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Conic Optimization Reformulation for the Continuous Center Location Problem under Uncertainty

Ali Jamalian*

Department of Computer Science, Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran

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ABSTRACT

In this paper, we consider the Euclidean continuous minimax location problem under uncertainty. We consider the single-facility and the multi-facility case with uncertain location of demand points and uncertain transportation costs. We study these two problems under two kinds of uncertainty, the interval and the ellipsoidal uncertainty. Equivalent formulations of robust counterparts of the single facility and multi facility Euclidean continuous minimax location problems under interval and ellipsoidal uncertainty are given as conic optimization problems.

1. Introduction

Location planning is one of the strategic and long-term decisions which decision makers are faced and concerned with finding a good location for one or several new facilities with respect to a given set of existing facilities. There are different kinds of facility location problem, capacitated and uncapacitated facility location problem, p-median problem, p-center problem, covering location problem, Weber location problem, minimax location problem, hub location problem, *p*-maxian location problem are the well-known among them [1-4]. Minimax location problem (MLP) is a type of facility location problems, in which new facilities are located under the minimax criteria. Applications of MLP are concerned with situations in which the new facility or facilities must be placed so that the largest weighted distance between existing and new facilities is minimized. The location of police and fire stations [5], the placement of detection stations [6], and siting service facilities in plants, offices, or warehouses are other applications. In some applications, MLP is considered with Euclidean distances [7-9] while in some others with rectangular distances [10, 11]. One may consult [5, 6, 12-18] for further related results.

* Corresponding author.
E-mail addresses: a.jamalian.math@gmail.com (A. Jamalian)

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Due to the uncertainty of some problem parameters, in recent years, many researchers have focused on the robust optimization method which handles uncertainty of the parameters. Snyder [19], describes two types of uncertainty in location problems in his review paper, stochastic location problems and robust location problems. Baron et. al [20], applied robust optimization approach to the multi-period fixed-charge network location problem with uncertain demand over multiple periods. They used both box and ellipsoidal uncertainties on demands. Jamalian and Salahi [21], considered the multi-facility Weber location problem with uncertain location of demand points and transportation costs. They studied the problem with both the Euclidean and block norms and interval and ellipsoidal uncertainty sets. Equivalent formulations of robust counterparts of the problem is given as conic linear optimization problems. Nikoofal and Sadjadi [22], considered the p-median problem with interval uncertainty on edge lengths and proposed a model to obtain robust solution. Averbakh and Berman [23], studied a minimax-regret formulation of the weighted p-center problem on a network with interval uncertainty on weights. They also considered the minimax-regret 1-center problem with interval uncertainty on node weights and edge lengths [24]. In [25] they further considered the minimax-regret 1-median problem with interval uncertainty on demands and presented an efficient approach to solve it. Burkard and Dollani considered robust 1-median problem on a tree network with uncertain or dynamically changing edge lengths and node weights which can take negative values [26]. Carrizosa and Nickel [27] used an alternative definition of robustness in positioning a facility on the plane, which is defined as the minimal changes in the uncertain parameters such that a solution location becomes inadmissible with respect to a total cost constraint.

In this paper, we present robust formulations for MLP with Euclidean norm in two cases, single facility and multi facility problems with uncertain costs and locations of existing facilities. The rest of the paper is organized as follows. In Section 2, we study the single facility MLP and give its robust counterparts for both interval and ellipsoidal uncertainty sets. In Section 3, the robust counterparts of multi facility MLP are given for both uncertainty sets. Finally, we present some conclusions and future research directions.

2. The Single Facility MLP

In this section, we study the single facility MLP with Euclidean norm as distance function. Let us first define some notations and state the problem modeling:

n: number of existing facilities

w_i: nonnegative weight between new facility and existing facility i by a unit distance

$\| x - P_i \|$: Euclidean distance between the location of new facility and existing facility i

$P_i = \binom{a_i}{b_i}$: the location coordinates of existing facility i

$x = \binom{x_1}{x_2}$: the location coordinates of new facility.

Let n existing facilities be located at the known distinct points $P_1, \dots, P_n$, in the plane. In single facility MLP, the optimal location of new facility, x, is sought with respect to the set of existing facilities. The total transportation cost associated with new facility located at x is given by

$$f(x) = \max_{i=1,\dots,n} w_i \parallel x - P_i \parallel. \quad \textbf{(1)}$$

The MLP can be stated as the selection of x^* for new facility such that total cost in (1) is minimized as follows [2-4]:

$$\min \quad \max_{i=1,\dots,n} w_i \|x - P_i\|, \quad \textbf{(2)}$$

where w_i's are nonnegative. The problem (2) can be written as

$$\begin{aligned} &\min \ z \\ &s.t. \ \ z \geq w_i \|x - P_i\|, \quad i = 1, \dots, n. \end{aligned} \quad \textbf{(3)}$$

In what follows, we consider two types of uncertainties on w_i and P_i for $i = 1, \dots, m$.

***Theorem* 1.** *The robust counterpart of problem (3) with interval uncertainty on* w, $w \in U = [\underline{w}, \overline{w}]$, *and bounded uncertainty on* P_i*'s,* $P_i \in U'_i = \{P_i^0 + \Delta P_i : \|\Delta P_i\| \leq \rho_i\}$, *for* $i = 1, \dots, n$, *is equivalent to*

$$\begin{aligned} &\min \quad z \\ &s.t. \quad z \geq \overline{w}_i (t_i + t'_i), \ \ i = 1, \dots, n, \\ &\qquad\ \ t_i \geq \left\|x - P_i^0\right\|, \quad i = 1, \dots, n, \\ &\qquad\ \ t'_i / \rho_i \geq \left\|\begin{pmatrix} x \\ 1 \end{pmatrix}\right\|, \ \ i = 1, \dots, n. \end{aligned} \quad \textbf{(4)}$$

Proof. The uncertain single MLP with uncertainty sets U, U'_i is as following:

$$\begin{aligned} &\min \ \ z \\ &s.t. \ \ z \geq w_i \|x - P_i\|, \ \ i = 1, \dots, n, \\ &\qquad w \in U, P_i \in U'_i, \ \ i = 1, \dots, n. \end{aligned}$$

To have $w_i \|x - P_i\| \leq z, \ \ \forall P_i \in U_i$ for $i = 1, \dots, n$, it is sufficient to have

$$\max\{w_i \|x - P_i\| : \ P_i \in U_i\} \leq z, \quad i = 1, \dots, n.$$

From the triangular inequality, for uncertainty on P_is we have

$$\begin{aligned} \left\|x - (P_i^0 + \Delta P_i)\right\| \ &= \left\|(x - P_i^0) + (-\Delta P_i)\right\| \\ &\leq \left\|x - P_i^0\right\| + \left\|(0, -\Delta P_i) \begin{pmatrix} x \\ 1 \end{pmatrix}\right\| \\ &\leq \left\|x - P_i^0\right\| + \|(0, -\Delta P_i)\|_F \left\|\begin{pmatrix} x \\ 1 \end{pmatrix}\right\| \\ &= \left\|x - P_i^0\right\| + \|-\Delta P_i\| \left\|\begin{pmatrix} x \\ 1 \end{pmatrix}\right\| \\ &\leq \left\|x - P_i^0\right\| + \ \rho_i \left\|\begin{pmatrix} x \\ 1 \end{pmatrix}\right\|. \end{aligned} \quad \textbf{(5)}$$

Now, for each $i = 1, \dots, n$, let

$$[0, -\Delta P_i] = u_i v^T, \quad \textbf{(6)}$$

where $v = (0 \quad 0 \quad 1)^T$ and

$$u_i = \begin{cases} \rho_i \dfrac{x - P_i^0}{\|x - P_i^0\|} & x - P_i^0 \neq 0, \\ \eta & o.w., \end{cases} \tag{7}$$

where η is any vector in $\mathbb{R}^2$ with $\| \eta \| = \rho_i$. The term $\|x - (P_i^0 + \Delta P_i)\|$ which is bounded from above, takes its maximum with this rank one choice of $[0, -\Delta P_i]$, and $\|[0,\ -\Delta P_i]\|_F = \|[0,\ -\Delta P_i]\| = \rho_i$. Therefore, we can conclude that

$$\max_{\|\Delta P_i\| \leq \rho_i} \|x - (P_i^0 + \Delta P_i)\| = \|x - P_i^0\| + \rho_i \left\|\begin{pmatrix} x \\ 1 \end{pmatrix}\right\|. \tag{8}$$

Hence, the robust counterpart of problem ***Eq.*** (3) using ***Eq.*** (8) is

$$\begin{aligned} &\min\ \ z \\ &s.t.\ \ \ z \geq w_i \left(\|x - P_i^0\| + \rho_i \left\|\begin{pmatrix} x \\ 1 \end{pmatrix}\right\|\right),\ \ i = 1, \dots, n, \\ &\qquad\ \ w \in U, \end{aligned}$$

or equivalently

$$\begin{aligned} &\min\ \ z \\ &s.t.\ \ \ z \geq w_i(t_i + t'_i),\ \ i = 1, \dots, n, \\ &\qquad\ \ t_i \geq \|x - P_i^0\|,\ \ \ \ i = 1, \dots, n, \\ &\qquad\ \ t'_i/\rho_i \geq \left\|\begin{pmatrix} x \\ 1 \end{pmatrix}\right\|,\ \ i = 1, \dots, n, \\ &\qquad\ \ w \in U. \end{aligned} \tag{9}$$

Moreover, to have $w_i(t_i + t'_i) \leq z,\ \ \forall w \in U$ for $i = 1, \dots, n$, it is sufficient to have:

$$\max \{w_i(t_i + t'_i):\ \ w_i \in [\underline{w_i}, \overline{w_i}]\} \leq z,\ \ \ i = 1, \dots, n. \tag{10}$$

Since, t_i and t'_i are nonnegative, ***Eq.*** (10) is equivalent to

$$\overline{w}_i(t_i + t'_i) \leq z \tag{11}$$

By substituting ***Eq.*** (11) in ***Eq.*** (9), we get the model (4). ■

Now, we consider ellipsoidal uncertainties on w_i, P_i's for $i = 1, \dots, n$, in model (4) as follows:

$$\begin{aligned} &U = \{w^0 + Qu, \|u\| \leq 1\}, \\ &U'_i = \{P_i^0 + Q'_i u, \|u\| \leq 1\}, \end{aligned} \tag{12}$$

where $w_i = w_i^0 + q_i^T u,\ Q = [q_1, \dots, q_n],\ Q'_i = [q'^i_1, \dots, q'^i_L]$. Q, Q'_i's are given matrices in $\mathbb{R}^{n \times n}$ and $\mathbb{R}^{2 \times L}$, respectively and w_i^0, P_i^0's are the nominal values. The following technical lemma is crucial for the next theorem.

Lemma **1.** Uncertain inequality $\|Ax - b\| \leq \lambda,\ \forall (A, b) \in \boldsymbol{\varepsilon} = \{(A, b) = (A^0, b^0) + \sum_{k=1}^m u_k(A^k, b^k): \|u\| \leq 1\}$, is equivalent to

$$\begin{pmatrix} \lambda-\mu & 0 & \dots & 0 & (A^0x-b^0)^T \\ 0 & \mu & \dots & 0 & (A^1x-b^1)^T \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & \mu & (A^mx-b^m)^T \\ A^0x-b^0 & A^1x-b^1 & \dots & A^mx-b^m & \lambda I \end{pmatrix} \succcurlyeq 0$$

where $(\lambda, \mu, x) \in \mathbb{R}^{n+2}$.

Proof. See [28].

***Theorem* 2.** *The robust counterpart of problem (3) with ellipsoidal uncertainty sets (12) is equivalent to the following conic optimization problem:*

$$\begin{aligned} & min \quad z \\ & s.t. \quad (w_i^0 + \|q_i\|)t_i \le z, \qquad\qquad i = 1, \dots, n, \\ & \begin{pmatrix} t_i-\mu_i & 0 & \dots & 0 & (x-P_i^0)^T \\ 0 & \mu_i & \dots & 0 & (-q'^i_1)^T \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & \mu_i & (-q'^i_L)^T \\ x-P_i^0 & -q'^i_1 & \dots & -q'^i_L & t_iI \end{pmatrix} \succcurlyeq 0, \quad i = 1, \dots, n. \end{aligned} \tag{13}$$

Proof. The robust counterpart of problem (3) can be written as follows:

$$\begin{aligned} & min \quad z \\ & s.t. \quad z \ge w_it_i, \qquad i = 1, \dots, n, \\ & \qquad t_i \ge \|x-P_i\|, \quad i = 1, \dots, n, \\ & \qquad w \in U, P_i \in U'_i \quad i = 1, \dots, n. \end{aligned}$$

Let $t \in \mathbb{R}^n$ is given, then to have $w_it_i \le z, \ \ \forall w \in U, \ i = 1, \dots, n,$ it is sufficient to have

$$max\{w_it_i: \ w \in U\} \le z. \tag{14}$$

Since $w_it_i = (w_i^0)t_i + (q_i^Tu)t_i$, thus we have

$$\max_{w \in U} \quad w_it_i = (w_i^0)t_i + \|q_i\|t_i. \tag{15}$$

Therefore, ***Eq.*** (14) holds if

$$(w_i^0 + \|q_i\|)t_i \le z. \tag{16}$$

Moreover, by *Lemma* 1, $\|x-P_i\| \le t_i, \quad \forall P_i \in U'_i$ is equivalent to

$$\begin{pmatrix} t_i-\mu_i & 0 & \dots & 0 & (x-P_i^0)^T \\ 0 & \mu_i & \dots & 0 & (-q'^i_1)^T \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & \mu_i & (-q'^i_2)^T \\ x-P_i^0 & -q'^i_1 & \dots & -q'^i_2 & t_iI \end{pmatrix} \succcurlyeq 0, \quad i = 1, \dots, n. \tag{17}$$

Thus, with ***Eq.*** (16) and ***Eq.*** (17), we get model (13). ■

3. The Multi Facility MLP

In this section, we study the multi facility MLP. Let us first define some notations and state the problem modeling:

$P_i = \binom{a_i}{b_i}$: The location coordinates of existing facility i, $(i = 1, \dots, n)$

$x_j = \binom{x_{1j}}{x_{2j}}$: The location coordinates of new facility j, $(j = 1, \dots, m)$

w_{ij} : The nonnegative weight between new facility j and existing facility i by a unit distance, $(i = 1, \dots, n, j = 1, \dots, m)$

v_{jk} : The nonnegative weight between new facilities j and k by a unit distance, $(j, k = 1, \dots, m)$

$\| x_j - P_i \|$: The Euclidean distance between the location of new facility j and existing facility i

$\| x_j - x_k \|$: The Euclidean distance between the location of new facilities j and k

Let n existing facilities be located at the known distinct points $P_1, \dots, P_n$, in the plane. The problem is to select $x = (x_1, \dots, x_m)$ to minimize the largest weighted distance between facilities [29, 30]. The transportation costs between facilities are

$$f_1(x_1, \dots, x_m) = max\,\{w_{ij} \| x_j - P_i \| : \ i = 1, \dots, n, j = 1, \dots, m\},$$
$$f_2(x_1, \dots, x_m) = max\,\{v_{jk} \| x_j - x_k \| : \ 1 \le j < k \le m\}.$$

The total cost to be minimized is given by

$$f(x_1, \dots, x_m) = max\{f_1(x_1, \dots, x_m), f_2(x_1, \dots, x_m)\}. \tag{18}$$

Thus the problem is

$$min\, f(x_1, \dots, x_m), \tag{19}$$

which can be reformulated as follows:

$$\begin{aligned} & min \ \ z \\ & s.t. \ \ z \ge w_{ij} \|x_j - P_i\|, \quad i = 1, \dots, n, j = 1, \dots, m, \\ & \qquad\ z \ge v_{jk} \|x_j - x_k\|, \quad 1 \le j < k \le m. \end{aligned} \tag{20}$$

In the sequel, we consider various uncertainties on w_{ij}, v_{jk} and P_i for $i = 1, \dots, n, 1 \le j < k \le m$ and give the robust counterparts of problem (20).

***Theorem* 3.** *The robust counterpart of problem (20) with interval uncertainties on w and v, $w \in U_1 = [\underline{w}, \overline{w}]$, $v \in U_2 = [\underline{v}, \bar{v}]$ and bounded uncertainty on P_i's, $P_i \in U'_i = \{P_i^0 + \Delta P_i : \|\Delta P_i\| \le \rho_i\}$, for $i = 1, \dots, n$, is equivalent to*

$$min \quad z \tag{21}$$

$$
\begin{aligned}
s.t.\quad & z \geq \bar{w}_{ij}(t_{ij} + t''_{ij}), && i = 1, \dots, n, j = 1, \dots, m,\\
& z \geq \bar{v}_{jk} t'_{jk}, && 1 \leq j < k \leq m,\\
& t_{ij} \geq \|x_j - P_i^0\|, && i = 1, \dots, n, j = 1, \dots, m,\\
& t'_{jk} \geq \|x_j - x_k\|, && 1 \leq j < k \leq m,\\
& t''_{ij}/\rho_i \geq \left\|\begin{pmatrix} x_j \\ 1 \end{pmatrix}\right\|, && i = 1, \dots, n, j = 1, \dots, m.
\end{aligned}
$$

Proof. The uncertain multi-facility MLP with uncertainty sets U_1, U_2, U'_i is as follows:

$$
\begin{aligned}
& \min\ z\\
s.t.\quad & z \geq w_{ij}\|x_j - P_i\|, && i = 1, \dots, n, j = 1, \dots, m,\\
& z \geq v_{jk}\|x_j - x_k\|, && 1 \leq j < k \leq m,\\
& w \in U_1, v \in U_2, P_i \in U'_i, && i = 1, \dots, n.
\end{aligned}
$$

To have $w_{ij}\|x_j - P_i\| \leq z,\ \forall P_i \in U'_i$ for $i = 1, \dots, n$, it is sufficient to have:

$$
\max\{w_{ij}\|x_j - P_i\|:\ P_i \in U'_i\} \leq z, \quad i = 1, \dots, n, j = 1, \dots, m.
$$

Similar to the interval uncertainty case in the single facility MLP, from the triangular inequality for uncertainty on P_is we have

$$
\|x_j - (P_i^0 + \Delta P_i)\| \leq \|x_j - P_i^0\| + \rho_i \left\|\begin{pmatrix} x_j \\ 1 \end{pmatrix}\right\|. \tag{22}
$$

For each $i = 1, \dots, n$, let

$$
[0, -\Delta P_i] = u_i v^T, \tag{23}
$$

where $v = (0 \quad 0 \quad 1)^T$ and

$$
u_i = \begin{cases} \rho_i \dfrac{x_j - P_i^0}{\|x_j - P_i^0\|} & x_j - P_i^0 \neq 0,\\ \eta & o.w., \end{cases} \tag{24}
$$

where η is any vector $\in \mathbb{R}^2$ of norm ρ_i. The term $\|x_j - (P_i^0 + \Delta P_i)\|$ takes its maximum with this rank one choice of $[0, -\Delta P_i]$. Therefore, we can conclude that

$$
\max_{\|\Delta P_i\| \leq \rho_i} \|x_j - (P_i^0 + \Delta P_i)\| = \|x_j - P_i^0\| + \rho_i \left\|\begin{pmatrix} x_j \\ 1 \end{pmatrix}\right\|. \tag{25}
$$

Hence, the robust counterpart of problem (20) using ***Eq.*** (25) is

$$
\begin{aligned}
& \min\ z\\
s.t.\quad & z \geq w_{ij}\left(\|x_j - P_i^0\| + \rho_i \left\|\begin{pmatrix} x_j \\ 1 \end{pmatrix}\right\|\right), && i = 1, \dots, n, j = 1, \dots, m\\
& z \geq v_{jk}\|x_j - x_k\|, && 1 \leq j < k \leq m,\\
& w \in U_1, v \in U_2,
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
& \min\ z\\
s.t.\quad & z \geq w_{ij}(t_{ij} + t''_{ij}), \quad i = 1, \dots, n, j = 1, \dots, m,
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
& z \geq v_{jk} t'_{jk}, && 1 \leq j < k \leq m,\\
& t_{ij} \geq \|x_j - P_i^0\|, && i = 1, \dots, n, j = 1, \dots, m,\\
& t'_{jk} \geq \|x_j - x_k\|, && 1 \leq j < k \leq m,\\
& t''_{ij}/\rho_i \geq \left\|\begin{pmatrix} x_j \\ 1 \end{pmatrix}\right\|, && i = 1, \dots, n, j = 1, \dots, m,\\
& w \in U_1, v \in U_2.
\end{aligned}
$$

Moreover, to have $w_{ij}(t_{ij} + t''_{ij}) \leq z,\ \ \forall w \in U_1$ and $v_{jk} t'_{jk} \leq z,\ \forall v \in U_2$ for all i, j and k, it is sufficient to have

$$max\left\{w_{ij}(t_{ij} + t''_{ij}):\ w \in [\underline{w}_{ij}, \overline{w}_{ij}]\right\} \leq z, \quad i = 1, \dots, n, j = 1, \dots, m, \tag{27}$$

$$max\left\{v_{jk} t'_{jk}:\ v \in [\underline{v}_{jk}, \overline{v}_{jk}]\right\} \leq z, \quad 1 \leq j < k \leq m. \tag{28}$$

Since, t_{ij}, t'_{jk} and t''_{ij} are nonnegative, ***Eq.*** (27) and ***Eq.*** (28) are equivalent to

$$\overline{w}_{ij}(t_{ij} + t''_{ij}) \leq z, \tag{29}$$

$$\overline{v}_{jk} t'_{jk} \leq z. \tag{30}$$

By substituting ***Eq.*** (29) and ***Eq.*** (30) in (26), we get model (21). ■

Now, we consider ellipsoidal uncertainties on w_{ij}, v_{jk}, P_i's for $i = 1, \dots, n, j = 1, \dots, m, k = 1, \dots, m$ in problem (20) with the following uncertainty sets:

$$
\begin{aligned}
& U = \{w^0 + Qu, \|u\| \leq 1\},\\
& U' = \{v^0 + Q'u, \|u\| \leq 1\},\\
& U''_i = \{P_i^0 + Q''_i u, \|u\| \leq 1\},
\end{aligned} \tag{31}
$$

where $w_{ij} = w_{ij}^0 + q_{ij}^T u,\ Q = [q_1, \dots, q_{mn}],\ v_{jk} = v_{jk}^0 + q'^T_{jk} u,\ Q' = [q'_1, \dots, q'_{m^2}]$ and $Q''_i = [q''^i_1, \dots, q''^i_L]$. We assume that w_{ij} is the $(im - m + j)$-th element of w and v_{jk} is the $(jm - m + k)$-th row of vector v. Q, Q' and Q''_i's are given matrices in $\mathbb{R}^{mn \times mn}, \mathbb{R}^{m^2 \times m^2}$ and $\mathbb{R}^{2 \times L}$, respectively and w_{ij}^0 and v_{jk}^0, P_i^0's are the nominal values.

***Theorem* 4.** *The robust counterpart of problem (20) with ellipsoidal uncertainty sets (31) is equivalent to the following conic optimization problem:*

$$
\begin{aligned}
min \quad & z\\
s.t. \quad & (\| q_{ij} \| + w_{ij}^0) t_{ij} \leq z, && i = 1, \dots, n, j = 1, \dots, m,\\
& (\| q'_{jk} \| + v_{jk}^0) t'_{jk} \leq z, && j, k = 1, \dots, m,\\
& \| x_j - x_k \| \leq t'_{jk}, && j, k = 1, \dots, m,\\
& \begin{pmatrix}
t_{ij} - \mu_{ij} & 0 & \dots & 0 & (x_j - P_i^0)^T\\
0 & \mu_{ij} & \dots & 0 & (-q''^i_1)^T\\
\vdots & & \ddots & & \vdots\\
0 & 0 & \dots & \mu_{ij} & (-q''^i_L)^T\\
x_j - P_i^0 & -q''^i_1 & \dots & -q''^i_L & t_{ij} I
\end{pmatrix} \succcurlyeq 0, && \begin{array}{l} i = 1, \dots, n,\\ j = 1, \dots, m. \end{array}
\end{aligned} \tag{31}
$$

Proof. Robust counterpart of problem (20) can be written as follows

$$
\begin{aligned}
min \quad & z\\
s.t. \quad & z \geq w_{ij} t_{ij}, && i = 1, \dots, n, j = 1, \dots, m,
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
& z \geq v_{jk} t'_{jk}, && j,k = 1,\dots,m, \\
& t_{ij} \geq \|x_j - P_i\|, && i = 1,\dots,n, j = 1,\dots,m, \\
& t'_{jk} \geq \|x_j - x_k\|, && j,k = 1,\dots,m, \\
& w \in U, v \in U', P_i \in U''_i, && i = 1,\dots,n.
\end{aligned}
$$

Let $t, t' \in \mathbb{R}^{mn}$ and $\mathbb{R}^{m^2}$ are given, then to have $w_{ij} t_{ij} \leq z, \ \ \forall w \in U$, and $v_{jk} t'_{jk} \leq z, \ \ \forall v \in U'$, it is sufficient to have

$$\max_{w \in U} \ w_{ij} t_{ij} \leq z, \tag{33}$$

$$\max_{v \in U'} \ v_{jk} t'_{jk} \leq z. \tag{34}$$

Since

$$w_{ij} t_{ij} = w_{ij}^0 t_{ij} + (q_{ij}^T u) t_{ij}, \tag{35}$$

$$v_{jk} t'_{jk} = v_{jk}^0 t'_{jk} + (q'^T_{jk} u) t'_{jk}, \tag{36}$$

thus we have

$$
\begin{aligned}
& \max_{w \in U} \quad w_{ij} t_{ij} = w_{ij}^0 t_{ij} + \| \, q_{ij}^T t_{ij} \, \|, \\
& \max_{v \in U'} \quad v_{jk} t'_{jk} = v_{jk}^0 t'_{jk} + \| \, q'^T_{jk} t'_{jk} \, \|.
\end{aligned} \tag{37}
$$

Therefore, ***Eq.*** (33) and ***Eq.*** (34) hold if

$$
\begin{aligned}
& \| \, q_{ij} t_{ij} \, \| \leq z - w_{ij}^0 t_{ij}, \\
& \| \, q'_{jk} t'_{jk} \, \| \leq z - v_{jk}^0 t'_{jk}.
\end{aligned} \tag{38}
$$

Moreover, by *Lemma* 1, $\|x_j - P_i\| \leq t_{ij}, \quad \forall P_i \in U''_i$ is equivalent to

$$
\begin{pmatrix}
t_{ij} - \mu_{ij} & 0 & \dots & 0 & (x_j - P_i^0)^T \\
0 & \mu_{ij} & \dots & 0 & (-q''^i_1)^T \\
\vdots & & \ddots & & \vdots \\
0 & 0 & \dots & \mu_{ij} & (-q''^i_L)^T \\
x_j - P_i^0 & -q''^i_1 & \dots & -q''^i_L & t_{ij} I
\end{pmatrix} \succcurlyeq 0, \quad
\begin{aligned} & i = 1,\dots,n, \\ & j = 1,\dots,m. \end{aligned} \tag{39}
$$

Thus, by ***Eq.*** (38) and ***Eq.*** (39), we get model (31). ■

Conclusions

In this paper, the robust counterpart of single facility and multi-facility MLP for both interval and ellipsoidal uncertainty sets are discussed. Depending on the uncertainty sets, it leads to an equivalent conic optimization problem with second-order cone constraints or positive semi-definite constraints. The extension of our approach to the other kinds of MLPs like multi-period and capacitated problems are left for interested readers.

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