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$P-$REGULAR AND $P-$LOCAL RINGS

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Abstract. This paper is a continuation of study rings relative to right ideal, where we study the concepts of regular and local rings relative to right ideal. We give some relations between $P-$local ($P-$regular) and local (regular) rings. New characterization obtained include necessary and sufficient conditions of a ring R to be regular, local ring in terms $P-$regular, $P-$local of matrices ring $M_2(R)$. Also, We proved that every ring is local relative to any maximal right ideal of it.

1. Introduction

V. A. Andrunakievich and Yu. M. Ryabukhin in 1987 [2], were the first who introduced the concept of rings relative to right ideal. They studied the concepts of quasi-regularity and primitivity of rings relative to right ideal. Later, V. A. Andrunakievich and A. V. Andrunakievich in 1991 [1], studied the concept of regularity of rings relative to right ideal as generalization of (Von Neumann) regular ring (also known as $P-$regular rings). A number of interesting papers have been published on this concept in recent years, e.g., [1], [2], [3], [5]. In 2011, P. Dheena and S. Manivasan [3] studied quasi-ideals of a $P-$regular near-rings. In [5], H. Hakmi continue study $P-$regular and $P-$potent rings. In section 2 of this paper, we study idempotent elements relative to right ideal and investigate its properties. We proved that an element e of a ring R is an idempotent if and only if for every $x, y \in R$ the element

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$\begin{bmatrix} x & y \\ 0 & e \end{bmatrix}$ is an idempotent relative to some right ideal in the ring of matrices $M_2(R)$. In section 3, we study regular rings relative to right ideal, we proved that an element a of a ring R is regular if and only if for every $x \in R$ the element $\begin{bmatrix} x & 0 \\ 0 & a \end{bmatrix}$ is a regular relative to some right ideal. In addition to that we proved that a ring R is regular if and only if the ring S_0 is regular relative to some right ideal of $M_2(R)$. In section 4, we study semi-potent ring relative to right ideal. we proved that an element a of a ring R is semi-potent if and only if for every $x \in R$ the element $\begin{bmatrix} x & 0 \\ 0 & a \end{bmatrix}$ is a semi-potent relative to some right ideal. In addition to that we proved that a ring R is semi-potent and $J(R) = 0$ if and only if the ring S_0 is semi-potent relative to some right ideal of $M_2(R)$. Finally, we study in section 4, the concept of local ring relative to right ideal. We proved that a ring R is local relative to right ideal P of R if and only if R contains exactly one maximal right ideal containing P. In addition to that, we proved that every ring R is local relative to any maximal right ideal of it. The connection between the local ring R and the ring of matrices $M_2(R)$ is obtained.

Throughout in this article, rings R, are associative with identity unless otherwise indicated. We denote the Jacobson radical of a ring R by $J(R)$. Also, for any ring R, we use the dotation: $R_2 = M_2(R)$ the ring of 2×2 matrices over a ring R. Then

$$P = \{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in R\}, \quad Q = \{\begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} : a, b \in R\}$$

are right ideals in R_2 such that $P \neq R_2$ and $Q \neq R_2$.

Let R be a ring and $P \neq R$ be a right ideal of R. Recall that an element $e \in R$ is an idempotent relative to right ideal P or ($P-$idempotent for short)[1], if

$$e^2 - e \in P$$

$$eP \subseteq P$$

Note that in any ring R, elements $0, 1 \in R$ are idempotents relative to every right ideal $P \neq R$ of R. Also, if $P = 0$, then an element $e \in R$ is $P-$idempotent if and only if e is idempotent.

An element a in a ring R is called regular if there exists $b \in R$ such that $a = aba$. A ring R is called regular if every element of R is regular, [4]. Let R be a ring and $P \neq R$ be a right ideal in R. An element $a \in R$ is called regular relative to right ideal P, or $P-$regular for short, if there exists $b \in R$ such that

$$aba - a \in P$$

$$abP \subseteq P$$

A ring R is called $P-$regular if every element $a \in R$ is $P-$regular, [1].

Note that in previous definition, it is easy to see that every $P-$ idempotent is an $P-$regular element. In particular, $0, 1 \in R$ are $P-$regular elements. In addition to that, in previous definition we can see that for $P = 0$, a ring R is $P-$regular if and only if R is regular. A ring R is called a semi-potent ring or an I_0-ring, if for every $a \in R$, $a \notin J(R)$ there exists $b \in R$ such that $bab = b$, [6]. It easy to see that any regular ring R is a semi-potent ring with $J(R) = 0$. Also, every $\pi-$regular, strongly $\pi-$regular ring are semi-potent.

2. Idempotents Relative to Right Ideal

Lemma 2.1. *Let R be a ring and $P \neq R$ be a right ideal of R. Then for every $P-$idempotent $e \in R$ the following hold:*
(1) *An element $1 - e \in R$ is $P-$idempotent.*
(2) *An element $e^2 \in R$ is $P-$idempotent.*
(3) *An element $1 - e^2 \in R$ is $P-$idempotent.*
(4) *If $e \in J(R)$, then $e \in P$.*
(5) *If e has a right inverse, then $1 - e \in P$.*
(6) *If $1 - e$ has a right inverse, then $e \in P$.*

Proof. Suppose that $e \in R$ is $P-$idempotent, then $e^2 - e \in P$ and $eP \subseteq P$. So $e^2 = e + p_0$ for some $p_0 \in P$.
(1) We have

$$(1 - e)^2 = (1 - e)(1 - e) = 1 - 2e + e + p_0 = 1 - e + p_0$$

So $(1-e)^2-(1-e) = p_0 \in P$. Also, for every $t \in P$, $(1-e)t = t-et \in P$, so $(1 - e)P \subseteq P$.
(2) We have

$$(e^2)^2 = (e + p_0)(e + p_0) = e^2 + ep_0 + p_0e + p_0^2$$

Thus

$$(e^2)^2 - e^2 = ep_0 + p_0e + p_0^2 \in eP + PR + P \subseteq P$$

So $e^2P = e(eP) \subseteq eP \subseteq P$. (3) Obvious by (1) and (2).
(4) Suppose that $e \in J(R)$, then $1-e$ has an inverse in R, so $(1-e)a = 1$ for some $a \in R$ and so $e = (e - e^2)a \in PR \subseteq P$.
(5) Suppose that e has a right inverse in R. Then $ea = 1$ for some $a \in R$ and

$$e = e^2a = (e + p_0)a = ea + p_0a = 1 + p_0a \in 1 + P$$

Thus $1-e\in P$.
(6) Suppose that $1-e$ has a right inverse in R. Then $(1-e)b=1$ for some $b\in R$ and $e=(e-e^2)b\in PR\subseteq P$. □

From Lemma 2.1 and for $P=0$ we obtain the following:

Corollary 2.2. *Let R be a ring and $e\in R$ be an idempotent. Then:*
(1) *Elements e^2, $1-e$ are idempotents in R.*
(2) *If $e\in J(R)$, then $e=0$.*
(3) *If e has a right inverse in R, then $e=1$.*
(4) *If $1-e$ has a right inverse in R, then $e=0$.*

Proposition 2.3. *Let R be a ring and $P\neq R$ be a right ideal of R. For every $P-$ idempotent $e\in R$ the following statements hold:*
(1) *For every integer $k\geq 1$, an element e^k is $P-$idempotent.*
(2) *For every integer $k\geq 1$, an element $1-e^k$ is $P-$idempotent.*
(3) *For every integer $k\geq 1$, an element $(1-e)^k$ is $P-$idempotent.*

Proof. (1) Suppose that e is an $P-$idempotent, then $e^2-e\in P$ and $eP\subseteq P$, so $e^2=e+p_0$ for some $p_0\in P$. Proof by induction on k. For $k=1,2$ the assertion holds by assumption and Lemma 2.1. Suppose that e^{k-1} is an $P-$idempotent, then

$$(e^{k-1})^2-e^{k-1}\in P$$
$$e^{k-1}P\subseteq P$$

So $(e^{k-1})^2=e^{k-1}+p_1$ for some $p_1\in P$. Thus

$$(e^k)^2=(e^{k-1})^2e^2=(e^{k-1}+p_1)(e+p_0)=$$
$$=e^k+e^{k-1}p_0+p_1e+p_1p_0$$

therefore $(e^k)^2-e^k=p$, where $p=e^{k-1}p_0+p_1e+p_1p_0\in P$. This shows that

$$(e^k)^2-e^k\in P$$
$$e^kP=ee^{k-1}P\subseteq eP\subseteq P$$

Thus, e^k is an $P-$idempotent. (2) and (3) By (1) and Lemma 2.1. □

Proposition 2.4. *Let R be a ring and $P\neq R$ be a right ideal in R. If $e,g\in R$ such that $e-g\in P$, then g is $P-$idempotent if and only if e is $P-$idempotent.*

Proof. Suppose that $e-g\in P$, then $e=g+p_1$ for some $p_1\in P$. Assume that g is $P-$idempotent, then $g^2-g\in P$, $gP\subseteq P$. So $g^2=g+p_0$ and

$$e^2=(g+p_1)(g+p_1)=g^2+gp_1+p_1g+p_1p_1=$$
$$=g+p_0+gp_1+p_1g+p_1p_1=$$
$$=g+p_1+(-p_1+p_0+gp_1+p_1g+p_1p_1)$$

For

$$p' = -p_1 + p_0 + gp_1 + p_1g + p_1p_1 \in P$$

We have $e^2 - e = p' \in P$ and $eP \subseteq gP + p_1P \subseteq P$. This shows that e is $P-$ idempotent. Similarly, we can prove conversely. □

Next, we present an example of an $P-$idempotent elements.

Example 2.5. Let $\mathbb{Z}$ be the ring of integers and let $R_2 = M_2(\mathbb{Z})$ be the ring of all 2×2 matrices over the ring of integers $\mathbb{Z}$. Let

$$e = \begin{bmatrix} n & m \\ 0 & 1 \end{bmatrix} \in R_2$$

Where $n, m \in \mathbb{Z}$, then e is $P-$idempotent in R_2, but not idempotent. And,

$$e^2 = \begin{bmatrix} n^2 & (n+1)m \\ 0 & 1 \end{bmatrix}$$

is $P-$idempotent in R_2. Also, for every positive integer k the element:

$$e^k = \begin{bmatrix} n^k & m\Sigma_{t=0}^{k-1} n^t \\ 0 & 1 \end{bmatrix}$$

is $P-$idempotent in R_2. In addition to that the element:

$$1 - e = \begin{bmatrix} 1-n & -m \\ 0 & 0 \end{bmatrix}$$

is $P-$idempotent in R_2 and the element:

$$1 - e^2 = \begin{bmatrix} 1-n^2 & -(n+1)m \\ 0 & 0 \end{bmatrix}$$

is $P-$idempotent in R_2. Also, for every positive integer k the element:

$$1 - e^k = \begin{bmatrix} 1-n^k & -m\Sigma_{t=0}^{k-1} n^t \\ 0 & 0 \end{bmatrix}$$

is $P-$idempotent in R_2 and the element:

$$(1-e)^2 = \begin{bmatrix} (1-n)^2 & -(1-n)m \\ 0 & 0 \end{bmatrix}$$

is $P-$idempotent in R_2. Also, for every positive integer k the element:

$$(1-e)^k = \begin{bmatrix} (1-n)^k & -m(1-n)^{k-1} \\ 0 & 0 \end{bmatrix}$$

is $P-$idempotent in R_2.

The connection between the idempotent elements in a ring R and $P-$idempotent ($Q-$idempotent) elements in R_2 we provide in the following:

Theorem 2.6. *For any element $e \in R$ the following hold:*
(1) *If e is idempotent in R, then for every $x, y \in R$, the element $e_0 = \begin{bmatrix} x & y \\ 0 & e \end{bmatrix}$ is $P-$idempotent in R_2.*
(2) *If for some $x, y \in R$, the element $e_0 = \begin{bmatrix} x & y \\ 0 & e \end{bmatrix}$ is $P-$idempotent in R_2, then e is idempotent in R.*
(3) *If e is idempotent in R, then for every $x, y \in R$, the element $e_0 = \begin{bmatrix} e & 0 \\ x & y \end{bmatrix}$ is $Q-$idempotent in R_2.*
(4) *If for some $x, y \in R$, the element $e_0 = \begin{bmatrix} e & 0 \\ x & y \end{bmatrix}$ is $Q-$idempotent in R_2, then e is an idempotent in R.*

Proof. (1) Suppose that e is an idempotent in R, then for every $x, y \in R$,

$$e_0^2 - e_0 = \begin{bmatrix} x^2 & xy + ye \\ 0 & e^2 \end{bmatrix} - \begin{bmatrix} x & y \\ 0 & e \end{bmatrix} = \begin{bmatrix} x^2 - x & xy + ye - y \\ 0 & e^2 - e \end{bmatrix} \in P$$

For every $p = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \in P$, where $a, b \in R$, $e_0 p = \begin{bmatrix} xa & xb \\ 0 & 0 \end{bmatrix} \in P$. So $e_0 P \subseteq P$, this shows that e_0 is $P-$idempotent in R_2.
(2) Let $x, y \in R$ such that $e_0 = \begin{bmatrix} x & y \\ 0 & e \end{bmatrix}$ is $P-$idempotent in R_2. Since $e_0^2 - e_0 \in P$,

$$\begin{bmatrix} x^2 - x & xy + ye - y \\ 0 & e^2 - e \end{bmatrix} = \begin{bmatrix} a' & b' \\ 0 & 0 \end{bmatrix}$$

for some $a', b' \in R$, so $e^2 = e$. This shows that e is idempotent in R.
(3) Similarly as in (1). (4) Similarly as in (2). □

From Theorem 2.6 we can obtain the following:

Corollary 2.7. *For any ring R the following hold:*
(1) *An element $e \in R$ is idempotent in R if and only if there exists $x, y \in R$ such that the element $\begin{bmatrix} x & y \\ 0 & e \end{bmatrix}$ is $P-$idempotent in R_2.*
(2) *An element $e \in R$ is idempotent in R if and only if there exists $x \in R$ such that the element $\begin{bmatrix} x & 0 \\ 0 & e \end{bmatrix}$ is $P-$idempotent in R_2.*
(3) *An element $e \in R$ is idempotent in R if and only if there exists $x, y \in R$ such that the element $\begin{bmatrix} e & 0 \\ x & y \end{bmatrix}$ is $Q-$idempotent in R_2.*

(4) *An element $e \in R$ is idempotent in R if and only if there exists $x \in R$ such that the element* $\begin{bmatrix} e & 0 \\ 0 & x \end{bmatrix}$ *is $Q-$idempotent in R_2.*

3. Regular Rings Relative to Right Ideal

We start this section with the following:

Lemma 3.1. *Let R be a ring and $P \neq R$ be a right ideal in R. If $a, b \in R$ such that $b - a \in P$, then a is $P-$regular if and only if b is $P-$regular.*

Proof. Suppose that $b - a \in P$, then $b = a + p_0$ for some $p_0 \in P$.
($\Rightarrow$) If a is $P-$regular, then there exists $x \in R$ such that $axa - a \in P$ and $axP \subseteq P$. So

$$bxb - b = (a + p_0)x(a + p_0) - (a + p_0) =$$
$$= (ax + p_0x)(a + p_0) - (a + p_0) =$$
$$= (axa - a) + axp_0 + p_0xa + p_0xp_0 - p_0 \in P$$

and $bxP = (a + p_0)xP \subseteq axP + p_0xP \subseteq P$ this shows that b is $P-$idempotent. Similarly, we can prove conversely. □

The connection between the regular elements in R and $P-$regular ($Q-$regular) elements in R_2 we provide in the following:

Proposition 3.2. *For any element $a \in R$ the following hold:*
(1) *If a is a regular element in R, then for every $x \in R$, the elements:*

$$\alpha = \begin{bmatrix} x & 0 \\ 0 & a \end{bmatrix}, \alpha' = \begin{bmatrix} 0 & x \\ a & 0 \end{bmatrix}$$

are $P-$regular elements in R_2.
(2) *If for some $x \in R$, the element* $\alpha = \begin{bmatrix} x & 0 \\ 0 & a \end{bmatrix}$ *is $P-$regular in R_2, then a is regular in R.*
(3) *If for some $x \in R$, the element* $\alpha = \begin{bmatrix} 0 & x \\ a & 0 \end{bmatrix}$ *is $P-$regular in R_2, then a is regular in R.*
(4) *If a is a regular element in R, then for every $x \in R$, the elements:*

$$\alpha = \begin{bmatrix} a & 0 \\ 0 & x \end{bmatrix}, \alpha' = \begin{bmatrix} 0 & a \\ x & 0 \end{bmatrix}$$

are $Q-$regular elements in R_2.
(5) *If for some $x \in R$, the element* $\alpha = \begin{bmatrix} a & 0 \\ 0 & x \end{bmatrix}$ *is $Q-$regular in R_2, then a is regular in R.*

(6) *If for some* $x \in R$, *the element* $\alpha = \begin{bmatrix} 0 & a \\ x & 0 \end{bmatrix}$ *is* $Q-$*regular in* R_2, *then* a *is regular in* R.

Proof. (1) Suppose that a is a regular element in R, then $a = aba$ for some $b \in R$. For every $x_1, y_1 \in R$,

$$\beta = \begin{bmatrix} x_1 & y_1 \\ 0 & b \end{bmatrix} \in R_2$$

such that

$$\alpha\beta\alpha - \alpha = \begin{bmatrix} xx_1x & xy_1a \\ 0 & aba \end{bmatrix} - \begin{bmatrix} x & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} xx_1x - x & xy_1a \\ 0 & aba - a \end{bmatrix} \in P$$

and for every $t = \begin{bmatrix} a' & b' \\ 0 & 0 \end{bmatrix} \in P$ where $a', b' \in R$

$$\alpha\beta t = \begin{bmatrix} xx_1 & xy_1 \\ 0 & ab \end{bmatrix} \begin{bmatrix} a' & b' \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} xx_1a' & xx_1b' \\ 0 & 0 \end{bmatrix} \in P$$

this shows that $\alpha\beta P \subseteq P$. Thus, α is an $P-$regular element in R_2. Similarly, we can prove that for every $x', y' \in R$, the element:

$$\beta' = \begin{bmatrix} 0 & b \\ x_1 & y_1 \end{bmatrix} \in R_2$$

such that $\alpha'\beta'\alpha' - \alpha' \in P$ and $\alpha'\beta' P \subseteq P$. i.e., α' is an $P-$regular element in R_2.
(2) Suppose that α is $P-$regular in R_2, then there exists

$$\beta = \begin{bmatrix} y & z \\ r & b \end{bmatrix} \in R_2$$

where $y, z, r, b \in R$ such that $\alpha\beta\alpha - \alpha \in P$, so

$$\begin{bmatrix} xyx & xza \\ arx & aba \end{bmatrix} - \begin{bmatrix} x & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} xyx - x & xza \\ arx & aba - a \end{bmatrix} \in P$$

This shows that $aba = a$. i.e., An element a is regular. (3) Similarly, as in (2). (4) Similarly, as in (1). (5) and (6) Similarly, as in (2) and (3). □

From Proposition 3.2 we can obtain of the following:

Corollary 3.3. *For any ring* R *the following conditions are equivalent:*
(1) *A ring* R *is regular.*
(2) *For every* $a \in R$ *there exists* $x \in R$ *such that the element* $\begin{bmatrix} x & 0 \\ 0 & a \end{bmatrix}$ *is* $P-$*regular in* R_2.

(3) *For every* $a \in R$ *there exists* $x \in R$ *such that the element* $\begin{bmatrix} 0 & x \\ a & 0 \end{bmatrix}$ *is* $P-$*regular in* R_2.
(4) *For every* $a \in R$ *there exists* $x \in R$ *such that the element* $\begin{bmatrix} a & 0 \\ 0 & x \end{bmatrix}$ *is* $Q-$*regular* R_2.
(5) *For every* $a \in R$ *there exists* $x \in R$ *such that the element* $\begin{bmatrix} 0 & a \\ x & 0 \end{bmatrix}$ *is* $Q-$*regular in* R_2.

We again use the notation, let R be a ring and $R_2 = M_2(R)$ be the ring of all 2×2 matrices over a ring R. It is clear that the set:

$$S_0 = \{\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} : x, y \in R\}$$

is a subring in R_2 with identity element. Also, the sets:

$$P_0 = \{\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in R\}, \quad Q_0 = \{\begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} : a \in R\}$$

are right ideals in S_0 and $P_0 \neq S_0$, $Q_0 \neq S_0$. Then we have the following:

Theorem 3.4. *For any ring* R *the following hold:*
(1) *A ring* R *is regular if and only if the ring* S_0 *is* P_0-*regular.*
(2) *A ring* R *is regular if and only if the ring* S_0 *is* Q_0-*regular.*

Proof. (1) Suppose that a ring R is regular. Let $\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \in S_0$, where $x, y \in R$. Since y is regular in R, by Lemma 3.1 the element $\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$ is P_0-regular. Conversely, let $a \in R$, then for every $x \in R$ the element $\begin{bmatrix} x & 0 \\ 0 & a \end{bmatrix}$ is P_0-regular in S_0, so a is regular in R. Similarly, we can proof (2). □

4. Semi-potent Rings Relative to Right Ideal

Definition 4.1. Let R be a ring and $P \neq R$ be a right ideal in R. We say that an element $a \in R$ is semi-potent relative to right ideal P or $P-$semi-potent for short, if there exists $x \in R$ such that

$$bxb - b \in P$$

$$bxP \subseteq P$$

Also, we say that a ring R is a semi-potent ring relative to right ideal P or an $P-$semi-potent ring for short, if every element $a \in R$ is $P-$semi-potent.

From previous definition, it is easy to see that $0, 1 \in R$ are $P-$semi-potent elements. In addition to that, we have the following:

Lemma 4.2. *Let R be a ring and $P \neq R$ be a right ideal in R. Every $P-$idempotent element is $P-$semi-potent.*

Proof. Let $e \in R$ is an $P-$idempotent, then $e^2 - e \in P$ and $eP \subseteq P$, so $e^2 = e + p_0$ for some $p_0 \in P$. For $b = e$ we have

$$e^3 - e = ee^2 - e = e(e + p_0) - e = (e^2 - e) + ep_0 \in P + eP \subseteq P$$

$$e^2P = e(eP) \subseteq eP \subseteq P$$

□

Next, we present an example of an $P-$semi-potent elements:

Example 4.3. Let $\mathbb{Z}$ be the ring of integers and let $R_2 = M_2(\mathbb{Z})$ be the ring of all 2×2 matrices over the ring of integers $\mathbb{Z}$. For every $n \in \mathbb{Z}$ the element $a = \begin{bmatrix} n & 0 \\ 0 & 1 \end{bmatrix}$ is $P-$semi-potent in R_2, hence for $b = \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}$, where $x, y \in \mathbb{Z}$ such that $bab - b \in P$ and $baP \subseteq P$. Also, the element $a' = \begin{bmatrix} 0 & 1 \\ n & 0 \end{bmatrix}$, where $n \in \mathbb{Z}$ is an $P-$semi-potent element in R, hence for $b' = \begin{bmatrix} x & y \\ 1 & 0 \end{bmatrix}$, where $x, y \in \mathbb{Z}$ such that $b'a'b' - b' \in P$ and $b'a'P \subseteq P$. In addition to that, for every $n, m \in \mathbb{Z}$ the element $\alpha = \begin{bmatrix} n & m \\ 0 & 1 \end{bmatrix}$ is $P-$semi-potent, hence for every $x, y \in \mathbb{Z}$ the element $\beta = \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} \in R_2$ such that

$$\beta\alpha\beta - \beta = \begin{bmatrix} xnx & xny + xmy \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} xnx - x & xny + xmy - y \\ 0 & 0 \end{bmatrix} \in P$$

For every $p = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \in P$, where $a, b \in \mathbb{Z}$. We have:

$$\beta\alpha p = \begin{bmatrix} xn & xm + y \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} xna & xnb \\ 0 & 0 \end{bmatrix} \in P$$

i.e., $\beta\alpha P \subseteq P$. On the other hand, for every $n \in \mathbb{Z}$ the element $a = \begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix}$ is $Q-$semi-potent in R_2, hence for $b = \begin{bmatrix} 1 & 0 \\ x & y \end{bmatrix} \in R_2$, where $x, y \in \mathbb{Z}$ such that $bab - b \in Q$ and $baQ \subseteq Q$. Also, the element $a' = \begin{bmatrix} 0 & n \\ 1 & 0 \end{bmatrix}$, where $n \in \mathbb{Z}$ is $Q-$semi-potent element in R, hence for $b' = \begin{bmatrix} 0 & 1 \\ x & y \end{bmatrix} \in R_2$, where $x, y \in \mathbb{Z}$ such that $b'a'b' - b' \in Q$ and $b'a'Q \subseteq Q$. In addition to that, for every $n, m \in \mathbb{Z}$ the element $\alpha' = \begin{bmatrix} 1 & 0 \\ m & n \end{bmatrix} \in R_2$ is $Q-$semi-potent, hence for every $x, y \in \mathbb{Z}$ the element $\beta' = \begin{bmatrix} 1 & 0 \\ x & y \end{bmatrix} \in R_2$ such that

$$\beta'\alpha'\beta' - \beta' = \begin{bmatrix} 1 & 0 \\ x^2 + my + nxy & ny^2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ x & y \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 \\ my + nxy & ny^2 - y \end{bmatrix} \in Q$$

For every $p = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} \in Q$, where $a, b \in \mathbb{Z}$. We have:

$$\beta'\alpha' p = \begin{bmatrix} 1 & 0 \\ x + my & ny \end{bmatrix} \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ (x + my)a & nyb \end{bmatrix} \in Q$$

i.e., $\beta'\alpha' Q \subseteq Q$.

The connection between the semi-potent elements in R and $P-$semi-potent ($Q-$semi-potent) elements in R_2 we provide in the following:

Proposition 4.4. *For any element $a \in R$ the following hold:*
(1) *If a is a semi-potent element in R, then for every $x \in R$, the elements:*

$$\alpha = \begin{bmatrix} x & 0 \\ 0 & a \end{bmatrix}, \alpha' = \begin{bmatrix} 0 & x \\ a & 0 \end{bmatrix}$$

are $P-$semi-potent elements in R_2.
(2) *If for some $x \in R$, the element:*

$$\alpha = \begin{bmatrix} x & 0 \\ 0 & a \end{bmatrix}$$

is $P-$semi-potent in R_2, then a is semi-potent in R.
(3) *If a is a semi-potent element in R, then for every $x \in R$, the elements:*

$$\alpha = \begin{bmatrix} a & 0 \\ 0 & x \end{bmatrix}, \alpha' = \begin{bmatrix} 0 & a \\ x & 0 \end{bmatrix}$$

are $Q-$semi-potent elements in R_2.
(4) *If for some $x \in R$, the element:*

$$\alpha = \begin{bmatrix} a & 0 \\ 0 & x \end{bmatrix}$$

is $Q-$semi-potent in R_2, then a is semi-potent in R.

Proof. (1) Suppose that a is a semi-potent element in R, then $b = bab$ for some $b \in R$. For every $x_1, y_1 \in R$, $\beta = \begin{bmatrix} x_1 & y_1 \\ 0 & b \end{bmatrix} \in R_2$ such that

$$\beta\alpha\beta - \beta = \begin{bmatrix} x_1xx_1 & x_1xy_1 + y_1ab \\ 0 & bab \end{bmatrix} - \begin{bmatrix} x_1 & y_1 \\ 0 & b \end{bmatrix} =$$

$$= \begin{bmatrix} x_1xx_1 - x_1 & x_1xy_1 + y_1ab - y_1 \\ 0 & bab - b \end{bmatrix} \in P$$

For every $t = \begin{bmatrix} a' & b' \\ 0 & 0 \end{bmatrix} \in P$, where $a', b' \in R$

$$\beta\alpha t = \begin{bmatrix} x_1x & y_1a \\ 0 & ba \end{bmatrix} \begin{bmatrix} a' & b' \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x_1xa' & x_1xb' \\ 0 & 0 \end{bmatrix} \in P$$

This shows that $\beta\alpha P \subseteq P$. Thus, α is an $P-$semi-potent element in R_2. Similarly, we can prove that for every $x', y' \in R$, the element $\beta' = \begin{bmatrix} 0 & b \\ x_1 & y_1 \end{bmatrix} \in R_2$ such that $\beta'\alpha'\beta' - \beta' \in P$ and $\beta'\alpha' P \subseteq P$. i.e., α' is an $P-$semi-potent element in R_2.
(2) Suppose that α is $P-$semi-potent in S, then there exists $\beta = \begin{bmatrix} y & z \\ r & b \end{bmatrix} \in R_2$ where $y, z, r, b \in R$ such that $\beta\alpha\beta - \beta \in P$. Since $\beta\alpha P \subseteq P$, implies that $r = 0$, so

$$\beta\alpha\beta - \beta = \begin{bmatrix} yxy & yxz + zab \\ 0 & bab \end{bmatrix} - \begin{bmatrix} y & z \\ 0 & b \end{bmatrix} =$$

$$= \begin{bmatrix} yxy - y & yxz + zab - z \\ 0 & bab - b \end{bmatrix} \in P$$

This shows that $bab = b$. i.e., an element a is semi-potent. (3) Similarly, as in (1). (4) Similarly, as in (2). □

From Proposition 4.4 we can obtain of the following:

Corollary 4.5. *For any ring R the following conditions are equivalent:*
(1) *A ring R is semi-potent with $J(R) = 0$.*
(2) *For every $a \in R$ there exists $x \in R$ such that the element $\begin{bmatrix} x & 0 \\ 0 & a \end{bmatrix}$ is*

$P-$semi-potent in R_2.

(3) *For every $a \in R$ there exists $x \in R$ such that the element $\begin{bmatrix} 0 & x \\ a & 0 \end{bmatrix}$ is $P-$semi-potent in R_2.*

(4) *For every $a \in R$ there exists $x \in R$ such that the element $\begin{bmatrix} a & 0 \\ 0 & x \end{bmatrix}$ is $Q-$semi-potent in R_2.*

(5) *For every $a \in R$ there exists $x \in R$ such that the element $\begin{bmatrix} 0 & a \\ x & 0 \end{bmatrix}$ is $Q-$semi-potent in R_2.*

Also, in view of the hypothesis of Proposition 4.4 we can obtain the following:

Theorem 4.6. *For any element $a \in R$ the following hold:*
(1) *If a is a semi-potent element in R, then for every $n, m \in R$, the elements $\alpha = \begin{bmatrix} n & m \\ 0 & a \end{bmatrix}$ is an $P-$semi-potent elements in R_2.*
(2) *If for some $n, m \in R$, the element $\alpha = \begin{bmatrix} n & m \\ 0 & a \end{bmatrix}$ is $P-$semi-potent in R_2, then the element a is a semi-potent in R.*
(3) *If a is a semi-potent element in R, then for every $n, m \in R$, the elements $\alpha = \begin{bmatrix} n & m \\ 0 & a \end{bmatrix}$ is $Q-$semi-potent elements in R_2.*
(4) *If for some $n, m \in R$, the element $\alpha = \begin{bmatrix} a & 0 \\ n & m \end{bmatrix}$ is $Q-$semi-potent in R_2, then the element a is semi-potent in R.*

Proof. (1) Suppose that a is a semi-potent element in R, then there exists $b \in R$ such that $bab = b$. So for every $x, y \in R$ the element $\beta = \begin{bmatrix} x & y \\ 0 & b \end{bmatrix} \in R_2$ such that:

$$\beta\alpha\beta - \beta = \begin{bmatrix} xnx & xny + (xm + ya)b \\ 0 & bab \end{bmatrix} - \begin{bmatrix} x & y \\ 0 & b \end{bmatrix} =$$

$$= \begin{bmatrix} xnx - x & xny + (xm + ya)b - y \\ 0 & bab - b \end{bmatrix} \in P$$

For every $p = \begin{bmatrix} a' & b' \\ 0 & 0 \end{bmatrix} \in P$, where $a', b' \in R$, $\beta\alpha p = \begin{bmatrix} xna' & xnb' \\ 0 & 0 \end{bmatrix} \in P$ this shows that $\beta\alpha P \subseteq P$,i.e., the element α is $P-$semi-potent in R_2.
(2) Suppose that α is an $P-$semi-potent in R_2, then there exists $\beta = \begin{bmatrix} x & y \\ z & r \end{bmatrix} \in R_2$, where $x, y, z, r \in R$ such that $\beta\alpha\beta - \beta \in P$ and $\beta\alpha P \subseteq P$.

Since $\beta\alpha P \subseteq P$ implies that $z = 0$. Also, since $\beta\alpha P \subseteq P$,

$$\beta\alpha\beta - \beta = \begin{bmatrix} xnx - x & xny + (xm + ya)b - y \\ 0 & bab - b \end{bmatrix} \in P$$

so $bab - b = 0$. Thus a is semi-potent in R. (3) Similarly, as in (1). (4) Similarly, as in (2). □

Theorem 4.7. *For any ring R the following hold:*
(1) *A ring R is semi-potent with $J(R) = 0$ if and only if the ring S_0 is $P-$semi-potent relative to right ideal P_0.*
(2) *A ring R is semi-potent with $J(R) = 0$ if and only if the ring S_0 is $P-$semi-potent relative to right ideal Q_0.*

Proof. (1) Suppose that a ring R is $P-$semi-potent. Let $\alpha = \begin{bmatrix} x & 0 \\ 0 & a \end{bmatrix} \in S_0$ where $x, a \in R$. Since a is $P-$semi-potent in R, there exists $b \in R$ such that $bab = b$. Let $\beta = \begin{bmatrix} x & 0 \\ 0 & b \end{bmatrix}$, then $\beta \in S_0$ such that

$$\beta\alpha\beta - \beta = \begin{bmatrix} x^3 & 0 \\ 0 & bab \end{bmatrix} - \begin{bmatrix} x & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} x^3 - x & 0 \\ 0 & bab - b \end{bmatrix} \in P_0$$

and for every $p_0 = \begin{bmatrix} a_0 & 0 \\ 0 & 0 \end{bmatrix} \in P_0$ we have

$$\beta\alpha p_0 = \begin{bmatrix} x^2 & 0 \\ 0 & ba \end{bmatrix} \begin{bmatrix} a_0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x^2 a_0 & 0 \\ 0 & 0 \end{bmatrix} \in P_0$$

this shows that $\beta\alpha P_0 \subseteq P_0$. Thus, the ring S_0 is P_0-semi-potent. Conversely, let $a \in R$, then by assumption for every $x \in R$ the element $\gamma = \begin{bmatrix} x & 0 \\ 0 & a \end{bmatrix}$ is $P-$semi-potent relative to right ideal P_0 in S_0, so there exists $\delta = \begin{bmatrix} y & 0 \\ 0 & b \end{bmatrix} \in S_0$ such that $\delta\gamma\delta - \delta \in P_0$. Thus

$$\delta\gamma\delta - \delta = \begin{bmatrix} yxy & 0 \\ 0 & bab \end{bmatrix} - \begin{bmatrix} y & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} yxy & 0 \\ 0 & bab - b \end{bmatrix} \in P_0$$

This shows that $bab = b$,.i.e, R is a semi-potent ring and $J(R) = 0$. Similarly, we can proof (2). □

5. Local Rings Relative to Right Ideal

Lemma 5.1. [7, Proposition 1. P.75]. *Let $R \neq 0$ be a ring. The following conditions are equivalent:*
(1) *A ring R has exactly one maximal right ideal.*

(2) *The nonunit elements in R form a proper right ideal.*
(3) *For every element $r \in R$, either r or $1-r$ has a right inverse in R.*

Recall that a ring R is local if R satisfies the equivalent conditions in Lemma 5.1, [7].

Definition 5.2. Let R be a ring and $P \neq R$ be a right ideal of R. We say that an element $a \in R$ has a right inverse relative to right ideal P or has a right $P-$inverse if $R = aR + P$.

From previous definition, it is easy to see that, for every $a \in R$, $R = aR + P$ if and only if there exists $x \in R$ such that $1 - ax \in P$.

Next, we present an example of elements have a right $P-$inverse. In view hypothesis of Example 2.5 we can obtain the following:

Example 5.3. Let $\mathbb{Z}$ be the ring of integers and let $R_2 = M_2(Z)$ be the ring of all 2×2 matrices over the ring of integers $\mathbb{Z}$.
(1) For every $n, m \in \mathbb{Z}$, the element $\alpha = \begin{bmatrix} n & m \\ 0 & 1 \end{bmatrix}$ has a right $P-$inverse. Because for $\beta = \begin{bmatrix} u & v \\ 0 & 1 \end{bmatrix} \in R_2$, where $u, v \in \mathbb{Z}$,

$$1 - \alpha\beta = \begin{bmatrix} 1 - nu & -(nv + m) \\ 0 & 0 \end{bmatrix} \in P$$

(2) For every $n, m \in \mathbb{Z}$, the element $\alpha' = \begin{bmatrix} n & m \\ 1 & 0 \end{bmatrix}$ has a right $P-$inverse. Because for $\beta' = \begin{bmatrix} 0 & 1 \\ 1 & u \end{bmatrix} \in R_2$, where $u \in \mathbb{Z}$,

$$1 - \alpha'\beta' = \begin{bmatrix} 1 - m & -(n + mu) \\ 0 & 0 \end{bmatrix} \in P$$

(3) For every $n \in \mathbb{Z}$, the element $\gamma = \begin{bmatrix} n & 0 \\ 1 & 0 \end{bmatrix}$ has a right $P-$inverse. Because for $\lambda = \begin{bmatrix} 0 & 1 \\ u & v \end{bmatrix} \in R_2$, where $u, v \in \mathbb{Z}$, $1 - \gamma\lambda = \begin{bmatrix} 1 & -n \\ 0 & 0 \end{bmatrix} \in P$.
(4) For every $n \in \mathbb{Z}$, the element $\gamma' = \begin{bmatrix} 0 & n \\ 0 & 1 \end{bmatrix}$ has a right $P-$inverse. Because for $\lambda' = \begin{bmatrix} u & v \\ 0 & 1 \end{bmatrix} \in R_2$, where $u, v \in \mathbb{Z}$, $1 - \gamma'\lambda' = \begin{bmatrix} 1 & -n \\ 0 & 0 \end{bmatrix} \in P$.

Lemma 5.4. *Let R be a ring and $P \neq R$ be a right ideal of R. The set*

$$K = \{a : a \in R; R \neq aR + P\}$$

satisfies the following:
(1) $K \neq \phi$ *and* $K \neq R$.
(2) $P \subseteq K$.

Proof. (1) It is clare that $K \neq \phi$, hence $R \neq P = 0R + P$, so $0 \in K$. Since $R = R + P$, then $1 \notin K$, so $K \neq R$.
(2) Since for every $p \in P$, $pR + P = P \neq R$, so $P \subseteq K$. □

Lemma 5.5. *Let* R *be a ring and* $P \neq R$ *be a right ideal of* R*. Then for every right ideal* A *of* R *such that* $R \neq A+P$ *there exists a maximal right* M *of* R *such that* $A + P \subseteq M$.

Proof. It is clear, hence $A+P$ is a right ideal in R and $R \neq A+P$. □

Theorem 5.6. *Let* R *be a ring and* $P \neq R$ *be a right ideal of* R*. Suppose that*

$$K = \{a : a \in R; R \neq aR + P\}$$

Then the following conditions are equivalent:
(1) K *is closed under addition.*
(2) K *is a right ideal in* R *such that* $K \neq R$.
(3) K *is a largest right ideal in* R.
(4) *In* R *there exists a largest right ideal* N *such that* $N \neq R$ *and* $P \subseteq N$.
(5) *For every* $r \in R$*, either* r *or* $1 - r$ *has a right* $P-$*inverse.*
(6) R *has exactly one maximal right ideal* M *such that* $P \subseteq M$.

Proof. (1) $\Rightarrow$ (2) Let $a \in K$, then $R \neq aR + P$. So for every $x \in R$, $R \neq axR + P$, hence if $R = axR + P \subseteq aR + P \subseteq R$ implies that $R = aR + P$ a contradiction, so $ax \in K$, thus K is a right ideal in R.
(2) $\Rightarrow$ (3) Let B be a right ideal in R such that $B \neq R$ and $P \subseteq B$, then for every $b \in B$, $R \neq bR + P$, because if for some $b \in B$, $R = bR + P$, then $R = bR + P \subseteq B + P \subseteq B \subseteq R$, so $B = R$ a contradiction. Thus, $B \subseteq K$. (3) $\Rightarrow$ (4) It is clear.
(4) $\Rightarrow$ (5) Suppose that N be a largest right ideal in R such that $N \neq R$ and $P \subseteq N$. Let $x \in R$, suppose $R \neq xR + P$ and $R \neq (1 - x)R + P$, then $xR + P \subseteq N$ and $(1 - x)R + P \subseteq N$, this shows that $x \in N$ and $1 - x \in N$, since $1 = x + (1 - x) \in N$, $N = R$ a contradiction.
(5) $\Rightarrow$ (1) Let $a, b \in K$, then $aR + P \neq R$ and $bR + P \neq R$. Assume that $a + b \notin K$, then $(a + b)R + P = R$, so $(a + b)x + p_0 = 1$ for some $x \in R$, $p_0 \in P$ and $1 - bx = ax + p_0$. Is is clear that $1 - bx \in K$ because if $1 - bx \notin K$ we have

$$R = (1 - bx)R + P = (ax + p_0)R + P \subseteq axR + P \subseteq aR + P = R$$

So $R = aR+P$ a contradiction. This shows that $bx \in K$ and $1-bx \in K$, i.e. bx and $1 - bx$ has not right $P-$inverse, a contradiction. Thus, K

is a closed under addition.
(3) $\Rightarrow$ (6) Suppose that K is a largest right ideal in R. Since by Lemma 5.4 $K \neq R$, there a exists maximal right ideal M in R such that $K \subseteq M$. Since K is largest, $K = M$. This shows that R has exactly one maximal right ideal M such that $P \subseteq M$. (6) $\Rightarrow$ (3) It is clear. □

Definition 5.7. Let R be a ring and $P \neq R$ be a right ideal of R. We say that a ring R is local relative to right ideal P or $P-$local for short, if R satisfies the equivalent condition in Theorem 5.6.

Next, we present an example of $P-$local elements. In view hypothesis of Example 2.5 we can obtain the following:

Example 5.8. Let $\mathbb{Z}$ be the ring of integers and let $R_2 = M_2(Z)$ be the ring of all 2×2 matrices over the ring of integers $\mathbb{Z}$.
(1) Let $\gamma = \begin{bmatrix} 3 & 4 \\ 2 & 2 \end{bmatrix} \in R_2$. It is easy to see that the element γ has no a right $P-$inverse in R_2. But $1 - \gamma = \begin{bmatrix} -2 & -4 \\ -2 & -1 \end{bmatrix} \in R_2$ has a right $P-$inverse. Because for $\beta = \begin{bmatrix} u & 0 \\ -2u & -1 \end{bmatrix} \in R_2$, where $u \in \mathbb{Z}$, $(1-\gamma)\beta = \begin{bmatrix} 6u & 4 \\ 0 & 1 \end{bmatrix}$. Thus, $1 - (1-\gamma)\beta = \begin{bmatrix} 1-6u & -4 \\ 0 & 0 \end{bmatrix} \in P$. This shows that the element $\gamma \in R_2$ is $P-$local in R_2.
(2) It is easy to see that for every $n, m \in \mathbb{Z}$, then element $\alpha = \begin{bmatrix} 1-n & -m \\ 0 & 0 \end{bmatrix} \in R_2$ has no a right $P-$inverse in R_2. But $1 - \alpha = \begin{bmatrix} n & m \\ 0 & 1 \end{bmatrix} \in R_2$ has a right $P-$inverse by Example 5.3. This shows that the element α ia $P-$local.

We again use the notation, let R be a ring and $R_2 = M_2(R)$ be the ring of all 2×2 matrices over a ring R. Let

$$P = \{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in R\}, \quad Q = \{\begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} : a, b \in R\}$$

are right ideals in R_2 such that $P \neq R_2$, $Q \neq R_2$. The connection between the local rings and $P-$local ($Q-$local) rings we provide in the following:

Theorem 5.9. *Let R be a ring and $a \in R$. Then the following hold:*
(1) If a has a right inverse in R, then for every $x \in R$ the element $\alpha = \begin{bmatrix} x & 0 \\ 0 & a \end{bmatrix}$ has a right $P-$inverse in R_2.

(2) *If for some element* $x \in R$*, the element* $\alpha = \begin{bmatrix} x & 0 \\ 0 & a \end{bmatrix}$ *has a right* $P-$*inverse in* R_2*, then* a *has a right inverse in* R*.*
(3) *If* a *has a right inverse in* R*, then for every* $x \in R$ *the element* $\alpha = \begin{bmatrix} a & 0 \\ 0 & x \end{bmatrix}$ *has a right* $Q-$*inverse in* R_2*.*
(4) *If for some element* $x \in R$*, the element* $\alpha = \begin{bmatrix} a & 0 \\ 0 & x \end{bmatrix}$ *has a right* $Q-$*inverse in* R_2*, the* a *has a right inverse in* R*.*

Proof. (1) Assume that a has a right inverse in R, then $ab = 1$ for some $b \in R$. Let $u, v \in R$, then $\beta = \begin{bmatrix} u & v \\ 0 & b \end{bmatrix} \in R_2$ such that for every $x \in R$

$$1 - \alpha\beta = \begin{bmatrix} 1 - xu & -xv \\ 0 & 1 - ab \end{bmatrix} = \begin{bmatrix} 1 - xu & -xv \\ 0 & 0 \end{bmatrix} \in P$$

so α has a right $P-$inverse.
(2) Let $x \in R$ such that $\alpha = \begin{bmatrix} x & 0 \\ 0 & a \end{bmatrix}$ has a right $P-$inverse in R_2, then there exists $\beta = \begin{bmatrix} u & v \\ w & r \end{bmatrix} \in R_2$ where $u, v, w, r \in R$ such that $1 - \alpha\beta \in P$. Thus,

$$1 - \alpha\beta = \begin{bmatrix} 1 - xu & -xv \\ -aw & 1 - ar \end{bmatrix} \in P$$

this shows that $ar = 1$, i.e., a has a right inverse in R. (3) Similarly, as in (1). (4) Similarly, as in (2). □

In view hypothesis of Theorem 3.4 and Theorem 5.9 we can obtain the following:

Corollary 5.10. *For any ring* R *the following hold:*
(1) *A ring* R *is local if and only if the ring* S_0 *is* $P-$*local relative to right ideal* P_0*.*
(2) *A ring* R *is local if and only if the ring* S_0 *is* $P-$*local relative to right ideal* Q_0*.*

Theorem 5.11. *Every ring is local relative to any maximal right ideal of it.*

Proof. Let R be a ring and M be a maximal right ideal of R. Let $a \in R$, we discus two cases $a \in M$ and $a \notin M$.
I- If $a \in M$, then $1 - a \notin M$, so $R = (1 - a)R + M$. This shows that $1 - a$ has a right $M-$inverse.

II- If $a \notin M$, then $R = aR + M$, so a has a right $M-$inverse. Thus, R is $M-$local ring. □

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References

1. A. V. Andrunakievich and V. A. Andrunakievich, *Rings that are regular relative to right ideal*, Mat. Zametki. **49**, (3) (1991), 3–11.
2. V. A. Andrunakievich and Yu. M. Ryabukhin, *Quasiregularity and Pimitivity Relative to Right Ideals of a Ring*, Mat. Sb. **134**, (4) (1987), 451–471.
3. P. Deena and S. Manivasan, *uas-ideal of a regular rings*, International Journal of Algebra, **5** (20) (2011), 1005–1010.
4. K. R. Goodearl, *Von Neumann regular rings*, Pitman. London. (1979).
5. H. Hamza, *On Some Characterizations of Regular and Potent Rings Relative to Right Ideals*, Novi Sad J. Math. **48**, (1) (2018), 1–7.
6. H. Hamza, I_0-*Rings and* I_0-*Modules*, Okayama Math. J. **40** (1998), 91–97.
7. J. Lambek, *Lectures on Rings and Modules*, Blaisdell, Mass. 1966.

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