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ON β-TOPOLOGICAL RINGS

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ABSTRACT. In this paper, we introduce a generalized form of the class of topological rings, namely β-topological rings, by using β-open sets which itself is a generalized form of open sets. Translation of open(closed) sets and multiplication by invertible elements of open(closed) sets of the β-topological rings are investigated. Some other useful results on β-topological rings are also given. Examples of β-topological rings which fails to be topological rings are also provided. We further define β-topological rings with unity in the sequel and presented some results on it.

1. Introduction

The class of $\beta-$topological rings is the extension of the class of topological rings. For better understanding of topological rings, one should know topological groups first. Let G be an abelian group together with some topology τ defined on it, then G is said to be an additive topological group [7] if the following maps

$$\varphi : G \times G \to G,\ (x, y) \mapsto x + y$$
$$\psi : G \to G,\ x \mapsto -x$$

are continuous.
Topological ring [3, 4] is a ring $\mathbf{R}$ with some topology defined on it making it an additive topological group such that the multiplication map $\mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is also continuous, where $\mathbf{R} \times \mathbf{R}$ has product topology. Since 1930s, a lot of work has been done on topological rings. Because of its vast properties and its applications in various disciples

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of mathematics, it is still an interesting area of research for mathematicians. The introduction of generalized forms of open sets like semi-open sets [5], α-open sets [6], β-open sets [1], etc. yield possibility for extending the notion of topological rings to its generalized forms. Study the properties of these generalized forms and investigate their relation with the topological rings is very exciting. The introduction of irresolute topological rings [9] and $\alpha-$irresolute topological rings [8] are recent advancements in this direction.
In this paper, we are introducing one such generalized form of topological rings, titled β-topological rings and its structure is based on β-open sets and β-continuous mappings [1]. This paper has four sections and the first one is introductory while the section 2 contains some preliminary information about the concepts that are used in this paper. Section 3 includes the definition of β-topological rings with some examples on it. It also includes results on translation of open(closed) sets in β-topological rings. Section 4 comprises of some more results on β-topological rings. Also, in this section, we introduce β-topological rings with unity and then present some results based on invertible elements.

2. Preliminaries

Throughout this paper, X and Y mean topological spaces unless stated otherwise. For any $A \subseteq X$, $Cl(A)$ and $Int(A)$ represent the closure of A and the interior of A respectively. The notation ϵ denotes negligibly small positive numbers and ϕ denotes the empty set.

Definition 2.1. Let X be a topological space then $A \subseteq X$ is called β-open [1] if $A \subseteq Cl(Int(Cl(A)))$.

From definition of β-open sets, it is confirm that every open set is a β-open set. But the converse need not be true. For, consider X be set of real numbers $\mathbb{R}$ with usual topology and $A = (3, 4) \cap \mathbb{Q}$, where $\mathbb{Q}$ denotes the set of rational numbers. Then A is β-open in X but it is not open.

The complement of a β-open set in X is called β-closed set. It is also proven in [1] that a subset A of a topological vector space X is β-closed if $Int(Cl(Int(A))) \subseteq A$.

For $A \subseteq X$, the union of all β-open sets in X that are contained in A is called β-interior of A [2] and is denoted by $\beta Int(A)$. On the other hand, the intersection of all β-closed sets containing $A \subseteq X$ is called β-closure of A [2] and is denoted by $\beta Cl(A)$. It is also known that $A \subset X$ to be β-closed in X if and only if $\beta Cl(A) = A$. A point

$x \in X$ is called β-interior point of a subset A if there exists a β-open set U in X containing x such that $x \in U \subseteq A$. The set of all β-interior points of A comprises $\beta Int(A)$. A point $x \in \beta Cl(A)$ if and only if for each β-open set U in X containing x, we have $A \cap U \neq \phi$. The family of all β-open sets in X is denoted by $\beta O(X)$ and that of β-closed sets in X is denoted by $\beta C(X)$.

Definition 2.2. A mapping $f : X \to Y$ from a topological space X to a topological space Y is said to be β-continuous [1] if for each $x \in X$ and each open set V in Y containing $f(x)$, there exists a β-open set U in X containing x such that $f(U) \subseteq V$.

3. β-TOPOLOGICAL RINGS

In this section, we define β-topological rings with some examples on it. We make use of the following notations. We use the standard notations $\mathbb{R}$ and $\mathbb{C}$ for the set of real numbers and complex numbers respectively. We simply denote a ring $(\mathbf{R}, +, .)$ by $\mathbf{R}$. Note that the ring $\mathbf{R}$ is without unity, unless it is stated explicitly.

Definition 3.1. Let $\mathbf{R}$ be a ring and τ be some topology induced on it such that the following conditions are satisfied:

(1)For each $x, y \in \mathbf{R}$ and each open set $O \subseteq \mathbf{R}$ containing $x + y$, there exist β-open sets U and V in $\mathbf{R}$ containing x and y respectively such that $U + V \subseteq O$,

(2)For each $x \in \mathbf{R}$ and each open set O in $\mathbf{R}$ containing $-x$, there exists a β-open set U in $\mathbf{R}$ containing x such that $-U \subseteq O$, and

(3)For each $x, y \in \mathbf{R}$ and each open set $O \subseteq \mathbf{R}$ containing xy, there exist β-open sets U and V in $\mathbf{R}$ containing x and y respectively such that $U.V \subseteq O$.

Then the pair $(\mathbf{R}, \tau)$ is called a β-topological ring.

Examples of β-topological rings:

Example 3.2. Consider $\mathbf{R}$ be the ring of real numbers with usual topology $\mathcal{U}$. Then $(\mathbf{R}, \mathcal{U})$ is a β-topological ring.

Example 3.3. Let $\mathbf{R}$ be any ring with the discrete topology $\mathcal{D}$ on it, then $(\mathbf{R}, \mathcal{D})$ is a β-topological ring.

Now, we discuss translation and negation of open sets and closed sets in β-topological rings.

Theorem 3.4. *Let A be an open set in a β-topological ring $\boldsymbol{R}$ and $x \in \boldsymbol{R}$ be arbitrary. Then, $-A$ and $x + A$ are β-open sets in $\boldsymbol{R}$.*

Proof. Let $y \in -A$, then $-y \in A$. By definition of a β-topological ring, there exists $U \in \beta O(\mathbf{R})$ such that $y \in U$ and $-U \subseteq A$. Thus $y \in U \subseteq -A$ implies that $y \in \beta Int(-A)$. Hence, $-A \in \beta O(\mathbf{R})$.
Next, let $y \in x + A$. Since A is open in $\mathbf{R}$ with $-x + y \in A$, then by definition, there exist $U, V \in \beta O(\mathbf{R})$ such that $-x \in U$, $y \in V$ and $U + V \subseteq A$. Thus, we have $-x + V \subseteq A \Rightarrow V \subseteq x + A \Rightarrow y \in \beta Int(x + A)$. Thus, $x + A \in \beta O(\mathbf{R})$. □

Corollary 3.5. *Let $\boldsymbol{R}$ be a β-topological ring. Then, for any open subset A of $\boldsymbol{R}$, we have:*
(1) $-A \subseteq Cl(Int(Cl(-A)))$, and
(2) $x + A \subseteq Cl(Int(Cl(x + A))), \forall x \in \boldsymbol{R}$.

Theorem 3.6. *Let B be any closed set in a β-topological ring $\boldsymbol{R}$ and $x \in \boldsymbol{R}$, then $-B$ and $x + B$ are β-closed sets in $\boldsymbol{R}$.*

Proof. For proving $-B \in \beta C(\mathbf{R})$, let $y \in \beta Cl(-B)$ and W be any open neighborhood of $-y$ in $\mathbf{R}$. Then, there is $U \in \beta O(\mathbf{R})$ such that $y \in U$ and $-U \subseteq W$. Since $y \in \beta Cl(-B)$, then $-B \cap U \neq \phi$, that is, there exists some $h \in (-B) \cap U$ which implies $-h \in B \cap (-U) \subseteq B \cap W$. Now $B \cap W \neq \phi$ implies $-y \in Cl(B) = B$. That is, $y \in -B$. Thus $-B$ is β-closed in $\mathbf{R}$.
Next, let $y \in \beta Cl(x + B)$. Consider $z = -x + y$ and let W be an open neighborhood of z in $\mathbf{R}$. Since $\mathbf{R}$ is β-topological ring, there exist $U, V \in \beta O(\mathbf{R})$ such that $-x \in U$, $y \in V$ and $U + V \subseteq W$. Since $y \in pCl(x + B)$, there is some $h \in \mathbf{R}$ such that $h \in (x + B) \cap V$. Then $-x + h \in B \cap (-x + V) \subseteq B \cap W$. Thus $-x + y \in Cl(B) = B$, that is, $y \in x + B$. Hence, $x + B$ is β-closed in $\mathbf{R}$. □

Corollary 3.7. *Let $\boldsymbol{R}$ be a β-topological ring and $B \subseteq \boldsymbol{R}$ be any closed subset of $\boldsymbol{R}$. Then the following inclusions hold:*
(1) $Int(Cl(Int(-B))) \subseteq -B$, and
(2) $Int(Cl(Int(x + B))) \subseteq x + B, \forall x \in \boldsymbol{R}$.

Since every open set is β-open, it is obvious that every topological ring is a β-topological ring but the converse is not true in general. This shows that the class of topological rings resides completely inside the class of β-topological rings. Now, we will present some examples of β-topological ring which fails to be a topological ring.

Example 3.8. Consider $\mathbf{R}$ be the ring of real numbers with the topology $\tau = \{\phi, \mathcal{I}, \mathbb{R}\}$, where $\mathcal{I}$ denotes the set of irrational numbers. Then the pair $(\mathbf{R}, \tau)$ is a β-topological ring. But it fails to be a topological ring. For, consider $\mathcal{I}$ be an open set containing $0 + \sqrt{3} = \sqrt{3}$, there do not exist open sets O_1 containing 0 and O_2 containing $\sqrt{3}$ in $(\mathbf{R}, \tau)$ satisfying $O_1 + O_2 \subseteq \mathcal{I}$.

Example 3.9. Let $\mathbf{R}$ be the ring of real numbers and τ be the topology on $\mathbf{R}$ generated by the base

$$\mathcal{B} = \{(a, b), [c, d) : a, b, c, d \in \mathbb{R}, \ \textit{and} \ 0 < c < d\}.$$

Then one can easily verify that $(\mathbf{R}, \tau)$ is a β-topological ring. It is not a topological ring because $[3, 4)$ be an open set in $(\mathbf{R}, \tau)$ containing $0+3 = 3$, but there do not exit open sets O_1 and O_2 in $(\mathbf{R}, \tau)$ containing 0 and 3 respectively, such that $O_1 + O_2 \subseteq [3, 4)$.

4. Characterizations

This section comprises of some basic properties of β-topological rings. It also includes the definition of β-topological rings with unity and some results on it.

Theorem 4.1. *Let $\boldsymbol{R}$ be a β-topological ring. Then the following mappings:*
(1) $\psi_x : \boldsymbol{R} \to \boldsymbol{R}$ defined by $\psi_x(y) = x + y$, for all $y \in \boldsymbol{R}$ ($x \in \boldsymbol{R}$ is fixed), and
(2) $\varphi : \boldsymbol{R} \to \boldsymbol{R}$ defined by $\varphi(x) = -x$, for all $x \in \boldsymbol{R}$
are β-continuous.

Proof. (1) To prove that ψ_x is β-continuous. Let O be any open set in $\mathbf{R}$. We have to show that $\psi_x^{-1}(O)$ is β-open in $\mathbf{R}$. We have $\psi_x^{-1}(O) = -x+O$. By Theorem 3.4, $-x+O$ is β-open. Hence ψ_x is β-continuous.
(2) Let $x \in \mathbf{R}$ and let W be an open neighborhood of $\varphi(x)$. Since $\mathbf{R}$ be a β-topological ring, there exists a β-open set U containing x such that $-U \subseteq W$. Thus $\varphi(U) \subseteq W$ and this proves that φ is β-continuous at x. Since x is chosen arbitrarily, hence φ is β-continuous. ☐

Now we define a β-topological ring with unity and make use of invertible elements of $\mathbf{R}$ in next results.

Definition 4.2. Let $(\mathbf{R}, \tau)$ be a β-topological ring. If $\mathbf{R}$ itself is a ring with unity, then $(\mathbf{R}, \tau)$ is said to be a β-topological ring with unity. Further, we use the standard notation $\mathbf{R}^*$ for the set of all invertible elements in $\mathbf{R}$.

Theorem 4.3. *For any open set A in a β-topological ring with unity $\boldsymbol{R}$, Ar and rA are β-open in $\boldsymbol{R}$ for each $r \in \boldsymbol{R}^*$.*

Proof. To show that Ar is β-open in $\mathbf{R}$. For, let $x \in Ar$ be arbitrary, then we have $xr^{-1} \in A$. Since $\mathbf{R}$ is a β-topological ring, then we have some β-open sets U, V in $\mathbf{R}$ containing x and r^{-1} respectively such that $U.V \subseteq A$. In particular, $U.r^{-1} \subseteq A$ implies that $U \subseteq Ar \Rightarrow x \in \beta Int(Ar)$. Thus Ar is β-open in $\mathbf{R}$.
Similarly, we can prove that rA is β-open in $\mathbf{R}$. □

Theorem 4.4. *Let $\boldsymbol{R}$ be a β-topological ring with unity and $F \subseteq \boldsymbol{R}$ be a closed set. Then rF and Fr are β-closed in $\boldsymbol{R}$ for each $r \in \boldsymbol{R}^*$.*

Proof. Firstly we will show that rF is β-closed in $\mathbf{R}$. For, let $x \in \beta Cl(rF)$ be arbitrary and W be any open neighborhood of $r^{-1}x$ in $\mathbf{R}$. Then, by definition of β-topological ring, there exist some β-open sets U and V in $\mathbf{R}$ containing r^{-1} and x respectively, such that $U.V \subseteq W$. Since $x \in \beta Cl(rF)$, there is some $h \in rF \cap V$. Now we have

$$r^{-1}h \in F \cap (r^{-1}.V) \Rightarrow r^{-1}h \in F \cap (U.V) \subseteq F \cap W \Rightarrow F \cap W \neq \phi.$$

Thus $r^{-1}x \in Cl(F)$. As F is closed, we have $r^{-1}x \in F$, which implies $x \in rF$. Therefore $\beta Cl(rF) \subseteq rF$. This implies that $rF = \beta Cl(rF)$. That is, rF is β-closed in $\mathbf{R}$.
The proof for Fr to be β-closed in $\mathbf{R}$ follows analogously from the proof of rF being β-closed in $\mathbf{R}$. □

Theorem 4.5. *Let $\boldsymbol{R}$ be a β-topological ring with unity and $A \subseteq \mathbb{R}$. Then, for each $r \in \boldsymbol{R}^*$, the following inclusions hold:*
(1) $r.\beta Cl(A) \subseteq Cl(rA)$.
(2) $r.Int(A) \subseteq \beta Int(rA)$.
(3) $\beta Cl(rA) \subseteq r.Cl(A)$.
(4) $Int(rA) \subseteq r.\beta Int(A)$.

Proof. (1) Let $y \in r.\beta Cl(A)$. Then $y = rx$ for some $x \in \beta Cl(A)$. Let W be an open neighborhood of $y = rx$ in $\mathbf{R}$. Then, we get $U, V \in \beta O(\mathbf{R})$ such that $r \in U$, $x \in V$ and $U.V \subseteq W$. Since $x \in \beta Cl(A)$, there is some $h \in A \cap V$. Now we have

$$rh \in (rA) \cap (U.V) \subseteq (rA) \cap W \Rightarrow (rA) \cap W \neq \phi.$$

By definition of closure of a set, we have $y \in Cl(rA)$. Hence, proved.
(2) Let $x \in r.Int(A)$. This implies $r^{-1}x \in Int(A)$. Since $Int(A)$ be open set in $\mathbf{R}$ containing $r^{-1}x$, we have $U, V \in \beta O(\mathbf{R})$ such that $r^{-1} \in U$, $x \in V$ and $U.V \subseteq Int(A)$. In particular, $r^{-1}.V \subseteq Int(A) \subseteq A$, that is, $V \subseteq rA$. Hence $x \in \beta Int(rA)$.
(3) Let $x \in \beta Cl(rA)$. Let W be an open neighborhood of $y = r^{-1}x$, we obtain β-open sets U containing r^{-1} and V containing x such that

$U.V \subseteq W$. By assumption, there must exist some $h \in rA \cap V$. This gives

$$r^{-1}h \in A \cap (U.V) \subseteq A \cap W \Rightarrow A \cap W \neq \phi.$$

We get $r^{-1}x \in Cl(A)$ and hence $x \in r.Cl(A)$.
(4) Let $x \in Int(rA)$. Then $x = ra$ for some $a \in A$. There exist β-open sets U and V in $\mathbf{R}$ containing r and a respectively, such that $U.V \subseteq Int(rA)$. Now $r.V \subseteq U.V \subseteq Int(rA) \subseteq rA$ implies that $x \in r.\beta Int(A)$. Hence, the result holds. □

Theorem 4.6. *Let $\boldsymbol{R}$ be a β-topological ring with unity and $r \in \boldsymbol{R}^*$. Then the mapping $\varphi_r : \boldsymbol{R} \to \boldsymbol{R}$ defined by $\varphi_r(x) = rx$, for all $x \in \boldsymbol{R}$, is β-continuous.*

Proof. This is straightforward. Hence, omitted. □

Theorem 4.7. *For any set A in a β-topological ring $\boldsymbol{R}$, the following assertions hold:*
(1) $x + \beta Cl(A) \subseteq Cl(x + A)$ for each $x \in \boldsymbol{R}$.
(2) $\beta Cl(x + A) \subseteq x + Cl(A)$ for each $x \in \boldsymbol{R}$.
(3) $x + Int(A) \subseteq \beta Int(x + A)$ for each $x \in \boldsymbol{R}$.
(4) $Int(x + A) \subseteq x + \beta Int(A)$ for each $x \in \boldsymbol{R}$.

Proof. (1) Let $z \in x + \beta Cl(A)$ be arbitrary and W be an open neighborhood of z in $\mathbf{R}$. Since $z \in x + \beta Cl(A)$, then $z = x + y$ for some $y \in \beta Cl(A)$. As W is open neighborhood of $x + y$ in $\mathbf{R}$, there exist β-open sets U and V in $\mathbf{R}$ containing x and y respectively, such that $U + V \subseteq W$. Since $y \in pCl(A)$, then, there is some $a \in A \cap V$. It gives $x + a \in (x + A) \cap U + V \subseteq (x + A) \cap W$. Thus $(x + A) \cap W \neq \phi$. Hence, $z \in Cl(x + A)$.
(2) Let $y \in \beta Cl(x + A)$ be arbitrary. Let W be open neighborhood of $-x + y$ in $\mathbf{R}$. By definition of β-topological ring, we get $U, V \in \beta O(\mathbf{R})$ such that $-x \in U$, $y \in V$ and $U + V \subseteq W$. Since $y \in \beta Cl(x + A)$, then $(x + A) \cap V \neq \phi$. Let $(x + A) \cap V$ has a common element, say h. Now $-x + h \in A \cap (-x + V) \subseteq A \cap (U + V) \subseteq A \cap W$. Thus, $-x + y \in Cl(A)$. Hence, $y \in x + Cl(A)$.
(3) Let $y \in x + Int(A)$. Then we have β-open sets U and V in $\mathbf{R}$ containing $-x$ and y respectively, such that $U + V \subseteq Int(A) \subseteq A$. In particular, $-x+V \subseteq A$ which implies $V \subseteq x+A$. Thus $y \in \beta Int(x+A)$.
(4) Let $y \in Int(x + A)$. Since $y \in Int(x + A)$, then $y = x + a$ for some $a \in A$. Thus, there exist β-open sets U and V in $\mathbf{R}$ such that $x \in U$, $a \in V$ and $U + V \subseteq Int(x + A) \subseteq (x + A)$. Hence we conclude that $y = x + a \in x + \beta Int(A)$. This completes the proof. □

By using similar arguments as in the above result, we derive the following result.

Theorem 4.8. *Let $\boldsymbol{R}$ be a β-topological ring and A be a subset of $\boldsymbol{R}$. Then we have:*
(1) $-\beta Cl(A) \subseteq Cl(-A)$.
(2) $\beta Cl(-A) \subseteq -Cl(A)$.
(3) $-Int(A) \subseteq \beta Int(-A)$.
(4) $Int(-A) \subseteq -\beta Int(A)$.

Theorem 4.9. *Let A be a subset of a β-topological ring $\boldsymbol{R}$ and $x \in \boldsymbol{R}$. Then we have:*
(1) $x + Int(Cl(Int(A))) \subseteq Cl(x + A)$.
(2) $Int(Cl(Int(x + A))) \subseteq x + Cl(A)$.
(3) $x + Int(A) \subseteq Cl(Int(Cl(x + A)))$.
(4) $Int(x + A) \subseteq x + Cl(Int(Cl(A)$.

Proof. (1) We know $Cl(x + A)$ is closed in $\mathbf{R}$. By Theorem 3.6, $-x + Cl(x + A)$ is β-closed in $\mathbf{R}$. Then $Int(Cl(Int(-x + Cl(x + A)))) \subseteq -x + Cl(x + A)$. Thus $Int(Cl(Int(A))) \subseteq -x + Cl(x + A)$. Hence the assertion follows.
(2) As a consequence of Theorem 3.6, $x+Cl(A)$ is β-closed in $\mathbf{R}$. Then $Int(Cl(Int(x + A))) \subseteq Int(Cl(Int(x + Cl(A)))) \subseteq x + Cl(A)$. Thus the assertion holds.
(3) Since $Int(A)$ is open in $\mathbf{R}$, by Theorem 3.4, $x+Int(A)$ is β-open in $\mathbf{R}$. Thus $x+Int(A) \subseteq Cl(Int(Cl(x+Int(A))))$. We know $x+Int(A) \subseteq x + A$, it gives $x + Int(A) \subseteq Cl(Int(Cl(x + A)))$. Hence proved.
(4) By Theorem 3.4, $-x+Int(A)$ is β-open in $\mathbf{R}$. Consequently, $Int(x+A) \subseteq x + Cl(Int(Cl(A)))$. □

The next theorem is analog of the Theorem 4.9.

Theorem 4.10. *Let $\boldsymbol{R}$ be a β-topological ring and A be a subset of $\boldsymbol{R}$. Then we have:*
(1) $-Int(Cl(Int(A))) \subseteq Cl(-A)$.
(2) $Int(Cl(Int(-A))) \subseteq -Cl(A)$.
(3) $-Int(A) \subseteq Cl(Int(Cl(-A)))$.
(4) $Int(-A) \subseteq -Cl(Int(Cl(A)))$.

Theorem 4.11. *Let A and B be subsets of a β-topological ring $\boldsymbol{R}$. Then $\beta Cl(A) + \beta Cl(B) \subseteq Cl(A + B)$.*

Proof. Let $z \in \beta Cl(A) + \beta Cl(B)$, then $z = x + y$ for some $x \in \beta Cl(A)$ and some $y \in \beta Cl(B)$. Let W be an open neighborhood of $z = x + y$

in **R**. Since **R** is a β-topological ring, then there exist β-open sets U containing x and V containing y such that $U + V \subseteq W$. We have $x \in \beta Cl(A)$ and $y \in \beta Cl(B)$, so there is some $g \in A \cap U$ and $h \in B \cap V$. This gives $g+h \in (A+B) \cap (U+V) \subseteq (A+B) \cap W$. Thus, $z \in Cl(A+B)$. Hence proved. □

Theorem 4.12. *Consider* $\boldsymbol{R}$ *to be a* β*-topological ring and* $\boldsymbol{S}$ *be a topological ring. If a ring homomorphism* $f : \boldsymbol{R} \to \boldsymbol{S}$ *is continuous at zero, then* f *is* β*-continuous.*

Proof. Let W be an open neighborhood of $f(x)$ in **S**, where $x \in$ **R**. This implies $W - f(x)$ is open neighborhood of $0 = f(0)$ in **S**. Since f is continuous at zero, there exists an open neighborhood U of 0 in **R** such that $f(U) \subseteq W - f(X)$. This gives $f(x + U) \subseteq W$. By Theorem 3.4, $x + U$ is β-open in $\mathbb{R}$ and hence f is β-continuous at x. Thus, f is β-continuous. □

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