Journal of Algebra and Related Topics
Vol. 9, No 1, (2021), pp 21-29

SOME CLASSES OF PERFECT ANNIHILATOR-IDEAL GRAPHS ASSOCIATED WITH COMMUTATIVE RINGS

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ABSTRACT. Let R be a commutative ring and let $\mathbb{A}(R)$ be the set of all ideals of R with nonzero annihilator. The annihilator-ideal graph of R is defined as the graph $A_I(R)$ with the vertex set $\mathbb{A}(R)^* = \mathbb{A}(R) \setminus \{0\}$ and two distinct vertices I and J are adjacent if and only if $\mathrm{Ann}_R(IJ) \neq \mathrm{Ann}_R(I) \cup \mathrm{Ann}_R(J)$. In this paper, perfectness of $A_I(R)$ for some classes of rings is investigated.

1. Introduction

Recently, the assignment of a graph to algebraic structures has been considered by many researchers. When a graph is assigned to a ring, many parameters such as the clique number, the chromatic number and the independence number of the graph, are considered. The relationship between these parameters and the structural properties of the ring is very important. For this reason, translating of the structural properties of the ring with the graph language is very interesting and in some cases it helps to solve problems in the ring more easily. There are a lot of papers which apply combinatorial methods to obtain algebraic results in ring theory; for instance see [2, 4, 5, 9].

We use the standard terminology for graphs following [7, 13]. Let $G = (V, E)$ be a graph, where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges. By $\overline{G}$, we mean the complement graph of G. We write $u - v$, to denote an edge with ends u, v. A graph $H = (V_0, E_0)$ is called a subgraph of G if $V_0 \subseteq V$ and $E_0 \subseteq E$. Moreover,

MSC(2010): Primary: 13A15, 13B99; Secondary: 05C99, 05C25.
Keywords: Annihilator-ideal graph, Perfect graph, Chromatic number, Clique number.
Received: 27 July 2020, Accepted: 27 January 2021.
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H is called an induced subgraph by V_0, denoted by $G[V_0]$, if $V_0 \subseteq V$ and $E_0 = \{u - v \in E \mid u, v \in V_0\}$. Also G is called a null graph if it has no edge. A complete graph of n vertices is denoted by K_n. The join of G_1 and G_2, denoted by $G_1 \vee G_2$, is a graph with the vertex set $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{u - v \mid u \in V(G_1), v \in V(G_2)\}$. A clique of G is a complete subgraph of G and the number of vertices in the largest clique of G, denoted by $\omega(G)$, is called the clique number of G. For a graph G, let $\chi(G)$ denote the vertex chromatic number of G, i.e., the minimal number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. Note that for every graph G, $\omega(G) \leq \chi(G)$. A graph G is said to be weakly perfect if $\omega(G) = \chi(G)$. A perfect graph G is a graph in which the chromatic number of every induced subgraph equals the size of the largest clique of that subgraph.

Let R be a commutative ring with nonzero identity and let $\mathbb{A}(R)$ be the set of all ideals with nonzero annihilator. The annihilator graph of a ring R is defined as the graph $AG(R)$ with the vertex set $Z(R)^* = Z(R) \setminus \{0\}$, and two distinct vertices x and y are adjacent if and only if $\mathrm{Ann}_R(xy) \neq \mathrm{Ann}_R(x) \cup \mathrm{Ann}_R(y)$. This graph was first introduced in [4]. The annihilator-ideal graph of R is defined as the graph $A_I(R)$ with the vertex set $\mathbb{A}(R)^* = \mathbb{A}(R) \setminus \{0\}$ and two distinct vertices I and J are adjacent if and only if $\mathrm{Ann}_R(IJ) \neq \mathrm{Ann}_R(I) \cup \mathrm{Ann}_R(J)$. This graph was first introduced and investigated in [10] and many interesting properties of annihilator-ideal graph were studied, see for examples [1, 6, 12]. In this paper, the perfectness of $A_I(R)$ for some classes of rings R is investigated.

Throughout this paper, all rings are assumed to be commutative with nonzero identity. We denote by $\mathrm{Max}(R)$ and $\mathrm{Nil}(R)$, the set of all maximal ideals of R and the set of all nilpotent elements of R, respectively. For every ideal I of R, we denote the annihilator of I by $\mathrm{Ann}_R(I)$. A nonzero ideal I of R is called essential, if it has a nonzero intersection with any nonzero ideal of R. Some more definitions about commutative rings can be find in [3, 11, 14].

2. Perfectness of Annihilator-ideal graph of reduced rings

In this section, we show that if R is a finite direct product of integral domains, then $A_I(R)$ is perfect. In 2004 Chudnovsky et al. [7] settled a long standing conjecture regarding perfect graphs and provided a characterization of perfect graphs.

Theorem 2.1. [7, The Strong Perfect Graph Theorem] *A graph G is perfect if and only if neither G nor $\overline{G}$ contains an induced odd cycle of length at least 5.*

Example 2.2. (1) Every complete graph and complete bipartite graph are perfect.

(2) Let G_1 be a complete graph and $G_2 = C_n$ be a cycle of length at least 5 and let $G = G_1 \vee G_2$. If n is odd, then $\omega(G) = |G_1| + 2$ and $\chi(G) = |G_1| + 3$, i.e., G is not perfect. If n is even, then every induced subgraph of G is weakly perfect and thus G is perfect.

Lemma 2.3. *Let I and J be distinct annihilator ideals of R. Then the following statements are true.*

(i) [10, Lemma 2.1(iii)] *If $I - J$ is not an edge of $A_I(R)$, then* $\mathrm{Ann}_R(I) \subseteq \mathrm{Ann}_R(J)$ *or* $\mathrm{Ann}_R(J) \subseteq \mathrm{Ann}_R(I)$.

(ii) [8, Corollary 12] *If* $\mathrm{Ann}_R(I) \not\subseteq \mathrm{Ann}_R(J)$ *and* $\mathrm{Ann}_R(J) \not\subseteq \mathrm{Ann}_R(I)$, *then $I - J$ is an edge of $A_I(R)$. Moreover, if R is a reduced ring, then the converse is also true.*

Theorem 2.4. *Let $R = F_1 \times \cdots \times F_k$, where F_i is a field, for all $1 \leq i \leq k$. Then $\mathrm{A}_I(R)$ is perfect.*

Proof. By Theorem 2.1, it is enough to show that $A_I(R)$ and $\overline{A_I(R)}$ contain no induced odd cycle of length at least 5. Consider the following claims:

Claim 1. $A_I(R)$ contains no induced odd cycle of length at least 5. Assume to the contrary

$$I_1 - I_2 - \cdots - I_n - I_1$$

is an induced odd cycle of length at least 5 in $A_I(R)$. By Lemma 2.3, we have $\mathrm{Ann}_R(I_1) \subseteq \mathrm{Ann}_R(I_3)$ or $\mathrm{Ann}_R(I_3) \subseteq \mathrm{Ann}_R(I_1)$. Without loss of generality, we may assume that $\mathrm{Ann}_R(I_1) \subseteq \mathrm{Ann}_R(I_3)$. We continue the proof in the following steps.

Step 1. For every $3 \leq i \leq n-1$, $\mathrm{Ann}_R(I_1) \subseteq \mathrm{Ann}_R(I_i)$. We know that $\mathrm{Ann}_R(I_1) \subseteq \mathrm{Ann}_R(I_3)$. By Lemma 2.3, we have $\mathrm{Ann}_R(I_1) \subseteq \mathrm{Ann}_R(I_4)$ or $\mathrm{Ann}_R(I_4) \subseteq \mathrm{Ann}_R(I_1)$. If $\mathrm{Ann}_R(I_4) \subseteq \mathrm{Ann}_R(I_1)$, then $\mathrm{Ann}_R(I_4) \subseteq \mathrm{Ann}_R(I_3)$ as $\mathrm{Ann}_R(I_1) \subseteq \mathrm{Ann}_R(I_3)$ and this is a contradiction . So $\mathrm{Ann}_R(I_1) \subseteq \mathrm{Ann}_R(I_4)$. Again, we have $\mathrm{Ann}_R(I_1) \subseteq \mathrm{Ann}_R(I_5)$ or $\mathrm{Ann}_R(I_5) \subseteq \mathrm{Ann}_R(I_1)$. If $\mathrm{Ann}_R(I_5) \subseteq \mathrm{Ann}_R(I_1)$, then since $\mathrm{Ann}_R(I_1) \subseteq \mathrm{Ann}_R(I_4)$, we have $\mathrm{Ann}_R(I_5) \subseteq \mathrm{Ann}_R(I_4)$, a contradiction. So $\mathrm{Ann}_R(I_1) \subseteq \mathrm{Ann}_R(I_5)$. Similarly, for every $3 \leq i \leq n-1$, $\mathrm{Ann}_R(I_1) \subseteq \mathrm{Ann}_R(I_i)$.

Step 2. For every $4 \leq i \leq n$, $\mathrm{Ann}_R(I_2) \subseteq \mathrm{Ann}_R(I_i)$. By the Step 1, $\mathrm{Ann}_R(I_1) \subseteq \mathrm{Ann}_R(I_4)$. By Lemma 2.3, $\mathrm{Ann}_R(I_2) \subseteq \mathrm{Ann}_R(I_4)$ or $\mathrm{Ann}_R(I_4) \subseteq \mathrm{Ann}_R(I_2)$. If $\mathrm{Ann}_R(I_4) \subseteq \mathrm{Ann}_R(I_2)$, then by $\mathrm{Ann}_R(I_1) \subseteq \mathrm{Ann}_R(I_4)$, we have $\mathrm{Ann}_R(I_1) \subseteq \mathrm{Ann}_R(I_2)$, a contradiction. So $\mathrm{Ann}_R(I_2) \subseteq \mathrm{Ann}_R(I_4)$. Now, by a similar argument to that of the Step 1, we can easily get for every $4 \leq i \leq n$, $\mathrm{Ann}_R(I_2) \subseteq \mathrm{Ann}_R(I_i)$.

Step 3. For every $5 \leq i \leq n$, $\mathrm{Ann}_R(I_3) \subseteq \mathrm{Ann}_R(I_i)$. By Lemma 2.3, we have $\mathrm{Ann}_R(I_3) \subseteq \mathrm{Ann}_R(I_5)$ or $\mathrm{Ann}_R(I_5) \subseteq \mathrm{Ann}_R(I_3)$. If $\mathrm{Ann}_R(I_5) \subseteq \mathrm{Ann}_R(I_3)$, then $\mathrm{Ann}_R(I_2) \subseteq \mathrm{Ann}_R(I_3)$, a contradiction since $\mathrm{Ann}_R(I_2) \subseteq \mathrm{Ann}_R(I_5)$. So we have $\mathrm{Ann}_R(I_3) \subseteq \mathrm{Ann}_R(I_5)$. Similarly, for every $5 \leq i \leq n$, $\mathrm{Ann}_R(I_3) \subseteq \mathrm{Ann}_R(I_i)$. Now, in view of $\mathrm{Ann}_R(I_1) \subseteq \mathrm{Ann}_R(I_3)$ and $\mathrm{Ann}_R(I_3) \subseteq \mathrm{Ann}_R(I_i)$, for every $5 \leq i \leq n$, we have $\mathrm{Ann}_R(I_1) \subseteq \mathrm{Ann}_R(I_n)$, a contradiction. By a similar argument one can show that $\mathrm{Ann}_R(I_3) \subseteq \mathrm{Ann}_R(I_1)$ leads to a contradiction. So $k \leq 4$. Therefore, $A_I(R)$ contains no induced odd cycle of length at least 5.

Claim 2. $\overline{A_I(R)}$ contains no induced odd cycle of length at least 5. Assume to the contrary

$$I_1 - I_2 - \cdots - I_n - I_1$$

is an induced odd cycle of length at least 5 in $\overline{A_I(R)}$. By Lemma 2.3, we may assume that $\mathrm{Ann}_R(I_1) \subseteq \mathrm{Ann}_R(I_2)$. Now, if $\mathrm{Ann}_R(I_2) \subseteq \mathrm{Ann}_R(I_3)$, then $\mathrm{Ann}_R(I_1) \subseteq \mathrm{Ann}_R(I_3)$, a contradiction. Thus $\mathrm{Ann}_R(I_1) \subseteq \mathrm{Ann}_R(I_2)$ and $\mathrm{Ann}_R(I_3) \subseteq \mathrm{Ann}_R(I_2)$. If $\mathrm{Ann}_R(I_4) \subseteq \mathrm{Ann}_R(I_3)$, then $\mathrm{Ann}_R(I_4) \subseteq \mathrm{Ann}_R(I_2)$, a contradiction. So $\mathrm{Ann}_R(I_3) \subseteq \mathrm{Ann}_R(I_4)$. Since n is odd, if we continue this procedure, then we get $\mathrm{Ann}_R(I_{n-2}) \subseteq \mathrm{Ann}_R(I_{n-1})$ and $\mathrm{Ann}_R(I_n) \subseteq \mathrm{Ann}_R(I_{n-1})$. If $\mathrm{Ann}_R(I_1) \subseteq \mathrm{Ann}_R(I_n)$, then $\mathrm{Ann}_R(I_1) \subseteq \mathrm{Ann}_R(I_{n-1})$, a contradiction. Thus $\mathrm{Ann}_R(I_n) \subseteq \mathrm{Ann}_R(I_1)$. Now, by $\mathrm{Ann}_R(I_1) \subseteq \mathrm{Ann}_R(I_2)$, we deduce that $\mathrm{Ann}_R(I_n) \subseteq \mathrm{Ann}_R(I_2)$, a contradiction. Therefore, $\overline{A_I(R)}$ contains no induced odd cycle of length at least 5. By the Claims 1,2, $\mathrm{A}_I(R)$ is perfect. □

Let G be a graph and $x \in V(G)$ a vertex, and let G' be obtained from G by adding a vertex x' and joining it to x and all the neighbors of x. We say that G' is obtained from G by expanding the vertex x to an edge $x - x'$.

Lemma 2.5. [7, Lemma 5.5.5] *Any graph obtained from a perfect graph by expanding a vertex is again perfect.*

Remark 2.6. Let G be a graph $x \in V(G)$ and $A \subseteq V(G)$. Let for every $y \in A$, $N(x) \setminus \{y\} = N(y) \setminus \{x\}$. Then, according to Lemma 2.5, G is perfect if and only if $G \setminus \{A \setminus \{y\}\}$ is perfect.

Lemma 2.7. *Let R be a ring and $I, J \in V(A_I(R))$. If* $\mathrm{Ann}_R(I) = \mathrm{Ann}_R(J)$*, then* $N(I) = N(J)$.

Proof. Suppose that $K - I$ is an edge of $A_I(R)$. Then $\mathrm{Ann}_R(K) \cup \mathrm{Ann}_R(I) \subset \mathrm{Ann}_R(KI)$. So $\mathrm{Ann}_R(K) \cup \mathrm{Ann}_R(J) \subset \mathrm{Ann}_R(KI)$. By hypothesis it is easy to see that $\mathrm{Ann}_R(KI) = \mathrm{Ann}_R(KJ)$. Hence, $\mathrm{Ann}_R(K) \cup \mathrm{Ann}_R(J) \subset \mathrm{Ann}_R(KJ)$. This means that $K - J$ is an edge of $A_I(R)$ and so $N(I) \subseteq N(J)$. Similarly, $N(J) \subseteq N(I)$, as desired. □

Theorem 2.8. *Let $R \cong D_1 \times \cdots \times D_n$, where D_i is an integral domain, for all $1 \le i \le n$. Then $A_I(R)$ is perfect.*

Proof. Assume that $I = I_1 \times \cdots \times I_n$ and $J = J_1 \times \cdots \times J_n$ are vertices of $A_I(R)$. Define the relation $\sim$ on $V(A_I(R))$ as follows: $I \sim J$, whenever "$I_i = 0$ if and only if $J_i = 0$", for every $1 \le i \le n$. It is easily seen that $\sim$ is an equivalence relation on $V(A_I(R))$ thus $V(A_I(R)) = \cup_{i=1}^{2^n-2}[I]_i$, where $[I]_i$ is the equivalence class of I (note that the number of equivalence classes is $2^n - 2$). Let $[I]$ be the equivalence class of I and $J \in [I]$. Then $\mathrm{Ann}_R(I) = \mathrm{Ann}_R(J)$ thus by Lemma 2.7, $N(I) = N(J)$. This fact together with I is not being adjacent to J, implies that $A_I(R)$ is perfect if and only if $A_I(R) \setminus \{[I] \setminus \{J\}\}$ is perfect, by Remark 2.6. If we continue this procedure for every equivalence class $[I]$ ($2^n - 2$ times), then it concludes that $A_I(R)$ is perfect if and only if $A_I(R)[A]$ is perfect, where

$$A = \{I = I_1 \times \cdots \times I_n \in V(A_I(R)) |\ I_i \in \{0, D_i\},\ \text{for every } 1 \le i \le n\}.$$

It is straightforward to check that $A_I(R)[A] \cong A_I(S)$ where $S = F_1 \times \cdots \times F_n$ and F_i is a field, for every $1 \le i \le n$. Now, by Theorem 2.4, $A_I(S)$ is perfect and hence $A_I(R)$ is perfect. □

3. Perfectness of Annihilator-ideal graph of Artinian rings

The aim of this section is to show that $A_I(R)$ is perfect whenever R is Artinian.

Lemma 3.1. *Let R be a non-reduced ring and I be an ideal of R such that $I^n = 0$, for some positive integer n. Then* $\mathrm{Ann}_R(I)$ *is an essential ideal of R.*

Proof. Let J be a nonzero ideal of R. If $J \cap \mathrm{Ann}_R(I) \neq 0$ there is nothing to prove. Otherwise, we have $J \cap \mathrm{Ann}_R(I) = 0$ so $IJ \neq 0$. Let s be the least positive integer such that $JI^{s-1} \neq 0$ and $JI^s = 0$. Then $JI^{s-1} \subseteq J \cap \mathrm{Ann}_R(I)$ as desired. Hence, $\mathrm{Ann}_R(I)$ is an essential ideal of R. □

Lemma 3.2. *Let R be an Artinian ring and $I, J \in \mathbb{A}(R)^*$. Then the following statements are true.*

1. *If $I \subseteq J(R)$, then $\mathrm{Ann}_R(I) \neq \mathrm{Ann}_R(I^2)$.*
2. *$I - J$ is an edge of $A_I(R)$ if and only if $I \cap \mathrm{Ann}_R(J) \neq 0$ and $J \cap \mathrm{Ann}_R(I) \neq 0$.*
3. *If $I \nsubseteq J$, then $I \cap \mathrm{Ann}_R(J) \neq 0$. In particular, if $I \nsubseteq J$ and $J \nsubseteq I$, then $I - J$ is an edge of $A_I(R)$.*
4. *If $I \subseteq J$ and $I \cap \mathrm{Ann}_R(J) \neq 0$, then $I - J$ is an edge of $A_I(R)$.*

Proof. (1) By the assumption $I \subseteq J(R)$ there exists a least positive integer n such that $I^n = 0$. If $n = 2$, then clearly $\mathrm{Ann}_R(I) \neq \mathrm{Ann}_R(I^2)$. Let $n \geq 3$. Then $I^2 I^{n-2} = 0$ so $I^{n-2} \subseteq \mathrm{Ann}_R(I^2)$. If $\mathrm{Ann}_R(I) = \mathrm{Ann}_R(I^2)$, then $I^{n-2} \subseteq \mathrm{Ann}_R(I^2) = \mathrm{Ann}_R(I)$ and so $I^{n-2}I = I^{n-1} = 0$, a contradiction. Hence, $\mathrm{Ann}_R(I) \neq \mathrm{Ann}_R(I^2)$.

(2) $\Rightarrow$ is clear by [8, Lemma 10].

$\Leftarrow$ By [3, Theorem 8.7] we have $R \cong R_1 \times \cdots \times R_n$, where R_i is an Artinian local ring, for all $1 \leq i \leq n$. Let $I = I_1 \times \cdots \times I_n$ and $J = J_1 \times \cdots \times J_n$, where $I_i, J_i \in \mathbb{A}(R_i)$ for every $1 \leq i \leq n$. Suppose that $I \cap \mathrm{Ann}_R(J) \neq 0$ and $J \cap \mathrm{Ann}_R(I) \neq 0$. Then for some $1 \leq i, j \leq n$, $I_i \cap \mathrm{Ann}_R(J_i) \neq 0$ and $J_j \cap \mathrm{Ann}_R(I_j) \neq 0$. Consider the following cases:

Case 1. $i = j$. In this case $I_i \neq 0$, $J_i \neq 0$ and $I_i, J_i \subseteq J(R)$. If $I_i \neq J_i$, then by [8, Theorem 22], $I_i - J_i$ is an edge of $A_I(R_i)$ and so $\mathrm{Ann}_{R_i}(I_iJ_i) \neq \mathrm{Ann}_R(I_i) \cup \mathrm{Ann}_R(J_i)$. This fact implies that $\mathrm{Ann}_R(IJ) \neq \mathrm{Ann}_R(I) \cup \mathrm{Ann}_R(J)$. Thus $I - J$ is an edge of $A_I(R)$. If $I_i = J_i$, then by the Part (1) we can let $r \in \mathrm{Ann}_{R_i}(I_i^2) \backslash \mathrm{Ann}_{R_i}(I_i)$. Now, we can easily get $(0, \ldots, 0, r, 0, \ldots, 0) \in \mathrm{Ann}_R(IJ) \backslash \mathrm{Ann}_R(I) \cup \mathrm{Ann}_R(J)$. Hence, $I - J$ is an edge of $A_I(R)$.

Case 2. $i \neq j$. Without loss of generality, we may assume that $i = 1, j = 2$. $I_1 \cap \mathrm{Ann}_R(J_1) \neq 0$ and $J_2 \cap \mathrm{Ann}_R(I_2) \neq 0$. This implies that $I_1 \neq 0$ and $J_2 \neq 0$. We consider the following subcases:

Subcase 1. $I_1 = R_1$ and $J_2 = R_2$. In this subcase $J_1 \in \mathbb{A}(R_1)$ and $I_2 \in \mathbb{A}(R_2)$. This implies that $\mathrm{Ann}_{R_1}(J_1) \neq 0$ and $\mathrm{Ann}_{R_2}(I_2) \neq 0$. So if we pick $0 \neq a \in \mathrm{Ann}_{R_1}(J_1)$ and $0 \neq b \in \mathrm{Ann}_{R_2}(I_2)$, then we can easily get $(a, b, 0, \ldots, 0) \in \mathrm{Ann}_R(IJ) \setminus \mathrm{Ann}_R(I) \cup \mathrm{Ann}_R(J)$. Therefore, $I - J$ is an edge of $A_I(R)$.

Subcase 2. $I_1 \in \mathbb{A}(R_1)^*$ and $J_2 \in \mathbb{A}(R_2)^*$. In this subcase, we have $J_1 \in \mathbb{A}(R_1)$ and $I_2 \in \mathbb{A}(R_2)$. If $J_1 \neq 0$ or $I_2 \neq 0$, then by the Case 1, $I - J$ is an edge of $A_I(R)$. If $J_1 = 0$ and $I_2 = 0$, then we can easily get $(1, 1, 0, \ldots, 0) \in \mathrm{Ann}_R(IJ) \setminus \mathrm{Ann}_R(I) \cup \mathrm{Ann}_R(J)$. Therefore, $I - J$ is an edge of $A_I(R)$.

Subcase 3. $I_1 \in \mathbb{A}(R_1)^*$ and $J_2 = R_2$ or $I_1 = R_1$ and $J_1 \in \mathbb{A}(R_2)^*$. Suppose that $I_1 \in \mathbb{A}(R_1)^*$ and $J_2 = R_2$. If $J_1 \neq 0$, then by the

Case 1, $I - J$ is an edge of $A_I(R)$. If $J_1 = 0$, then we can easily get $(1, b, 0, \ldots, 0) \in \mathrm{Ann}_R(IJ) \setminus \mathrm{Ann}_R(I) \cup \mathrm{Ann}_R(J)$ where $0 \neq b \in \mathrm{Ann}_{R_2}(I_2)$. Therefore, $I - J$ is an edge of $A_I(R)$.

(3) Let $I = I_1 \times \cdots \times I_n$ and $J = J_1 \times \cdots \times J_n$. Since $I \nsubseteq J$ without loss of generality we may assume that $I_1 \nsubseteq J_1$. This implies that $J_1 \neq R_1$ and thus $\mathrm{Ann}(J_1)$ is an essential ideal of R_1, by Lemma 3.1. Hence, $I_1 \cap \mathrm{Ann}(J_1) \neq 0$. Let $0 \neq a_1 \in I_1 \cap \mathrm{Ann}(J_1)$. Then $(a_1, 0, \ldots, 0) \in I \cap \mathrm{Ann}_R(J)$ and so $I \cap \mathrm{Ann}_R(J) \neq 0$. The "in particular" statement is now clear.

(4) Since $I \cap \mathrm{Ann}_R(J) \neq 0$ we need only to show that $J \cap \mathrm{Ann}_R(I) \neq 0$. Let $0 \neq a \in I \cap \mathrm{Ann}_R(J)$. Then $a \in J$ since $I \subseteq J$. Also, $aJ = 0$ and $I \subseteq J$ imply that $aI = 0$. Thus $a \in J \cap \mathrm{Ann}_R(I)$ and so $I - J$ is an edge of $A_I(R)$. □

Theorem 3.3. *If R is an Artinian ring, then $A_I(R)$ is perfect.*

Proof. By [3, Theorem 8.7], $R \cong R_1 \times \cdots \times R_k$, where R_i is an Artinian local ring, for all $1 \leq i \leq k$. The argument here is a refinement of the proof of Theorem 2.4. By Theorem 2.1, it is enough to show that $A_I(R)$ and $\overline{A_I(R)}$ contains no induced odd cycle of length at least 5. Consider the following claims:

Claim 1. $A_I(R)$ contains no induced odd cycle of length at least 5. Assume to the contrary

$$I_1 - I_2 - \cdots - I_n - I_1$$

is an induced odd cycle of length at least 5 in $A_I(R)$. Since I_1 is not adjacent to I_3, by Lemma 3.2(3), $I_1 \subseteq I_3$ or $I_3 \subseteq I_1$. Without loss of generality, we may assume that $I_1 \subseteq I_3$. We have the following facts.

Case 1. $I_1 \subseteq I_i$, for every $3 \leq i \leq n-1$. Clearly, $I_1 \subseteq I_3$. Since I_1 is not adjacent to I_4, by Lemma 3.2, $I_1 \subseteq I_4$ or $I_4 \subseteq I_1$. If $I_4 \subseteq I_1$, then $I_4 \subseteq I_3$. Since $I_4 \cap \mathrm{Ann}(I_3) \neq 0$ and $I_4 \subseteq I_1$ so we have $I_1 \cap \mathrm{Ann}(I_3) \neq 0$. This fact together with Lemma 3.2(4), implies that I_1 is adjacent to I_3, a contradiction (note that $I_1 \subseteq I_3$). Thus $I_1 \subseteq I_4$. Again, by Lemma 3.2 we have $I_1 \subseteq I_5$ or $I_5 \subseteq I_1$. If $I_5 \subseteq I_1$, then by $I_1 \subseteq I_4$, we deduce that $I_5 \subseteq I_4$. Since $I_5 \cap \mathrm{Ann}(I_4) \neq 0$ and $I_5 \subseteq I_1$ so $I_1 \cap \mathrm{Ann}(I_4) \neq 0$. This fact together Lemma 3.2(4), imply that I_1 is adjacent to I_4, a contradiction (note that $I_1 \subseteq I_4$). Thus $I_1 \subseteq I_5$. Similarly, $I_1 \subseteq I_i$, for every $6 \leq i \leq n-1$.

Case 2. $I_2 \subseteq I_i$, for every $4 \leq i \leq n$. By the Case 1, $I_1 \subseteq I_4$ and by Lemma 3.2 we have $I_2 \subseteq I_4$ or $I_4 \subseteq I_2$. If $I_4 \subseteq I_2$, then we conclude that $I_4 \cap \mathrm{Ann}(I_2) \neq 0$ since $I_1 \subseteq I_4$ and $I_1 \cap \mathrm{Ann}(I_2) \neq 0$. This fact together Lemma 3.2(4), imply that I_2 is adjacent to I_4, a contradiction (note that $I_4 \subseteq I_2$). So $I_2 \subseteq I_4$. Next, we show that $I_2 \subseteq I_5$. If $I_5 \subseteq I_2$, then since $I_5 \cap \text{Ann}(I_4) \neq 0$, we get $I_2 \cap \text{Ann}(I_4) \neq 0$ also since $I_2 \subseteq I_4$, I_2 is adjacent to I_4, a contradiction. Hence, $I_2 \subseteq I_5$. Similarly, $I_2 \subseteq I_i$, for every $6 \leq i \leq n$.

Case 3. For every $7 \leq i \leq n$, $I_3 \subseteq I_i$. By Lemma 3.2, $I_3 \subseteq I_5$ or $I_5 \subseteq I_3$. If $I_5 \subseteq I_3$, then by $I_2 \cap \text{Ann}(I_3) \neq 0$ and $I_2 \subseteq I_5$ we get $I_5 \cap \text{Ann}(I_3) \neq 0$, a contradiction, by Lemma 3.2(4). Hence, $I_3 \subseteq I_5$. We show that $I_3 \subseteq I_6$. If $I_6 \subseteq I_3$, then by $I_2 \cap \text{Ann}(I_3) \neq 0$ and $I_2 \subseteq I_6$ we have $I_6 \cap \text{Ann}(I_3) \neq 0$, a contradiction, by Lemma 3.2(4). Hence, $I_3 \subseteq I_6$. Similarly, $I_3 \subseteq I_i$, for every $7 \leq i \leq n$. Thus $I_1 \subseteq I_3 \subseteq I_n$. Since $I_1 \cap \text{Ann}(I_n) \neq 0$, $I_3 \cap \text{Ann}(I_n) \neq 0$ we get a contradiction, by Lemma 3.2. By a similar argument one can show that $I_3 \subseteq I_1$ leads to a contradiction. So $k \leq 4$. Therefore, $A_I(R)$ contains no induced odd cycle of length at least 5.

Claim 2. $\overline{A_I(R)}$ contains no induced odd cycle of length at least 5. Assume to the contrary

$$I_1 - I_2 - \cdots - I_n - I_1$$

is an induced odd cycle of length at least 5 in $\overline{A_I(R)}$. By Lemma 3.2, we may assume that $I_1 \subseteq I_2$. If $I_2 \subseteq I_3$, then since $I_1 \cap \text{Ann}(I_3) \neq 0$, we conclude that $I_2 \cap \text{Ann}(I_3) \neq 0$ I_2 is adjacent to I_3 in $A_I(R)$, a contradiction. Thus $I_1 \subseteq I_2$ and $I_3 \subseteq I_2$. If $I_4 \subseteq I_3$, then $I_3 \cap \text{Ann}(I_2) \neq 0$. As $I_4 \cap \text{Ann}(I_2) \neq 0$ we deduce I_2 is adjacent to I_3 in $\text{A}_I(R)$, a contradiction. So $I_3 \subseteq I_4$. If $I_4 \subseteq I_5$, then the facts $I_3 \subseteq I_4$ and $I_3 \cap \text{Ann}(I_5) \neq 0$ imply that $I_4 \cap \text{Ann}(I_5) \neq 0$ thus I_4 is adjacent to I_5 in $A_I(R)$, a contradiction. Hence, $I_3 \subseteq I_4$ and $I_5 \subseteq I_4$. Since n is odd, by continuing this procedure $I_{n-2} \subseteq I_{n-1}$ and $I_n \subseteq I_{n-1}$. If $I_1 \subseteq I_n$, then $I_1 \cap \text{Ann}(I_{n-1}) \neq 0$ implies that $I_n \cap \text{Ann}(I_{n-1}) \neq 0$. So I_n is adjacent to I_{n-1} in $A_I(R)$, a contradiction. Hence, $I_n \subseteq I_1$ and so $I_n \cap \text{Ann}(I_2) \neq 0$. Thus $I_1 \cap \text{Ann}(I_2) \neq 0$. By Lemma 3.2, I_1 is adjacent to I_2 in $A_I(R)$, a contradiction. Therefore, $\overline{A_I(R)}$ contains no induced odd cycle of length at least 5. By the Claims 1,2 the proof is completed. □

Acknowledgments

The authors would like to thank the referee for a careful reading of our paper and insightful comments which saved us from several errors.

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