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N-FOLD OBSTINATE AND N-FOLD FANTASTIC (PRE)FILTERS OF EQ-ALGEBRAS

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Abstract. In this paper, the notions of n-fold obstinate and n-fold fantastic (pre)filter in EQ-algebras are introduced and the relationship among n-fold obstinate, maximal, n-fold fantastic and n-fold (positive) implicative prefilters are investigated. Moreover, the quotient EQ-algebra induced by an n-fold obstinate filter is studied and it is proved that the quotient EQ-algebra induced by an n-fold fantastic filter of a good EQ-algebra with bottom element 0 is an involutive EQ-algebra. Finally, the relationships between types of n-fold filters in residuated EQ-algebras is shown by diagrams.

1. Introduction

EQ-algebras were proposed by Novák and De Baets [13, 14]. One of the motivations was to introduce a special algebra as the correspondence of truth values for high-order fuzzy type theory (FTT)[12] that generalizes the system of classical type theory ([1]) in which the sole basic connective is equality. Analogously, the basic connective in (FTT) should be fuzzy equality. Another motivation is from the equational style of proof in logic. It has three connectives: meet $\wedge$, product $\odot$ and fuzzy equality $\sim$. The implication operation $\rightarrow$ is the derived of the fuzzy equality $\sim$ and it together with $\odot$ no longer strictly form the adjoint pair in general. EQ-algebras are interesting and important for studying and researching and residuated lattices are particular cases of

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EQ-algebras. In fact, EQ-algebras generalize non-commutative residuated lattices [4]. From the point of view of logic, the main diference between residuated lattices and EQ-algebras lies in the way the implication operation is obtained. While in residuated lattices it is obtained from (strong) conjunction, in EQ-algebras it is obtained from equivalence. The prefilter theory plays a fundamental role in the general development of EQ-algebras. From a logical point of view, various prefilters correspond to various sets of provable formulas. Some types of prefilters on EQ-algebras based on logical algebras have been widely studied [2, 5, 9, 16], and some important results have been obtained. Since residuated lattices (BL-algebras, $MV-$algebras, MTL-algebras, R_0-algebras) are EQ-algebras, it is natural to extend some notions of residuated lattices to EQ-algebras and study some of their properties.

For example, Haveshki and Eslami introduced the notions of n-fold implicative filters and n-fold positive implicative filters in BL-algebras and they prove some relations between these filters and construct quotient algebras via these filters in [6]. The fantastic filters are studied by Kondo and Dudek [7] and the notion of n-fold fantastic filters and n-fold BL-algebras are defined by Lele [8]. As a generalization of obstinate and fantastic filters in BL-algebras, the notion of n-fold obstinate and n-fold fantastic filters in BL-algebras were proposed by Motamed and Borumand Saeid [10].

So generalization existing results in BL-algebras and residuated lattices, to EQ-algebras is important tool for studying various algebraic and logical systems in special case EQ-algebras. In the theory of EQ-algebras, as various algebraic structures, the notion of (pre)filter is at the center and so the study of EQ-algebras has experienced a tremendous growth over resent years and the main focus has been on some types (pre)filter. This motivates us to extend different types of n-fold (obstinate, fantastic) (pre)filters to EQ-algebras. Hence, we introduce the notions n-fold obstinate and n-fold fantastic (pre)filters in EQ-algebras and investigate the properties and characterized them as it have done in residuated lattices.

This paper is organized as follows: in section 2, the basic definitions, properties and theorems of EQ-algebras are reviewed. In section 3, the notion of n-fold obstinate (pre)filter in EQ-algebras is introduced and several properties and equivalent conditions of them are provided. Moreover, the relationship among n-fold obstinate, maximal and n-fold (positive) implicative prefilters are investigated and the extension property for n-fold obstinate prefilters in EQ-algebras are studied. In section 4, the notion of n-fold fantastic (pre)filter in EQ-algebras is introduced and some characteristics of it is presented. Finally, it is proved that the quotient EQ-algebra induced by an n-fold fantastic filter of a good EQ-algebra with bottom element 0 is an involutive EQ-algebra and the relationships between types of n-fold filters in residuated EQ-algebras is shown by a diagram.

2. Preliminaries

In this section, we present some definitions and results about EQ-algebras that will be used in the sequal.

Definition 2.1. [5, 13] An EQ-algebra is an algebra $(L, \wedge, \odot, \sim, 1)$ of type $(2, 2, 2, 0)$ satisfying the following axioms:
$(E1)$ $(L, \wedge, 1)$ is a $\wedge$-semilattice with top element 1. We set $x \leq y$ if and only if $x \wedge y = x$,
$(E2)$ $(L, \odot, 1)$ is a commutative monoid and $\odot$ is isotone with respect to $\leq$,
$(E3)$ $x \sim x = 1$ (reflexivity axiom),
$(E4)$ $((x \wedge y) \sim z) \odot (s \sim x) \leq z \sim (s \wedge y)$ (substitution axiom),
$(E5)$ $(x \sim y) \odot (s \sim t) \leq (x \sim s) \sim (y \sim t)$ (congruence axiom),
$(E6)$ $(x \wedge y \wedge z) \sim x \leq (x \wedge y) \sim x$ (monotonicity axiom),
$(E7)$ $x \odot y \leq x \sim y$ (boundedness axiom),
For all $s, t, x, y, z \in L$.

EQ-algebra L is called with exchange principle, if $x \to (y \to z) = y \to (x \to z)$, for all $x, y, z \in L$. We denote $\tilde{x} = x \sim 1$ and $x \to y = (x \wedge y) \sim x$, for all $x, y \in L$.
If L contains a bottom element 0, then we may define the unary operation $\neg$ on L by $\neg x = x \sim 0$. Moreover, we denote $x \Leftrightarrow y := (x \to y) \wedge (y \to x)$ and $x \Leftrightarrow^{\circ} y := (x \to y) \odot (y \to x)$, for all $x, y \in L$.

Definition 2.2. [4, 13, 17] Let L be an EQ-algebra. Then we say that it is:
(i) Separated, if $x \sim y = 1$, then $x = y$, for all $x, y \in L$.
(ii) Good, if $\tilde{x} = x$, for all $x \in L$.
(iii) Residuated, if $(x \odot y) \wedge z = x \odot y$ if and only if $x \wedge ((y \wedge z) \sim y) = x$, for all $x, y, z \in L$.
(iv) Involutive(IEQ-algebra), if it contains a bottom element 0 and $\neg\neg x = x$, for all $x \in E$.
(iv) Multiplicatively relative, if $x \sim y \leq (x \odot z) \sim (y \odot z)$, for all $x, y, z \in L$.

Example 2.3. [9] Let $L = (L, \wedge, \vee, \odot, \Rightarrow, 0, 1)$ be a residuated lattice. For any $x, y \in L$, we define $x \sim y = (x \Rightarrow y) \wedge (y \Rightarrow x)$. Then $(L, \wedge, \odot, \sim, 1)$ is a residuated EQ-algebra. It is easily proved that $x \to y = (x \wedge y) \sim x = x \Rightarrow y$, for any $x, y \in L$.

Lemma 2.4. [5] *Let L be an EQ-algebra. Then the following properties hold for any $x, y, z \in L$:*
(i) $x \sim y = y \sim x$, $x \sim y \leq x \to y$, $x \odot y \leq x \wedge y \leq x, y$,
(ii) $x \leq 1 \sim x = 1 \to x \leq y \to x$,
(iii) $x \to y \leq (y \to z) \to (x \to z)$,
(iv) $x \to y \leq (z \to x) \to (z \to y)$,
(v) *If* $x \leq y$, *then* $x \to y = 1$, $x \sim y = y \to x$,
(vi) *If* $x \leq y$, *then* $z \to x \leq z \to x$, $y \to z \leq x \to z$,
(vii) *If L contains a bottom element* 0, *then* $\neg 0 = 1$, $\neg x = x \to 0$,
$(viii)$ *If L separated, then* $x \to y = 1$ *if and only if* $x \leq y$.

Theorem 2.5. [4] *Let L be a residuated EQ-algebra. Then* $(x \odot y) \to z = x \to (y \to z)$, *for any* $x, y, z \in L$.

Definition 2.6. [5] Let L be an EQ-algebra and $\emptyset \neq F \subseteq L$. Then F is called a prefilter of L, if it satisfies for any $x, y \in L$,
$(F1)$ $1 \in F$,
$(F2)$ If $x \in F$ and $x \to y \in F$, then $y \in F$.
A prefilter F is said to be a filter if it satisfies
$(F3)$ If $x \to y \in F$, then $(x \odot z) \to (y \odot z) \in F$, for any $x, y, z \in L$. It is proved in [5] that $\{1\}$ is a filter in any separated EQ-algebra and it is contained in any other filter.

Lemma 2.7. [5] *Let F be a prefilter of EQ-algebra L. Then the following hold, for any $x, y \in L$:*
(i) $x \in F$ *and* $x \leq y$, *imply* $y \in F$,
(ii) $x, x \sim y \in F$ *implies* $y \in F$.
(iii) $x, y \in F$, *implies* $x \odot y \in F$, *when F is a filter of L.*

Definition 2.8. [2, 9, 15, 16] Let L be an EQ-algebra and $\emptyset \neq F \subseteq L$. Then
(i) F is called an n-fold implicative prefilter, if
$(F1)$ $1 \in F$,
$(F7)$ $z \to ((x^n \to y) \to x) \in F$ and $z \in F$ imply $x \in F$, for any $x, y, z \in L$.
An n-fold implicative prefilter is called an n-fold implicative filter, if it satisfies $(F3)$. 1-fold implicative (pre)filter is called an implicative (pre)filter.
(ii) Perfilter F is called an n-fold positive implicative prefilter, if it satisfies:
$(F5)$ $x^n \to (y \to z) \in F$, $x^n \to y \in F$ imply $x^n \to z \in F$, for all $x, y, z \in L$.
If F is a filter and satisfies $(F5)$, then F is called an n-fold positive

implicative filter. 1-fold positive implicative (pre)filter is called a positive implicative (pre)filter.
(iii) Prefilter F of L is called an obstinate prefilter, if it satisfies:
$(F8)$ $x, y \notin F$ implies $x \to y \in F$ and $y \to x \in F$, for all $x, y \in L$.
If F is a filter and satisfies $(F8)$, then F is called an obstinate filter.
(iv) Prefilter F is called a fantastic prefilter of L if it satisfies:
$(F9)$ $y \to x \in F$ implies $((x \to y) \to y) \to x \in F$, for all $x, y \in L$.
If F is a filter and it satisfies $(F9)$, then it is called a fantastic filter of L.
(v) Prefilter F of L is called a maximal prefilter, if it is proper and no proper prefilter of L stricly contains F, that is, for each prefilter, $G \neq F$, if $F \subseteq G$, then $G = L$.

Theorem 2.9. [4] *Let F be a prefilter of EQ-algebra L. Define a relation $\equiv_F$ on L as follows:*

$$x \equiv_F y \;\; \text{if and only if} \;\; x \sim y \in F$$

It follows that $\equiv_F$ is an equivalence relation on L. Let $\frac{L}{F}$ denote the quotient algebra induced by F and $[x]_F$ denote the equivalence class of x with respect to $\equiv_F$. Moreover, if F is a filter, then $\equiv_F$ is a congruence relation on L and quotient algebra $\frac{L}{F}$ is a separated EQ-algebra.

From now on, we let $(L, \wedge, \odot, \sim, 1)$ denote an EQ-algebra, unless otherwise we state.

3. n-Fold Obstinate Prefilter in EQ-algebras

In this section we introduce the concept of n-fold obstinate prefilters in EQ-algebras and we give some related results.

Definition 3.1. A prefilter F of L is called an *n-fold obstinate prefilter* of L, if for all $x, y \in L$,
$(F10)$ $x, y \notin F$ implies $x^n \to y \in F$ and $y^n \to x \in F$.

If F is a filter and satisfies $(F10)$, then F is called an n-fold obstinate filter.

Example 3.2. [2] Let $L = \{0, a, b, c, 1\}$, such that $0 < a, b < c < 1$. The following binary operation $\odot$ and $\sim$ define an EQ-algebra on L

and we have the following $\rightarrow$:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

$\sim$	0	a	b	c	1
0	1	b	a	0	0
a	b	1	1	a	a
b	a	1	1	b	b
c	0	a	b	1	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Then $F = \{a, c, 1\}$ is a 2-fold obstinate prefilter of L.

Proposition 3.3. *Let F be a prefilter of L. Then for any $x, y \in L$:*
(i) If $x, y \notin F$ implies $x^n \sim y^n \in F$, then F is an n-fold obstinate prefilter of L,
(ii) If $x, y \notin F$ implies $x^n \sim y \in F$ and $y^n \sim x \in F$, then F is an n-fold obstinate prefilter of L.

Proof. (i) Suppose that $x, y \notin F$, implies $x^n \sim y^n \in F$, since by Lemma 2.4(i), $x^n \sim y^n \leq x^n \rightarrow y^n, y^n \rightarrow x^n$, so by Lemma 2.7$(i)$, we get that $x^n \rightarrow y^n \in F$ and $y^n \rightarrow x^n \in F$. Moreover, since $y^n \leq y$ and $x^n \leq x$, we get that by Lemma 2.4(vi), $x^n \rightarrow y^n \leq x^n \rightarrow y$ and $y^n \rightarrow x^n \leq y^n \rightarrow x$. Thus, by Lemma 2.7$(i)$, $x^n \rightarrow y \in F$ and $y^n \rightarrow x \in F$. Therefore, F is an n-fold obstinate prefilter of L.
(ii) Suppose that $x, y \notin F$, then $x^n \sim y \in F$ and $y^n \sim x \in F$, since by Lemma 2.4(i), $x^n \sim y \leq x^n \rightarrow y$ and $y^n \sim x \leq y^n \rightarrow x$, so by Lemma 2.7$(i)$, we get that $x^n \rightarrow y \in F$ and $y^n \rightarrow x \in F$. Therefore, F is an n-fold obstinate prefilter of L. □

Proposition 3.4. *Let F be a filter of L and $x^n \Leftrightarrow^\circ y^n \in F$, for all $x, y \in L - F$. Then F is a n-fold obstinate filter.*

Proof. Let $x^n \Leftrightarrow^\circ y^n \in F$ for $x, y \in L - F$. Then $(x^n \rightarrow y^n) \odot (y^n \rightarrow x^n) \in F$. Since by Lemma 2.4$(i)$ and (iv),

$$\begin{aligned}(x^n \rightarrow y^n) \odot (y^n \rightarrow x^n) &\leq (x^n \rightarrow y^n) \wedge (y^n \rightarrow x^n) \\ &\leq (x^n \rightarrow y) \wedge (y^n \rightarrow x) \\ &\leq (x^n \rightarrow y), (y^n \rightarrow x)\end{aligned}$$

By Lemma 2.7(i), we get that $x^n \to y \in F$ and $y^n \to x \in F$ and so F is an n-fold obstinate filter of L. □

Theorem 3.5. *Let L be an EQ-algebra with bottom element 0 and F be a proper prefilter of L. Then F is an n-fold obstinate prefilter if and only if $x \in L - F$, implies $(\neg x^n)^m \in F$, for some natural number m.*

Proof. Suppose that F is an n-fold obstinate prefilter of L and $x \in L$ such that $x \notin F$. Then $\neg x^n = x^n \to 0 \in F$. Hence, for $m = 1$, we have $(\neg x^n)^m \in F$. Conversely, let $x, y \notin F$. Then we show that $x^n \to y \in F$ and $y^n \to x \in F$. By the hypothesis, there are natural numbers m, s such that $(\neg x^n)^m \in F$ and $(\neg y^n)^s \in F$. By $(E2)$ and Lemma 2.4(vi), we have $(\neg x^n)^m \leq \neg x^n \leq x^n \to y$ and $(\neg y^n)^s \leq \neg y^n \leq y^n \to x$ and so by Lemma 2.7(i), we get $x^n \to y \in F$ and $y^n \to x \in F$. Therefore, F is an n-fold obstinate filter of L. □

Corollary 3.6. *Let L be an EQ-algebra with bottom element 0 and F be a proper prefilter of L. Then F is a n-fold obstinate prefilter of L if and only if $x \in F$ or $(\neg x^n)^m \in F$, for all $x \in L$ and for some natural numbers m.*

Definition 3.7. [17] EQ-algebra L is called a local EQ-algebra, if it has only one maximal filter. The order of an element x of a EQ-algebra with bottom element 0, in symbols $ord(x)$, is the least positive integer m such that $x^m = 0$. If no such m exists, then $ord(x) = \infty$. Moreover, We have the following notations:

$$D(L) = \{x \in L | ord(x) = \infty\}$$

Proposition 3.8. [17] *Let L be a multiplicatively relative EQ-algebra with bottom element 0. Then the following statements are equivalent:*
(i) *$D(L)$ is a filter,*
(ii) *$D(L)$ is a proper filter,*
(iii) *L is a local EQ-algebra,*
(iv) *$D(L)$ is a unique maximal filter.*

Theorem 3.9. *Let L be a good EQ-algebra with bottom element 0. Then*
(i) *If F is an n-fold obstinate filter of L, the every nonunit element of EQ-algebra $\frac{L}{F}$ has order n.*
(ii) *If L is a multiplicatively relative EQ-algebra, then F is an n-fold obstinate filter of L if and only if $\frac{L}{F}$ is local and $\{[1]\}$ is a unique maximal and n-fold obstinate filter of $\frac{L}{F}$.*

Proof. (i) Let L be a good EQ-algebra with bottom element 0 and $[x] \neq [1]$. Then $x = x \sim 1 \notin F$ and since F is an n-fold obstinate prefilter, we get that $x^n \sim 0 = x^n \to 0 \in F$ and so $[x]^n = [0]$. Therefore, $ord([x]) = n$, for any $[1] \neq [x] \in \frac{L}{F}$.
(ii) Let L be a multiplicatively relative EQ-algebra. If F is an n-fold obstinate filter of L, then it is clear $\frac{L}{F}$ is a multiplicatively relative EQ-algebra and by (i), we get that $D(\frac{L}{F}) = \{[1]\}$ and since $\frac{L}{F}$ is a separated EQ-algebra, we conclude that $\{[1]\}$ is a filter of $\frac{L}{F}$ and so by Proposition 3.8, $\frac{L}{F}$ is local and $\{[1]\}$ is unique maximal filter of $\frac{L}{F}$. Moreover, if $[1] \neq [x] \in \frac{L}{F}$, then $ord([x]) = n$ and so $[x]^n = [0]$. Hence, $([x]^n)^- = [1]$ and so by Theorem 3.5, $\{[1]\}$ is an n-fold obstinate filter of $\frac{L}{F}$. □

Proposition 3.10. *Every n-fold obstinate prefilter of L is an $(n+1)$-fold obstinate prefilter.*

Proof. Let F be an n-fold obstinate prefilter of L and $x, y \notin F$. Then $x^n \to y \in F$ and $y^n \to x \in F$. We must show that $x^{n+1} \to y \in F$ and $y^{n+1} \to x \in F$. Since $x^{n+1} \leq x^n$ and $y^{n+1} \leq y^n$, so by Lemma 2.4(vi), we get that $x^n \to y \leq x^{n+1} \to y$ and $y^n \to x \leq y^{n+1} \to x$. Hence, by Lemma 2.7$(i)$, we have $x^{n+1} \to y \in F$ and $y^{n+1} \to x \in F$. Therefore, F is an $(n+1)$-fold obstinate prefilter of L. □

By the following example we show that the converse of Proposition 3.10 is not correct in general.

Example 3.11. [2] Let $L = (\{0, a, b, c, d, 1\}, \wedge, \odot, \sim, 1)$ be an EQ-algebra, with $0 < a < b < c < d < 1$ and Cayley tables as follows:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	0	0	a	b
c	0	0	0	a	a	c
d	0	0	a	a	a	d
1	0	a	b	c	d	1

$\sim$	0	a	b	c	d	1
0	1	c	b	a	0	0
a	c	1	b	a	a	a
b	b	b	1	b	b	b
c	a	a	b	1	c	c
d	0	a	b	c	1	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	1	1	1	1	1	1
b	b	b	1	1	1	1
c	a	a	b	1	1	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

Then $\{c, d, 1\}$ is a 2-fold obstinate prefilter of L while it is not a 1-fold obstinate prefilter because $0, b \notin \{c, d, 1\}$ and $b \to 0 = b \notin \{c, d, 1\}$.

Theorem 3.12. *(Extension property)Let F, G be two prefilters of L such that $F \subseteq G$. If F is an n-fold obstinate prefilter of L, then so is G.*

Proof. Let F, G be two prefilter of L such that $F \subseteq G$ and F be an n-fold obstinate prefilter of L. Assume that $x \in L - G$, since $F \subseteq G$, we get that $x \notin F$, and since F is an n-fold obstinate prefilter, by Theorem 3.5, there exists $m \in \mathbb{N}$ such that $(\neg x^n)^m \in F$. Now, by $F \subseteq G$, we get that $(\neg x^n)^m \in G$ and so by Theorem 3.5, we conclude that G is an n-fold obstinate prefilter of L. □

[2] Let A and B be two EQ-algebras. A function $\phi : A \to B$ is a homomorphism of EQ-algebras, if it satisfies the following conditions, for any $x, y \in A$:

$$\phi(1) = 1,$$
$$\phi(x \odot y) = \phi(x) \odot \phi(y),$$
$$\phi(x \sim y) = \phi(x) \sim \phi(y),$$
$$\phi(x \wedge y) = \phi(x) \wedge \phi(y).$$

Note that $\phi(x \to y) = \phi(x) \to \phi(y)$ and for $X \subseteq A$ and $Y \subseteq B$, $\phi(X) = \{\phi(x) \mid x \in X\}$ and $\phi^{-1}(Y) = \{a \in A \mid \phi(a) \in Y\}$. The set of all homomorphisms from A into B is denoted by $Hom(A, B)$.

Proposition 3.13. *Let $\phi \in Hom(A, B)$ and F, G be two n-fold obstinate prefilters of A, B, respectively. Then*
(i) $\phi^{-1}(G)$ is an n-fold obstinate prefilter of A.
(ii) If ϕ is onto, then $\phi(F)$ is an n-fold obstinate prefilter of B.

Proof. Straightforward.

□

For brevity, we need the following notations for all $a, z \in L$ and natural number n:

$a \to^0 z = z$; $a \to^1 z = a \to z$; $a \to^2 z = a \to (a \to z)$; $a \to^n z = a \to (a \to^{n-1} z)$

Definition 3.14. [11] Let L be an EQ-algebra and $\emptyset \neq X \subseteq L$. Then a generated prefilter by X, is the smallest prefilter containing X and denoted by $< X >$. We have

$< X > := \{a \in L \mid \exists x_i \in X \, and \, m \geq 1 \, such \, that \, x_1 \to (x_2 \to \ldots \to (x_m \to a) \ldots) = 1\}$.

Moreover, for a prefilter F of L and $x \in L$,

$$F(x) :=< \{x\} \cup F >= \{a \in L \mid \exists m \geq 1 \, such \, that \, x \to^m a \in F\}$$

Proposition 3.15. *Let F be an n-fold positive implicative prefilter of L and $x \in L - F$. Then*

$$< \{x^n\} \cup F >= \{a \in L \mid x^n \to a \in F\}$$

Proof. Let F be an n-fold positive implicative prefilter of L and $x \in L - F$. Then by Definition 3.14,

$$< \{x^n\} \cup F >= \{a \in L \mid \exists m \geq 1 \, such \, that \, x^n \to^m a \in F\}$$

Since F is an n-fold positive implicative prefilter and $\overbrace{x^n \to (x^n \to \ldots \to x^n \to (x^n}^{m-times} \to a) \ldots) = x^n \to^m a \in F$ and $x^n \to x^n = 1 \in F$, we get that $\overbrace{x^n \to (x^n \to \ldots \to x^n \to (x^n}^{(m-1)-times} \to a) \ldots) = x^n \to^{(m-1)} a \in F$ and by continuing this process, we have $x^n \to a \in F$. Hence, $< \{x^n\} \cup F >= \{a \in L \mid x^n \to a \in F\}$. □

Theorem 3.16. *Let F be a maximal and n-fold positive implicative prefilter of L. Then F is an n-fold obstinate prefilter of L.*

Proof. Assume that F is a maximal and n-fold positive implicative prefilter of L and n is a natural number, $x, y \in L$ such that $x, y \notin F$. Then by Proposition 3.15, $< \{x^n\} \cup F >= \{z \in L | x^n \to z \in F\}$ and since $x \notin F$, we get that $x^n \notin F$. Since if $x^n \in F$, then by Lemma 2.4(i) and we have $x^n \leq x$ and so by Lemma 2.7(i), we get that $x \in F$, which is impossible. Hence, $F \subsetneq < \{x^n\} \cup F > \subseteq L$ and since F is a maximal prefilter of L, we conclude that $< \{x^n\} \cup F >= L$. Now, since $y \in L$, we get that $y \in < \{x^n\} \cup F >$. Thus, $x^n \to y \in F$ and by similarly, we can obtain $y^n \to x \in F$. Therefore, F is an n-fold obstinate prefilter of L. □

Theorem 3.17. *Let L be an EQ-algebra with bottom element 0. Then every n-fold obstinate proper filter of L is a maximal and n-fold positive implicative filter of L.*

Proof. Let F be an n-fold obstinate proper filter of L, G be a filter of L and $F \subseteq G \subseteq L$. If $F \neq G$, then there exists $x \in G$ such that $x \notin F$, so by Corollary 3.9, $(\neg x^n)^m \in F$, for some natural number m, and so we get that $(\neg x^n)^m \in G$. By Lemma 2.4(i), we have $(\neg x^n)^m \leq \neg x^n$ and so by Lemma 2.7(i), we conclude that $x^n \sim 0 = \neg x^n \in G$. Now, since $x \in G$, by Lemma 2.7(iii), we get that $x^n \in G$ and so by Lemma 2.7(i), $0 \in G$. Hence, $G = L$ and so F is a maximal filter of L. Now, let $x^n \to (y \to z) \in F$ and $x^n \to y \in F$, for $x, y \in L$. Then we consider two cases:
case (1): if $z \in F$, then by Lemma 2.4(ii), since by Lemma 2.7(i), $z \leq x^n \to z$, we conclude that $x^n \to z \in F$.
Case (2): If $z \notin F$, then we have two cases:
(i) if $x \notin F$, then since F is an n-fold obstinate filter of L, we conclude that $x^n \to z \in F$.
(ii) If $x \in F$, then by Lemma 2.7(iii), $x^n \in F$ and by $x^n \to y \in F$, we get that $y \in F$. Now, by $x^n \to (y \to z) \in F$ and $x^n, y \in F$, we conclude that $z \in F$, which is impossible. Therefore, F is an n-fold positive implicative filter of L. □

Theorem 3.18. *Every n-fold obstinate prefilter of L is an n-fold implicative prefilter of L.*

Proof. Let $(x^n \to y) \to x \in F$, for $x, y \in L$. We consider two cases:
case (1): If $y \in F$, then by Lemma 2.4(ii), we have $y \leq x^n \to y$ and so by Lemma 2.7(i), we conclude that $x^n \to y \in F$. Now, by hypothesis, $(x^n \to y) \to x \in F$ and since $x^n \to y \in F$, we obtain $x \in F$.
Case (2): If $x, y \notin F$, since F is an n-fold obstinate prefilter, we get that $x^n \to y \in F$ and so we conclude that $x \in F$ by hypothesis, which is contradiction. Therefore, F is an n-fold implicative prefilter of L. □

The following example shows that the converse of Theorem 3.18 is not correct in general.

Example 3.19. [2] Let $L = (\{0, a, b, 1\}, \wedge, \odot, \sim, 1)$ be a chain with Cayley tables as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	1
1	0	a	b	1

$\sim$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	1	1

Then L is an EQ-algebra and $\{b, 1\}$ is a 2-fold implicative prefilter, while it is not a 2-fold obstinate prefilter because $a, 0 \notin \{b, 1\}$ and $a^2 \rightarrow 0 = a \rightarrow 0 = 0 \notin \{b, 1\}$.

Theorem 3.20. [15] *Let F be an n-fold implicative filter of residuated EQ-algebra L. Then F is an n-fold positive implicative filter of L.*

As a direct consequence of Theorem 3.20 and Theorem 3.18, we get the following theorem:

Theorem 3.21. *Every n-fold obstinate prefilter of a residuated EQ-algebra L is an n-fold positive implicative prefilter of L.*

The following example shows that the converse of Theorem 3.21 is not correct in general.

Example 3.22. Let L be the EQ-algebra defined in Example 3.19. One can see that $\{b, 1\}$ is a 2-fold positive implicative prefilter, while it is not a 2-fold obstinate prefilter of L.

By Theorem 3.16, Theorem 3.17, Theorem 3.18, Theorem 3.20 and Theorem 3.21, we conclude the following corollary:

Corollary 3.23. *Let F be a proper filter of residuated EQ-algebra L. Then the following conditions are equivalent:*
(i) F is a maximal and n-fold implicative filter of L ,
(ii) F is a maximal and n-fold positive implicative filter of L ,
(iii) F is an n-fold obstinate filter of L.

By the following diagrams, we show summarizing the relations between different types of n-fold filters in residuated EQ-algebras.

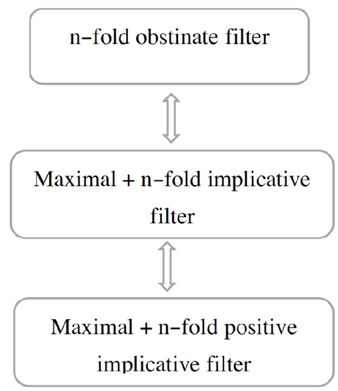

Figure 1: Diagram of n−fold filters in residuated EQ−algebras

4. n-Fold Fantastic Prefilter in EQ-algebras

In this section we introduce the concept of n-fold fantastic prefilters in EQ-algebras and we give some related results.

Definition 4.1. Let F be a prefilter of L. Then F is called an *n-fold fantastic prefilter*, if it satisfies for all $x, y \in L$:
$(F11)$ $y \to x \in F$ implies $((x^n \to y) \to y) \to x \in F$.
If F is a filter and satisfies $(F11)$, then F is called an n-fold fantastic filter. 1-fold fantastic (pre)filter is called a fantastic (pre)filter.

Example 4.2. [16] Let $L = \{0, a, b, 1\}$ be a chain with the following Cayley tables:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	a	b
1	0	a	b	1

$\sim$	0	a	b	1
0	1	b	a	0
a	b	1	b	a
b	a	b	1	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

Then $(L, \wedge, \odot, \sim, 1)$ is an EQ-algebra. One can check that $F = \{1\}$ is a 2-fold fantastic prefilter of L.

Theorem 4.3. [5] *Let L be an EQ-algebra. Then the following are equivalent:*
(i) L is good,
(ii) L is separated and satisfies exchange principle,
(iii) L is separated and satisfies $x \leq (x \to y) \to y$, for any $x, y \in L$.

Theorem 4.4. *Let L be a good EQ-algebra with bottom element 0 and F be a fantastic filter of L. Then $\frac{L}{F}$ is an involutive EQ-algebra.*

Proof. Let L be a good EQ-algebra with bottom element 0 and F be a fantastic filter of L. Then by Theorem 2.9, $\frac{L}{F}$ is an EQ-algebra. Now, since $0 \to x = 1 \in F$ and F is a fantastic filter, we get that $\neg\neg x \to x \in F$ and so $\neg\neg x \wedge x \sim \neg\neg x \in F$, for any $x \in L$. By Theorem 4.3, $x \leq \neg\neg x$, for any $x \in L$ and so $x \sim \neg\neg x = \neg\neg x \wedge x \sim \neg\neg x \in F$, for any $x \in L$ and so $[x] = [\neg\neg x] = \neg\neg[x]$, for any $x \in L$. Therefore, $\frac{L}{F}$ is an involutive EQ-algebra.

□

The following example shows that the converse of Theorem 4.4 is not correct in general.

Example 4.5. [13] Let $L = \{0, a, b, c, d, e, f, 1\}$ be such that $0 < a < c < d < e < 1$, $0 < b < c < d < f < 1$. Multiplication, fuzzy equality and implication on L are defined below.

$\odot$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	0	0	0	0	0	0	a
b	0	0	0	0	0	0	0	b
c	0	0	0	0	0	0	0	c
d	0	0	0	0	d	d	d	d
e	0	0	0	0	d	e	d	e
f	0	0	0	0	d	d	d	f
1	0	a	b	c	d	e	f	1

$\sim$	0	a	b	c	d	e	f	1
0	1	e	f	d	e	a	b	0
a	e	1	d	f	c	a	c	a
b	f	d	1	e	c	c	b	b
c	d	f	e	1	c	c	c	c
d	c	c	c	c	1	f	e	d
e	a	a	c	c	f	1	d	e
f	b	c	b	c	d	d	1	f
1	0	a	b	c	e	e	f	1

$\to$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	e	1	e	1	1	1	1	1
b	f	f	1	1	1	1	1	1
c	d	f	e	1	1	1	1	1
d	c	c	c	c	1	1	1	1
e	a	a	c	c	f	1	f	1
f	b	c	b	c	e	e	1	1
1	0	a	b	c	d	e	f	1

Then $(L, \wedge, \odot, \sim, 1)$ is a good and involutive EQ-algebra. It is clear that $\{1\}$ is a filter of L and $\dfrac{L}{\{1\}}$ is an involutive EQ-algebra, while $\{1\}$ is not a fantastic filter because $a \to e = 1 \in \{1\}$ and $((e \to a) \to a) \to e = e \notin \{1\}$.

Theorem 4.6. *(Extension property)Let L be an EQ-algebra with exchange principle and F, G be two prefilters of L, such that $F \subseteq G$. If F is an n-fold fantastic prefilter of L, then G is an n-fold fantastic prefilter of L.*

Proof. Let L be an EQ-algebra with exchange principle and F, G be two prefilters of L, such that $F \subseteq G$ and $y \to x \in G$. Then $y \to ((y \to x) \to x) = (y \to x) \to (y \to x) = 1 \in F$ and since F is an n-fold fantastic prefilter of L, we get that

$$\big(((((y \to x) \to x)^n \to y) \to y) \to ((y \to x) \to x)\big) \in F \subseteq G$$

Now, with exchange principle we have:
$(y \to x) \to (((((y \to x) \to x)^n \to y) \to y) \to x) = (((((y \to x) \to x)^n \to y) \to y)) \to ((y \to x) \to x) \in G$ and since $y \to x \in G$ and G is a prefilter of L, we conclude that

$$(((((y \to x) \to x)^n \to y) \to y) \to x) \in G$$

Moreover, by Lemma 2.4(i), (iii) and (v), we have

$$\begin{aligned}(((((y \to x) \to x)^n \to y) \to y) \to x) \to (((x^n \to y) \to y) \to x) &\geq \\ ((x^n \to y) \to y) \to ((((y \to x) \to x)^n \to y) \to y) &\geq \\ (((y \to x) \to x)^n \to y) \to (x^n \to y) &\geq \\ x^n \to ((y \to x) \to x)^n &= 1\end{aligned}$$

Hence, $(((((y \to x) \to x)^n \to y) \to y) \to x) \to (((x^n \to y) \to y) \to x) = 1 \in G$ and since G is a prefilter and $(((((y \to x) \to x)^n \to y) \to y) \to x) \in G$, we conclude that $((x^n \to y) \to y) \to x \in F$. Therefore, G is an n-fold fantastic prefilter of L.

□

Proposition 4.7. *Every n-fold fantastic prefilter of L is an $(n+1)$-fold fantastic prefilter of L.*

Proof. Let F be an n-fold fantastic prefilter of L and $y \to x \in F$, for $x, y \in L$. Then $((x^n \to y) \to y) \to x \in F$ and since by Lemma 2.4(i), $x^{n+1} \leq x^n$, by Lemma 2.4(vi), we get that $x^n \to y \leq x^{n+1} \to y$ and so $(x^{n+1} \to y) \to y \leq (x^n \to y) \to y$, hence, $((x^n \to y) \to y) \to x \leq ((x^{n+1} \to y) \to y) \to x$. Now, since $((x^n \to y) \to y) \to x \in F$, by Lemma 2.7($i$), we conclude that $((x^{n+1} \to y) \to y) \to x \in F$. Therefore, F is an $(n+1)$-fold fantastic prefilter of L. □

The following example shows that the converse of Proposition 4.7 may not be true.

Example 4.8. [9] Let $L = \{0, a, b, 1\}$ be a chain with Cayley tables as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	0	b
1	0	a	b	1

$\sim$	0	a	b	1
0	1	a	0	0
a	a	1	a	a
b	0	a	1	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Routine calculation shows that $(L, \wedge, \odot, \sim, 1)$ is a good EQ-algebra. It is easy to check that $F = \{1\}$ is a 2-fold fantastic prefilter but F is not a 1-fold fantastice prefilter because $a \to b = 1 \in F$ but $((b \to a) \to a) \to b = b \notin F$.

Theorem 4.9. [15] *Every n-fold implicative prefilter of L is a prefilter.*

Theorem 4.10. [15] *Let F be a prefilter of L. Then the following are equivalent:*
(i) F is an n-fold implicative prefilter of L,
(ii) $(x^n \to y) \to x \in F$ implies, $x \in F$ for any $x, y \in L$.

Theorem 4.11. *Let L be an EQ-algebra with exchange principle and F be an n-fold implicative prefilter of L. Then F is an n-fold fantastic prefilter of L.*

Proof. Let F be an n-fold implicative prefilter of L. Then by Theorem 4.9, F is a prefilter of L. Now, let $x, y \in L$ such that $y \to x \in F$. Then by Lemma 2.4(ii), we get that $x \leq ((x^n \to y) \to y) \to x$, and by the mathematical induction and by Lemma 2.4(i), we obtain $x^n \leq (((x^n \to y) \to y) \to x)^n$, hence, by Lemma 2.4($vi$), we get that

$$(((x^n \to y) \to y) \to x)^n \to y \leq x^n \to y$$

and so by Lemma 2.4(vi), $(x^n \to y) \to (((x^n \to y) \to y) \to x) \leq ((((x^n \to y) \to y) \to x)^n \to y) \to (((x^n \to y) \to y) \to x)$.

Moreover, by exchange principle property and by Lemma 2.4(iv) and (vi), we have:

$$\begin{aligned} y \to x &\leq ((x^n \to y) \to y) \to ((x^n \to y) \to x) \\ &= (x^n \to y) \to (((x^n \to y) \to y) \to x) \\ &\leq ((((x^n \to y) \to y) \to x)^n \to y) \to (((x^n \to y) \to y) \to x) \end{aligned}$$

and since F is a prefilter, we conclude that $((((x^n \to y) \to y) \to x)^n \to y) \to (((x^n \to y) \to y) \to x) \in F$. Now, since F is an n-fold implicative prefilter, by Theorem 4.10, we conclude that $((x^n \to y) \to y) \to x \in F$. Therefore, F is an n-fold fantastic prefilter of L. □

The following example shows that the converse of Theorem 4.11, is not correct in general.

Example 4.12. [3] Let $L = \{0, a, b, c, d, 1\}$ be a lattice such that $0 < a, b, d, c < 1$, a, b, c, d are pairwise incomparable. Define the operations $\odot$ and $\Rightarrow$ by the two tables.

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	b	d	d	a
b	0	b	b	0	0	b
c	0	d	0	d	d	c
d	0	d	0	d	d	d
1	0	a	b	c	d	1

$\Rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	a	1	b	c	c	1
b	c	a	1	c	c	1
c	b	a	b	1	a	1
d	b	a	b	a	1	1
1	0	a	b	c	d	1

Then $(L, \wedge, \odot, \Rightarrow, 1)$ is a residuated lattice and so $(L, \wedge, \odot, \sim, 1)$ which $x \sim y = (x \Rightarrow y) \wedge (y \Rightarrow x)$ and $\to = \Rightarrow$ is an EQ-algebra(See Example 2.3). We can see $\{1\}$ is a 2-fold fantastic prefilter, but it is not a 2-fold implicative prefilter. Since $(a^2 \to 0) \to a) = (a \to 0) \to a) = 1 \in \{1\}$ and $a \notin \{1\}$.

As a direct consequence of Theorem 3.18 and Theorem 4.11, we conclude the following corollary:

Corollary 4.13. *Let L be an EQ-algebra with exchange principle and F be an n-fold obstinate prefilter of L. Then F is an n-fold fantastic prefilter of L.*

Proposition 4.14. *Let $\phi \in Hom(A, B)$ and F, G be two n-fold fantastic prefilters of EQ-algebras A, B, respectively. Then*
(i) $\phi^{-1}(G)$ is an n-fold fantastic prefilter of A.
(ii) If ϕ is onto, then $\phi(F)$ is an n-fold fantastic prefilter of B.

Proof. Straightforward.

□

Theorem 4.15. [15] *Let F be a filter of residuated EQ-algebra L. Then F is an n-fold positive implicative filter of L if and only if $x^n \to x^{2n} \in F$, for any $x \in L$.*

Proposition 4.16. *Let F be an n-fold fantastic and n-fold positive implicative prefilter of residuated EQ-algebra L. Then F is an n-fold implicative prefilter.*

Proof. Let F be an n-fold fantastic and n-fold positive implicative prefilter of residuated EQ-algebra L and $(x^n \to y) \to x \in F$, for $x, y \in L$. Then by Definition 4.1, we have

$$((x^n \to (x^n \to y)) \to (x^n \to y)) \to x \in F.$$

Since by Lemma 2.4(iii), $(x^n \to x^{2n}) \leq (x^{2n} \to y) \to (x^n \to y)$, by Theorem 2.5, we get that $x^n \to x^{2n} \leq (x^n \to (x^n \to y)) \to (x^n \to y)$. Hence, by Lemma 2.4$(vi)$, we conclude that

$$((x^n \to (x^n \to y)) \to (x^n \to y)) \to x \leq (x^n \to x^{2n}) \to x.$$

Now, since F is a prefilter of L, by Lemma 2.7(i), we get that $(x^n \to x^{2n}) \to x \in F$ and since F is an n-fold positive implicative prefilter of L, by Theorem 4.15, we get $x^n \to x^{2n} \in F$ and so $x \in F$. Therefore, by Theorem 4.10, F is an n-fold implicative prefilter of L. □

As a direct consequence of Theorem 3.20, Theorem 4.11 and Theorem 4.16, we conclude the following corollary:

Corollary 4.17. *Let F be a prefilter of residuated EQ-algebra L. Then F is an n-fold implicative prefilter of L if and only if F is an n-fold positive implicative and n-fold fantastic prefilter.*

By the following diagrams, we show summarizing the relations between different types of n-fold filters in residuated EQ-algebras.

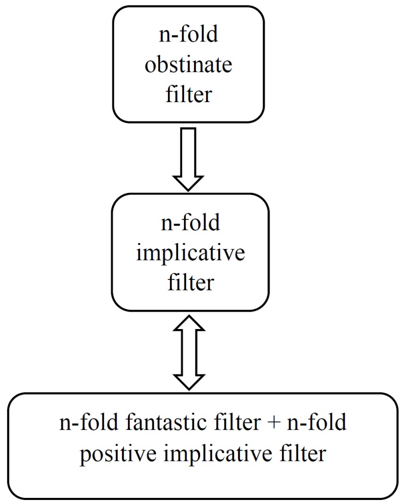

Figure 2: Diagram of n-fold filters in residuated EQ-algebras

5. Conclusion

In this paper, we introduced the notions of n-fold obstinate and n-fold fantastic (pre)filter in EQ-algebras and we studied several properties, characterizations and equivalent conditions of them. Moreover, we investigated the relationship among n-fold obstinate, maximal, n-fold fantastic and n-fold (positive) implicative prefilters and we provided the extension property for n-fold obstinate and n-fold fantastic prefilters in EQ-algebras. Also, we proved that the quotient EQ-algebra induced by an n-fold fantastic filter of a good EQ-algebra with bottom element 0 is an involutive EQ-algebra. Finally, we draw diagram summarizing the relations between different types of n-fold filters in EQ-algebras.

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