm{Rad}(M) = \bigcap_{L \in \mathrm{Max}(M)} L = \sum_{N \ll M} N$. If M does not have maximal submodules, we put $\mathrm{Rad}(M) = M$, see [4, Proposition 9.13].

(iv) The *socle* of M is defined to be the sum of the minimal non-zero submodules of M. It can be considered as a dual notion to that of the radical Jacbson of a module. In set notation, $\mathrm{soc}(M) = \sum_{N \in \mathrm{Min}(M)} N = \bigcap_{E \in ess(M)} E$, see [4, Proposition 9.7].

(v) A ring R is said to be a *CS-ring* if every complement ideal is a direct summand of R_R, also called *extending ring* [21, p.222].

Definition 2.4. Let M be an R-module.

(i) An ideal I of R is called an *M-cancellation ideal*, if for all submodules N and K of M, the equality $IN = IK$ implies that $N = K$. A ring R is called an *M-cancellation ideal ring*, if every non-zero ideal I of R is an M-cancellation ideal. In this case, we say that M is a *cancellation module*.

(ii) A non-zero submodule N of M is said to be *second* if for each $a \in R$, the homomorphism $N \xrightarrow{a} N$ is either surjective or zero. This implies that $\mathrm{Ann}_R(N)$ is a prime ideal of R [7, Definition 1.1].

(iii) The *M-radical* of a submodule N of M is the intersection of all prime submodules of M containing N, denoted by $\mathrm{rad}_M(N)$ or briefly, $\mathrm{rad}(N)$, i.e., $\mathrm{rad}(N) = \bigcap_{P \in V(N)} P$. If N is not contained in any prime submodule, then $\mathrm{rad}(N) = M$.

(iv) A submodule N of M is called *quasi-copure* if every proper prime submodule P containing N is a copure submodule of M [24, Definition 2.4].

Definition 2.5. Let M be an R-module.

(i) The submodule N of M is called *closed* if it has no proper essential extension in M, i.e. if $N \leq_e K$ for some $K \subseteq M$, then $K = N$ [19, page18]. Also N is *coclosed* in M, if whenever $N/K \ll M/K$ for some $K \subseteq M$ implies that $N = K$, i.e., N/K is not small in M/K for any proper submodule K of N [20].

(ii) M is said to be *Hopfian* if every surjective R-endomorphism $f \in \mathrm{End}_R(M)$ is an isomorphism. Also M is said to be *generalized Hopfian* (gH for short) if for every surjective R-endomorphism $f \in \mathrm{End}_R(M)$, we have $\ker f \ll M$ [17, p.325].

3. On the quotient of co-m modules

The aim of this section is to investigate some results on the quotient of co-m modules. For any unexplained notions or terminology please see [2, 4, 14, 19, 21]. Regular modules have been studied under different definitions. A module M is called *regular* if each submodule N of M is (Cohn) pure in M, i.e., the inclusion $0 \to A \to B$ remains exact upon tensoring by any (right) R-module. We recall that a ring R is *regular* if every element in R is von Neumann regular [19, p.10]. Equivalently, R is regular if it is regular as an R-module, i.e., every ideal I of R is pure.

Now we state and prove the following theorem, which is one of the main result of this section.

Theorem 3.1. *Let M be a co-m R-module. Then the following statements are true.*

(i) *If M is a cancellation module and $0 \neq L \in \mathrm{Min}(M)$, then M/L is also a co-m R-module.*
(ii) *If M is a cancellation module and $N \in \mathrm{Min}(M)$, then for every $\mathfrak{m} \in \mathrm{Max}(R)$, $M_{\mathfrak{m}}/N_{\mathfrak{m}}$ is a co-m $R_{\mathfrak{m}}$-module.*
(iii) *Assume that $S = \{K < M : K \neq N, N \cap K \neq 0\}$ for a submodule $N \in L^*(M)$. If $|S| = 0$, then M/N is a simple co-m R-module. Moreover, in this case $L^*(M) = \{N\}$ or M is the direct sum of two copure submodules.*
(iv) *If M is cancellation and $\mathfrak{m} \in \mathrm{Max}(R)$, then $M/(0 :_M \mathfrak{m})$ is a co-m R-module.*
(v) *If M is either semisimple or fully pure, then for every submodule N of M, M/N is also a co-m R-module.*

Proof. i) Let I be an M-cancellation ideal of R and L be a non-zero simple submodule of M. Clearly, $I(L :_M I) \subseteq L$. It follows that $I(L :_M I) = 0$ or $I(L :_M I) = L$.

Case i. Suppose that $I(L :_M I) = 0$, then $(L :_M I) \subseteq (0 :_M I) \subseteq L + (0 :_M I)$. Conversely, it is clear that $L + (0 :_M I) \subseteq (L :_M I)$.

Case ii. Let $I(L :_M I) = L$. We note that $I(L + (0 :_M I)) = IL \subseteq L$. If $IL = 0$, then $IL = I0 = 0$, and since I is an M-cancellation ideal, hence $L = 0$, which is impossible. It implies that $IL = L$ and so $I(L :_M I) = L = I(L + (0 :_M I)) = IL$. Since I is an M-cancellation ideal, hence $(L :_M I) = L + (0 :_M I)$. Moreover, in this case, if M is co-m, then by [25, Theorem 2.1], M/L is also a co-m R-module.

ii) We know that M is a cancellation R-module if and only if $M_{\mathfrak{m}}$ is a cancellation $R_{\mathfrak{m}}$-module for every maximal ideal $\mathfrak{m}$ of R. Moreover, since $N \in \mathrm{Min}(M)$ hence $N_{\mathfrak{m}} \in \mathrm{Min}(M_{\mathfrak{m}})$. By virtue of (i), the proof is complete.

iii) It is clear that N is a simple submodule of M and by part (i), N is a copure submodule of M. Since $|S| = 0$, hence M/N is also a simple R-module. By virtue of [25, Theorem 2.1], M/N is a simple co-m R-module. It follows that $N \in \mathrm{Min}(M) \cap \mathrm{Max}(M)$, therefore $L^*(M) = \{N\}$ otherwise, N is a simple submodule of M and $M = N \oplus K$ for a simple submodule K of M, where by part (i), N and K are copure submodules of M.

iv) Since $N = (0 :_M \mathfrak{m})$ is a simple submodule of M, hence the proof is completed by part (i). Furthermore, in virtue of [6] from the fact that every non-zero submodule N of M contains a simple submodule K of M, therefore M/K is a co-m R-module.

v) By [9, Lemma 3.11] and [9, Corollary 3.16 (b)], M is a fully copure module and by [25, Theorem 2.1] the proof is complete. □

Corollary 3.2. *Let R be a PID and M be a cyclic co-m module generated by a non-torsion element. Then for every $N \in \mathrm{Min}(M)$, M/N is a co-m R-module.*

Proof. Clearly every non-zero element of M is non-torsion. It implies that M is a cancellation module. The proof is complete by Theorem 3.1 (i). □

Corollary 3.3. *Let $M = N \oplus K$ be a co-m R-module and let N, K be cancellation R-modules where $\mathrm{Ann}(N) + \mathrm{Ann}(K) = R$. Then for every $L \in \mathrm{Min}(M)$, M/L is a co-m R-module.*

Proof. Since N, K are cancellation modules where $\mathrm{Ann}(N)+\mathrm{Ann}(K) = R$, hence for two submodules $L = L_1 \oplus L_2$ and $T = T_1 \oplus T_2$ of M and every ideal I of R the equality $IL = IT$ implies that $IL_1 = IL_2$ and also $IT_1 = IT_2$. By hypothesis, N, K are cancellation modules hence $L = T$. This conclude that M is a cancellation module and the proof is complete by Theorem 3.1 (i). □

Remark 3.4. Every submodule N of $\mathbb{Z}$-module $\mathbb{Z}_n$ is copure, where n is square-free. Therefore $\mathbb{Z}_n/N$ is a co-m $\mathbb{Z}$-module.

Theorem 3.5. *Let M and M' be two co-m R-modules. The following assertions hold.*

(i) *If M is a fully copure module, then for every $N \in L^*(M)$, M/N is a co-m R-module.*
(ii) *If R is a Noetherian ring and I is a pure ideal of R, then for every copure submodule N of M, $M/(N :_M I)$ is a co-m R-module.*

(iii) *For every copure submodule N of M and each $P \in \mathrm{Spec}(R)$, M_P/N_P is a co-m R_P-module. Furthermore, if K is a second submodule of M, then N_P is a copure submodule of M_P, where $P = \mathrm{Ann}(K)$.*

(iv) *If R is a* PID, *then for every pure submodule N of M, M/N is a co-m R-module.*

(v) *If R is a principal ideal regular domain, then for any submodule N of M, M/N is a co-m R-module.*

(vi) *If $f \in \mathrm{End}(M)$ is idempotent, then $Im(f)$ is a co-m R-module..*

Proof. i) It is clear by [25, Theorem 2.1].

ii) It follows by [8, Proposition 2.6] and [25, Theorem 2.1].

iii) It is clear since for every $P \in \mathrm{Spec}(R)$, M_P is a co-m R_P-module, and N_P is a copure submodule of M_P. The second part follows from this fact that since K is a second submodule of M, hence $\mathrm{Ann}(K)$ is a prime ideal of R.

iv) It is clear by [8, Theorem 2.12] and [25, Theorem 2.1].

v) By [16, Theorem 4], M is a regular module and hence every submodule of M is a pure submodule. Then the proof is clear by (iv).

vi) By [15, Proposition 2.4] since $f^2 = f$, hence $\ker(f)$ is a copure submodule of M and by [25, Theorem 2.1], $M/\ker(f) \cong Im(f)$ is a co-m R-module. □

Recall that a module M is called *cocyclic* if there exists an essential simple submodule in M. In fact, M is cocyclic, if and only if, M is a uniform module with non-zero socle, if and only if, M is isomorphic to a non-zero submodule of the injective hull of a simple module.

Corollary 3.6. *Let M be an R-module. Then the following statements are true.*

(i) *If M is a cancellation cocyclic module, then* $\mathrm{soc}(M)$ *is a copure submodule of M. In this case, if M is a co-m module, then $M/\mathrm{soc}(M)$ is also a co-m R-module.*

(ii) *If M is a cancellation Noetherian co-m module, then for every second submodule N of M the quotient module M/N is a co-m R-module.*

(iii) *If M is a semisimple co-m module, then every quotient module of M is also a co-m R-module.*

(iv) *If $N_1 \subseteq N_2 \subseteq \cdots$ is an ascending chain of copure submodules of a co-m module M, then $M/\cup_{i=1}^{\infty} N_i$ is a co-m R-module.*

Proof. i) We know that a cocyclic module M over an arbitrary ring R has a simple essential socle. It follows by Theorem 3.1, (i), that $\mathrm{soc}(M)$

is a copure submodule of M. Then by [25, Theorem 2.1], $M/\mathrm{soc}(M)$ is a co-m R-module.

ii) Suppose that M is a Noetherian co-m module. By [6, Theorem 3.1], M has a finite number of second submodules and every second submodule of M is a minimal submodule of M. By virtue of Theorem 3.1 (i), every second submodule is copure and by [25, Theorem 2.1], M/N is a co-m R-module.

iii) One can check that every direct summand of an R-module M is a copure submodule of M. Since M is semisimple, hence every submodule N of M is a direct summand of M. It implies that $M = N \oplus K$ for some $K \in L(M)$. It follows that $M/N \cong K$ is a co-m R-module.

iv) The proof is clear. □

Corollary 3.7. *Let M be an R-module and let $\{N_i\}_{i\in\Lambda}$ be a finite family of copure submodules. Then the following assertions hold.*

(i) *If M is s-co-m, then $M/\oplus_{i\in\Lambda} N_i$ is co-m.*
(ii) *If M is a distributive co-m module, then $M/\cap_{i\in\Lambda} N_i$ is co-m.*

Proof. i) Clearly the class of s-co-m modules is closed under sum of copure submodules, hence $\oplus_{i\in\Lambda}N_i$ is a copure submodule of M and the proof is complete by [25, Theorem 2.1].

ii) It follows by [24, Theorem 3.3 (ii)] and [25, Theorem 2.1]. □

The following theorem shows that if M is s-co-m on a regular ring R, then M/N is a co-m R-module for every submodule N of M.

Theorem 3.8. *Let M be a s-co-m R-module and $N \in L(M)$. Then the following assertions hold.*

(i) *M/N is a co-m R-module if and only if $(0 :_R N)(N :_R K) = (0 :_R K)$ for each submodule K of M with $N \subseteq K$.*
(ii) *If $\mathrm{Ann}_R(N)$ is a pure ideal of R, then M/N is a co-m R-module.*
(iii) *For every pure ideal I of R, the quotient module $M/\mathrm{Ann}_M(I)$ is a co-m R-module. Furthermore, M is a fully copure module if and only if R is a regular ring.*

Proof. i) It follows by [7, Theorem 2.5, (a)].

ii) By [7, Theorem 2.13, (a)], N is a copure submodule of M and the proof follows from [25, Theorem 2.1].

iii) It follows by [7, Theorem 2.13, (b)] and [25, Theorem 2.1]. For second part, since M is s-co-m, hence $N = (0 :_M I)$ for some ideal I of R. By virtue of [7, Theorem 2.13, (b)], an ideal I of R is pure if and only if $N = (0 :_M I)$ is a copure submodule of M. This implies that R

is a regular ring if and only if M is a fully copure module. In this case M/N is a co-m R-module for every $N \in L(M)$. □

Note that if R is an integral domain and there exists a faithful multiplication and co-m R-module M, then R is a field. In this case, M is simple, see [6, Theorem 3.3 (e)].

Theorem 3.9. *Let M be a non-zero co-m R-module. The following assertions are true.*

(i) *If N is a quasi-copure submodule of M, then for every $P \in \mathrm{V}(N)$, M/P is a co-m R-module.*
(ii) *If M is a distributive s-co-m module and N is a quasi-copure submodule of M with $|V(N)| < \infty$, then $M/\mathrm{rad}(N)$ is a co-m R-module.*
(iii) *If M is a finitely generated multiplication module and N is a quasi-copure primary submodule of M, then $M/\mathrm{rad}(N)$ is co-m. Furthermore, for every primary submodule $L \supseteq N$, $M/\mathrm{rad}(L)$ is a co-m R-module.*
(iv) *If M is a Noetherian multiplication module, then for every quasi-copure submodule N of M, $M/\mathrm{rad}(N)$ is a co-m R-module. Moreover, if R is an arithmetical ring, then for quasi-copure submodules N and K of M, $M/\mathrm{rad}(N \cap K)$ is a co-m R-module.*

Proof. i) The proof is clear by definition and [25, Theorem 2.1].

ii) By Corollary 3.7 (i), $\mathrm{rad}(N) = \cap_{P \in \mathrm{V}(N)} P$ is a copure submodule of M and the proof is complete.

iii) By [24, Corollary 3.4 (ii)], $\mathrm{rad}(N)$ is a prime submodule of M containing N. Since N is a quasi-copure submodule of M, therefore $\mathrm{rad}(N)$ is a copure submodule of M and by [25, Theorem 2.1], $M/\mathrm{rad}(N)$ is a co-m R-module. The second part is followed by [24, Theorem 3.3 (i)] and [25, Theorem 2.1].

iv) The proof is clear by [24, Corollary 3.4, (iv)] and part (i). □

4. Some results on radical and socle of a co-m module

In this section we investigate some results concerning socle and radical of co-m modules. Note that $Z_2(M)$ is a closed submodule of M and $Z(M) \leq_e Z_2(M)$. It is known that $Z_2(M)$ is the unique closure of $Z(M)$. It is easy to see that each of $Z(M)$ and $Z_2(M)$ is a fully invariant submodule of M. It can be verified that if $N \leq_e M$, then $Z(M/N) = M/N$. If R is a (right) nonsingular ring, i.e. R_R is nonsingular, then $Z(M) = Z_2(M)$ for every R-module M see ([2], p.153). It is well known that $M/Z_2(M)$ is a nonsingular module.

In the following theorem we investigate some various results related to the radical and socle of co-m modules.

Proposition 4.1. *Let M be a s-co-m R-module. Then the following assertions hold.*

(i) *If* $(0:_R \mathrm{Rad}(M)) = 0$*, then* $\mathrm{soc}(R) = 0$*.*
(ii) *If R is a CS-ring and cogenerates the simple R-modules, then* $\mathrm{Rad}(M)$ *is a small submodule of M and also M has at least a maximal submodule. In addition, if M is nonsingular, then M is semiprimitive.*

Proof. i) By [25, Proposition 2.4], for a submodule N of M, $N \ll M$ if and only if there exists $I \leq_e R$ such that $N = \mathrm{Ann}_M(I)$. According to [7], for a collection $\{M_\lambda\}_{\lambda\in\Lambda}$ of submodules of M the following equality holds $(0:_M \bigcap_{\lambda\in\Lambda} \mathrm{Ann}_R(M_\lambda)) = \sum_{\lambda\in\Lambda}(0:_M \mathrm{Ann}_R(M_\lambda))$. Therefore

$$\mathrm{Rad}(M) = \textstyle\sum_{N\ll M} N = \sum_{I\in F\subseteq ess(R)} \mathrm{Ann}_M(I) = (0:_M \bigcap_{I\in F\subseteq ess(R)} I).$$

We conclude that $(\bigcap_{I\in F\subseteq ess(R)} I)\mathrm{Rad}(M) = 0$. Since $(0:_R \mathrm{Rad}(M)) = 0$, therefore $\mathrm{soc}(R) \subseteq \bigcap_{I\in F\subseteq ess(R)} I = 0$. In this case, R have infinite number of essential ideals.

ii) We note that since R is a CS-ring and cogenerates the simple R-modules by [18, Corollary 2.7], R has a finite essential socle. Therefore $\mathrm{Rad}(M) = (0:_M \mathrm{soc}(R))$, where $\mathrm{soc}(R)$ is an essential ideal of R. By [25, Proposition 2.4], $\mathrm{Rad}(M)$ is a small submodule of M. Hence, $\mathrm{Rad}(M) \neq M$ and so M has at least a maximal submodule.

Furthermore, if M is a nonsingular module, then $\mathrm{soc}(R)\mathrm{Rad}(M) = 0$, and this implies that $\mathrm{Rad}(M) = 0$. Therefore M is semiprimitive. □

Proposition 4.2. *Let M be a faithful s-co-m R-module, and let* $\mathrm{soc}(M)$ *be a pure submodule of M. Then M is a semisimple module. Furthermore,* $\mathrm{Rad}(R) \subseteq \mathrm{Ann}(M)$*.*

Proof. Since M is s-co-m, hence M is finitely cogenerated and by [23, Theorem 1 (2)] for any finitely cogenerated module M, $\mathrm{soc}(M)$ is an essential submodule of M and also a finite direct sum of simple modules. By [25, Theorem 2.5], there exists $I \ll R$ such that $\mathrm{soc}(M) = \mathrm{Ann}_M(I)$. Moreover, since M has the property $(*)$, then by [25, Corollary 2.7], $\mathrm{soc}(M) = (0:_M \mathrm{Rad}(R))$. By virtue of [7, Proposition 2.11 (b)], since $\mathrm{soc}(M)$ is a pure submodule of the faithful s-co-m M, hence $\mathrm{soc}(M) = M$ i.e., M is semisimple. Moreover, it follows that $M = (0:_M \mathrm{Rad}(R))$ and hence $\mathrm{Rad}(R) \subseteq \mathrm{Ann}(M)$. □

Theorem 4.3. *Let M be a s-co-m R-module and let K be a copure submodule of M with $\mathrm{Ann}_R(M/K)=0$. Then for every small submodule N of M, where $N \supseteq K$ there exists $I \leq_e R$ such that $I(N/K)=K$. In particular, if $\mathrm{Ann}_M(\mathrm{soc}(R))=0$, then M/K is a semiprimitive R-module.*

Proof. First we prove that if M has property $(*)$ and K is a copure submodule of M, then M/K has also property $(*)$ as an R-module.

$$\begin{aligned} x+K \in (\bar{0}:_{M/K} I\cap J) &\Rightarrow (I\cap J)(x+K)=(I\cap J)x+K \subseteq K \\ &\Rightarrow (I\cap J)x \subseteq K \Rightarrow x \in (K:_M I\cap J) \\ &= K+(0:_M I\cap J) = K+(0:_M I)+(0:_M J) \\ &= K+(0:_M I)+K+(0:_M J) \\ &\Rightarrow x+K \in \frac{K+(0:_M I)}{K}+\frac{K+(0:_M J)}{K} \\ &= \frac{(K:_M I)}{K}+\frac{(K:_M J)}{K} \\ &= (\bar{0}:_{M/K} I)+(\bar{0}:_{M/K} J). \end{aligned}$$

The converse is clear. Let N be a small submodule of M such that $N \supseteq K$, then N/K is a small submodule of M/K. Because, if $N/K + L/K = M/K$, where $L/K \leq M/K$, then $(N+L)/K = M/K$, and so $N+L=M$ but N is a small submodule of M, hence $L=M$ and so $L/K = M/K$. Since K is copure and $\mathrm{Ann}_R(M/K)=0$, hence M/K is a faithful co-m R-module. Now for every small submodule N of M such that $N \supseteq K$ we have $N/K \ll M/K$ and by virtue of [25, Theorem 2.4], it is true if and only if there exists $I \leq_e R$ such that $N/K = (\bar{0}:_{M/K} I) = (K:_{M/K} I) = (K:_M I)/K = (K+(0:_M I))/K$. This implies that $I(N/K) = (IK+K)/K = K = \bar{0}$ for some ideal $I \leq_e R$. In particular, in this case we obtain that

$$\begin{aligned} \mathrm{Rad}(M/K) &= \sum_{N/K \ll M/K} N/K = \sum_{I\in F\subseteq ess(R)} (K:_M I)/K \\ &= (K+\sum_{I\in F\subseteq ess(R)} (0:_M I))/K \\ &= (K+(0:_M \bigcap_{I\in F\subseteq ess(R)} I))/K \\ &\subseteq (K+(0:_M \mathrm{soc}(R)))/K = K = 0_{M/K}. \end{aligned}$$

It follows that M/K is a semiprimitive R-module. □

Remark 4.4. Let M be a nonsingular R-module, and let $N \leq M$. The following assertions hold.

(i) N is an essential submodule of M if and only if M/N is singular [19, Proposition 1.21].
(ii) The class of all nonsingular R-modules is closed under submodules, direct products, essential extensions, and module extensions. The class of all singular R-modules is closed under submodules, factor modules, and direct sums [19, Proposition 1.22].
(iii) If M is a simple module, then M is a projective module [19, Proposition 1.24]. Therefore every nonsingular semisimple R-module is projective [19, Corollary 1.25].

Theorem 4.5. *Let M be a nonsingular s-co-m R-module. Then the following assertions hold.*

(i) $S(M) = \{0\}$ *and M is a co-semisimple module.*
(ii) *if $M/Z_2(M)$ is a faithful R-module and $Z_2(M)$ is a copure submodule of M, then $M/Z_2(M)$ contain no non-zero small submodule.*
(iii) *if $Z_2(M) \ll M$, then every submodule of M containing $Z_2(M)$ is coclosed.*
(iv) *$M/Z_2(M)$ is Hopfian.*

Proof. i) Since M is nonsingular hence for any essential ideal I of R and every non-zero element $m \in M$, $Im \neq 0$. If N is a small submodule of co-m M, then by [25, Proposition 2.4], there exists $I \leq_e R$ such that $N = \mathrm{Ann}_M(I)$. It follows that $N = 0$ and we conclude that $S(M) = \{0\}$ and so $\mathrm{Rad}(M) = 0$. It implies that M is a V-ring and also M is cogenerated by the class of simple modules hence M is co-semisimple.

ii) Now let $Z_2(M)$ be a copure submodule of M. By virtue of [19, Exercise 20, p.37], $Z(M/Z_2(M)) = 0$, i.e., $M/Z_2(M)$ is a nonsingular faithful co-m R-module and by (i), $S(M/Z_2(M)) = \{0\}$. Moreover, by [1, Corollary 1.7], $M/Z_2(M)$ is semisimple and projective.

In this case, by Remark 4.4 (i), $Z_2(M)$ is not an essential submodule of M, since $M/Z_2(M)$ is a nonsingular R-module and also $M/Z_2(M)$ is a fully copure R-module.

iii) By (i), there is no proper small submodule containing $Z_2(M)$. In particular, if $Z_2(M) \ll M$, then for every submodule $Z_2(M) \subseteq N$, $N/Z_2(M) \ll M/Z_2(M)$ implies that $N \ll M$ and hence $Z_2(M) = N$. This implies that N is coclosed in M.

iv) Let $f : M/Z_2(M) \to M/Z_2(M)$ be a epimorphism of R-modules. Then since $M/Z_2(M)$ is a faithful co-m R-module, hence $M/Z_2(M)$ is gH. It follows that $\ker f \ll M/Z_2(M)$. By (i), we must have $\ker f =$

$\{\bar{0}\}$ and so f is an R-module isomorphism. Therefore $M/Z_2(M)$ is Hopfian. □

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