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ON MAX-INJECTIVE MODULES

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ABSTRACT. Let R be a commutative ring and M an R-module. In this paper, we explore more properties of Max-injective modules and we study some conditions under which the maximal spectrum of M is a Max-spectral space for its Zariski topology.

1. INTRODUCTION

Throughout this article, R denotes a commutative ring with non zero identity and all modules are nonzero and unitary. For any ideal I of R containing $Ann_R(M)$, $\bar{R}$ and $\bar{I}$ denote $R/Ann(M)$ and $I/Ann(M)$, respectively. Further, $\mathbb{N}$ and $\mathbb{Z}$ denote respectively the set of positive integers and the ring of integers.

Definition 1.1. For M as an R-module and P, N its submodules, we recall

- The *colon ideal of M into N*, $(N :_R M) = \{r \in R | rM \subseteq N\} = Ann(M/N)$
(simply $(N : M)$).
- P is a *prime submodule or p-prime submodule* of M if $P \neq M$ and for $p = (P : M)$, whenever $re \in P$ for $r \in R$ and $e \in M$, we have $r \in p$ or $e \in P$. If Q is a maximal submodule of M, then Q is a prime submodule and $(Q : M) = m$ is a maximal ideal of R. In this case, we say Q is an m-maximal submodule of M [10] and [11].

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- The set of all maximal submodules of M is denoted by $Max_R(M)$. Moreover, if $p \in Spec(R)$ (resp. $m \in Max(R)$), then $Spec_p(M)$ (resp. $Max_m(M)$) is the set of all p-prime (resp. m-maximal) submodules of M. The *prime spectrum (or simply, the spectrum)* of M is the set of all prime submodules of M and denoted by $Spec_R(M)$.
- If $Spec_R(M) \neq \emptyset$ (resp. $Max_R(M) \neq \emptyset$), the mapping $\psi : Spec_R(M) \to Spec(\bar{R})$ (resp. $\phi : Max_R(M) \to Max(\bar{R})$ such that $\psi(P) = \overline{(P : M)}$ (resp. $\phi(Q) = \overline{(Q : M)}$) for every $P \in Spec_R(M)$ (resp. $Q \in Max_R(M)$), is called the *natural map* of $Spec_R(M)$ (resp. $Max_R(M)$) [12] and [3].
- M is said to be *primeful* if either $M = \mathbf{(0)}$ or $M \neq \mathbf{(0)}$ and the natural map of $X = Spec_R(M)$ is surjective [13].
- M is said to be X*-injective*, where $X = Spec_R(M)$, if either $X = \emptyset$ or $X \neq \emptyset$ and the natural map of X is injective [1].
- M is said to be a *Max-surjective* module (or an Ms-module for short) if either $M = \mathbf{(0)}$ or $M \neq \mathbf{(0)}$ and the natural map of $Max_R(M)$ is surjective [3].
- M is called Max-injective if $Max_R(M) = \emptyset$ or $Max_R(M) \neq \emptyset$ and the natural map of $Max_R(M)$ is injective [5].
- M is said to be *weak multiplication* (resp. *Max-weak multiplication*) module if either $Spec_R(M)$ (resp. $Max_R(M)) = \emptyset$ or $Spec_R(M)$ (resp. $Max_R(M)) \neq \emptyset$ and and for every prime (resp. maximal) submodule P of M, $P = IM$ for some ideal I of R . Every weak multiplication or X-injective module is Max-weak multiplication [4] and [6].
- The *Zariski topology* on $X = Spec_R(M)$ is the topology τ_M described by taking the set $Z(M) = \{V(N) | N$ is a submodule of $M\}$ as the set of closed sets of X, where $V(N) = \{P \in X | (P : M) \supseteq (N : M)\}$ [12].
- There exists a topology on $Max_R(M)$ having $Z^m(N) = \{V^m(N) | N$ is a submodule of $M\}$ as the set of closed sets of $Max_R(M)$, where $V^m(N) = \{Q \in Max_R(M) | (Q : M) \supseteq (N : M)\}$. This topology is denoted by τ_M^m. In fact this topology is the same as the subspace topology induced by τ_M on $Max_R(M)$ [5]. Also for every ideal I of R, we have

$$V_R^m(I) = \{q \in Max(R) | q \supseteq I\}.$$

H. Ansari-Toroghy, S. Keyvani, and S.S. Pourmortazavi introduced the concepts of Max-injective and Max-weak multiplication modules in [5] and [4], respectively. These modules contains the family of X-injective and weak multiplication modules properly. They developed some basic properties of X-injective and weak multiplication modules to these new classes of modules. In Theorem 3.3 of this paper, we prove that these classes of modules are the same. This theorem provides some useful characterizations for Max-injective modules and it helps us to study some algebraic properties of this class of modules. For example in Theorem 3.7, we investigate a necessary and sufficient condition for which a direct sum of Max-injective R-modules is Max-injective. Also we explore some conditions under which $Max(M)$ is a spectral space, where M is a Max-injective module (see 3.10).

2. Preliminaries

In this section we review some properties of prime submodules and maximal submodules.

Remark 2.1. Let M be an R-module.

(a) Let K be a submodule of M such that $(K : M)$ is a maximal ideal of R. Then K is a prime submodule of M.
(b) Let P be a prime submodule of M and S be a multiplicatively closed subset of R. Then $S^{-1}(P :_R M) = (S^{-1}P :_{S^{-1}R} S^{-1}M)$ [10].

Remark 2.2. [7, Lemma. 2.7]. Let M be an R-module and p is a maximal ideal of R.

(a) If N is a p-maximal submodule of M, then N_p is a maximal submodule of the R_p-module M_p.
(b) If M is finitely generated module and N is a maximal submodule of M_p, then $N \cap M$ is a p-maximal submodule of M.

Remark 2.3. [11, Proposition. 1]. Let S be a multiplicatively closed subset of R, p a prime ideal of R such that $p \cap S = \emptyset$, and M an R-module. Then there exists a one-to-one correspondence between the p-prime submodules P of M and the $S^{-1}p$-prime submodules W of $S^{-1}M$. This is such that when P and W correspond $W = S^{-1}P$ and $P = W \cap M$.

Remark 2.4. Let M be an R- module. Then the following statements are equivalent:

(a) $M = 0$;
(b) $\forall\ p \in Spec(R)$, $M_p = 0$;
(c) $\forall\ \underline{m} \in Max(R)$, $M_{\underline{m}} = 0$.

3. Main results

Remark 3.1. [5, Proposition. 3.19] Let M be an R-module. Then the following are equivalent.

(a) $Max_R(M)$ is a T_0-Space;
(b) $Max_R(M)$ is a T_1-Space;
(c) M is Max-injective.

Proposition 3.2. (a) Every homomorphic image of a Max-weak multiplication module is a Max-weak multiplication module.
(b) Every homomorphic image of a Max-injective module is a Max-injective module.
(c) If M is a free R-module, then M is Max-weak multiplication if and only if M is cyclic.
(d) If M is a free R-module, then M is Max-injective if and only if M is cyclic.
(e) Let M be a Max-weak multiplication R-module and I is an ideal of R containing $Ann(M)$. Then M is a Max-weak multiplication $\frac{R}{I}$-module.
(f) Let M be a Max-injective R-module and I is an ideal of R containing $Ann(M)$. Then M is a Max-injective $\frac{R}{I}$-module.

Proof. (a) See [4, Theorem. 3.13 (a)].
(b) This part is immediate by using the fact that

$$Max_R(\frac{M}{N}) = \{\frac{Q}{N} : Q \in Max_R(M), Q \supseteq N\}.$$

(c) The sufficiency is clear. To see the necessity let M be a Max-weak multiplication free R-module and let M is not cyclic. Without loss of generality, let $M = R \oplus R$ and $p \in Max(R)$. Then $\dfrac{M}{p \oplus R} \cong \dfrac{R}{p}$. This implies that $p \oplus R \in Max(M)$. Since M is a weak multiplication module, we have $p \oplus R = pM$, a contradiction.
(d) The sufficiency is clear. To see the necessity let M be a Max-injective free R-module and let M is not cyclic. We can choose $M = R \oplus R$. Then for a maximal prime ideal p of R, we have $\dfrac{M}{p \oplus R} \cong \dfrac{R}{p} \cong \dfrac{M}{R \oplus p}$. This implies that $\{p \oplus R, R \oplus p\} \subseteq Max(M)$. Since M is Max-injective and $(p \oplus R : M) = (M : R \oplus p) = p$, Hence $p \oplus R = R \oplus p$, a contradiction.
(e) Let $P \in Max_{\frac{R}{I}}(M)$. Then there exist $p \in Max(R) \cap V(Ann(M))$ such that $(P :_{\frac{R}{I}} M) = \dfrac{p}{I}$. Now we have $\frac{p}{I}M \subseteq P$. Thus

$pM \subseteq P$. This implies that $(P : M) = p$. Now since M is a Max-weak multiplication R-module, $P = pM = \frac{p}{I}M$.

(f) Let $Q, P \in Max_{\frac{R}{I}}(M)$ such that $(P :_{\frac{R}{I}} M) = (Q :_{\frac{R}{I}} M)$. Then there exist $p \in Max(R) \cap V(Ann(M))$ such that $(P :_{\frac{R}{I}} M) = (Q :_{\frac{R}{I}} M) = \dfrac{p}{I}$. Now we have $\frac{p}{I}M \subseteq P$ and $\frac{p}{I}M \subseteq Q$. Thus $pM \subseteq P, pM \subseteq Q$. This implies that $(P : M) = (Q : M)$. Now since M is a Max-injective R-module, $P = Q$.

□

The following theorem is one of the main results of this article. In this theorem, we characterize Max-injective modules. These statements are fundamental tools to investigate this class of modules.

Theorem 3.3. *Let M be an R-module. Then the following are equivalent:*

(a) *M is Max-injective;*
(b) *$|Max_P(M)| \leq 1$ for every $p \in Max(R)$;*
(c) *$(Max_R(M), \tau_M^m)$ is a T_0-Space;*
(d) *$(Max_R(M), \tau_M^m)$ is a T_1-Space;*
(e) *For every $p \in V_R^m(Ann(M)); Spec_p(M) \subseteq Max(M)$;*
(f) *For every $p \in V_R^m(Ann(M)); P \in Spec_p(M) \Rightarrow P = pM$;*
(g) *$Max(M) = \{pM \mid p \in V_R^m(Ann(M)), pM \neq M\}$;*
(h) *M is a Max-weak multiplication R-module;*
(i) *For every $p \in V_R^m(Ann(M))$; M_p is a Max-injective R-module.*

Proof. $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d)$. See [5, Proposition. 3.19].

$(a) \Rightarrow (e)$. Let $p \in V_R^m(Ann(M))$, and $P \in Spec_p(M)$. Then we have $(P : M) = p$ and therefore $pM \subseteq P$. But $\dfrac{M}{pM}$ is a Max-injective R-module by Proposition 3.2(b). This implies that $\dfrac{M}{pM}$ is a Max-injective $\dfrac{R}{p}$-module, by Proposition 3.2(f). Thus $\dfrac{M}{pM} \cong \dfrac{R}{p}$ by Proposition 3.2(d) and therefore $pM \in Max(M)$. It follows that $P = pM \in Max(M)$.

$(e) \Rightarrow (a)$. Let $P, Q \in Max(M)$ such that $(P : M) = (Q : M) = p \in Max(R)$. Then $p \subseteq (pM : M) \subseteq (P : M) = p$. Therefore $(pM : M) = p \in V_R^m(Ann(M))$. This implies that $pM \in Max(M)$. On the other hand $pM \subseteq P$ and $pM \subseteq Q$. Hence $pM = P = Q$.

$(e) \Rightarrow (f)$. Let $p \in V_R^m(Ann(M))$ and $P \in Spec_p(M)$. Then we have $p = (pM : M) = (P : M)$. This implies that $pM \in Spec_p(M)$. Since $pM \subseteq P$, we have $pM = P$.

$(f) \Rightarrow (g)$. Set

$$T = \{pM \mid p \in V_R^m(Ann(M)), pM \neq M\}.$$

Let $N \in Max(M)$. Then $(N : M) = q \in V_R^m(Ann(M))$. Thus $N \in Spec_q(M)$, and by hypothesis $N = qM$. Therefore $N \in T$. To prove the converse assume that $q \in V_R^m(Ann(M))$ and $qM \neq M$. It is enough to show that $qM \in Max(M)$. Let N be a submodule of M such that $qM \subseteq N \subset M$. Then $qM : M = N : M = q$. Hence by assumption $N \in Spec_q(M)$ and therefore $N = qM$. This implies that $qM \in Max(M)$.

$(g) \Rightarrow (e)$. Let $p \in V_R^m(Ann(M))$ and let $P \in Spec_p(M)$. Since $(P :_R M) = p$, then $pM \subseteq P$ and so $pM \in Max_R(M)$ by hypothesis. This implies that $pM = P$.

$(a) \Rightarrow (h)$. We proved that (a) and (g) are equivalent. Also $(g \Rightarrow h)$ is straightforward. Thus $(a) \Rightarrow (h)$ is true.

$(h) \Rightarrow (a)$. Let M be an Max-weak multiplication R-module and $Q_1, Q_2 \in Max(M)$ with $(Q_1 : M) = (Q_2 : M)$. Then for some maximal ideals I_1 and I_2 of R, we have $Q_1 = I_1M$ and $Q_2 = I_2M$. Hence $(Q_1 : M)M = (I_1M : M)M = I_1M = Q_1$ and $(Q_2 : M)M = (I_2M : M)M = I_2M = Q_2$. Therefore $Q_1 = (Q_1 : M)M = (Q_2 : M)M = Q_2$.

$(a) \Rightarrow (i)$. Let $W, W' \in Max_{R_p}(M_p)$ such that $(W :_{R_p} M_p) = (W' :_{R_p} M_p)$. By Remark 2.3, there exist $P, Q \in Spec_R(M)$ with $PR_p = W, QR_p = W'$. Therefore by Remark 2.1 (b) we have $(PR_p : M_p) = (P : M)_p$ and $(QR_p : M_p) = (Q : M)_p$. Thus $(P : M)_p = (Q : M)_p = pR_p$, whence $(P : M) = (Q : M) = p$. But by $(a) \Rightarrow (e)$, $P, Q \in Max(M)$. This implies that $P = Q$, so that $W = W'$.

$(i) \Rightarrow (a)$. Let $P, Q \in Max(M)$ such that $(P : M) = (Q : M)$. Then there exists $p \in Max(R) \cap V(Ann(M)$ such that $(P : M) = (Q : M) = p$. Therefore $(P : M)_p = (Q : M)_p$, and we have

$$(pR_p :_{R_p} M_p) = (QR_p :_{R_p} M_p) = pR_p.$$

Now since M_p is a Max-injective R_p-module, $PR_p = QR_p = pM_p$ by ((a) $\Leftrightarrow$ (f)). Hence $(\frac{Q}{pM})_p = 0$. Now set $M' = \frac{Q}{pM}$, then $Ann(M') = (pM : Q) = p$. Hence $M' = 0$ as an $\bar{R}$-module by Remark 2.4. This implies that $M' = 0$ as an R-module, so that $Q = pM$. By similar arguments we can show that $P = pM$. Thus $P = Q$ as desired and the proof is completed. □

Corollary 3.4. Let M be an R-module. Then the following hold.

(a) If $Spec_R(M) = Max_R(M)$, then M is weak multiplication if and only if M is Max-injective.

(b) If M is Max-injective and $Spec(R) = Max(R)$, then

$$Spec_R(M) = Max_R(M) = \{pM | p \in V_R^m(Ann(M)), pM \neq M\}$$

(c) If $Spec(R) = Max(R)$, then M is weak multiplication if and only if M is Max-injective.

Proof. (a) Follows from ((a) $\Leftrightarrow$ (h)) of Theorem 3.3.

(b) The claim is immediate by using ((a) $\Leftrightarrow$ (e) $\Leftrightarrow$ (g)) of Theorem 3.3.

(c) Use parts (a) and (b).

□

The following example shows that the condition $Spec_R(M) = Max_R(M)$ or $Spec(R) = Max(R)$ in the above theorem can not be dropped.

Example 3.5. Let $\{p_i\}_{i\in\mathbb{N}}$ be the set of all prime integers. Then $M = \prod_{i\in\mathbb{N}} \frac{\mathbb{Z}}{p_i\mathbb{Z}}$ is a Max-injective $\mathbb{Z}$-module which is not weak multiplication.

Proof. by [2, Table of example. 3.1], we have

$$Max(M) = \{p_iM : i \in \mathbb{N}\}, \quad Spec(M) = \{p_iM : i \in \mathbb{N}\} \cup \{\bigoplus_{i\in\mathbb{N}} \frac{\mathbb{Z}}{p_i\mathbb{Z}}\}.$$

Hence $\bigoplus_{i\in\mathbb{N}} \frac{\mathbb{Z}}{p_i\mathbb{Z}} \in Spec_0(M)$ while $\bigoplus_{i\in\mathbb{N}} \frac{\mathbb{Z}}{p_i\mathbb{Z}} \neq 0M$. Thus M is not weak multiplication. But by ((a) $\Leftrightarrow$ (g)) of Theorem 3.3, M is Max-injective. □

Definition 3.6. A family $(M_i)_{i\in I}$ of R-modules is said to be *Max-compatible* if for all $i \neq j$ in I, there does not exist a maximal ideal p in R with $Max_p(M_i)$ and $Max_p(M_j)$ both non-empty.

Theorem 3.7. *Let $(M_i)_{i\in I}$ be a family of R-modules and let $M = \bigoplus_{i\in I} M_i$. Then*

(a) *M is a Max-injective R-module if and only if $(M_i)_{i\in I}$ is the family of Max-compatible Max-injective modules.*
(b) *If $M = \bigoplus_{i\in I} M_i$ is a Max-injective R-module, then*

$$Max_R(M) = \{pM_j \oplus (\bigoplus_{j\neq i\in\Lambda} M_i) | p \in V_R^m(Ann(M)), pM_j \neq M_j\}.$$

Proof. (a) If M is a Max-injective R-module, then each M_i is Max-injective by Proposition 3.2 (b). We prove that M_k and M_j are Max-compatible for all $k \neq j$ in I. To see this, let $p \in Max(R)$ and assume that $Max_p(M_k)$ and $Max_p(M_j)$ are both non-empty. Choose $Q_k \in Max_p(M_k)$ and $Q_j \in Max_p(M_j)$. Then we have

$$\frac{M}{Q_k \oplus (\bigoplus_{k\neq i\in\Lambda} M_i)} \cong \frac{M_k}{Q_k}, \quad \frac{M}{Q_j \oplus (\bigoplus_{j\neq i\in\Lambda} M_i)} \cong \frac{M_j}{Q_j}.$$

This implies that $Q_k \oplus (\bigoplus_{k\neq i\in\Lambda} M_i), Q_j \oplus (\bigoplus_{i\neq j\in\Lambda} M_i) \in Max_p(M)$, a contradiction.

To see the revers implication, let $p \in V_R^m(Ann(M))$ and $P \in Spec_p(M)$. Since $(P : M) = p$, we have $\bigoplus_{i\in\Lambda} pM_i \subseteq P$. But M_i's are Max-compatible. Hence there exists a unique element $j \in \Lambda$ such that $pM_j \neq M_j$. This in turn implies that $\frac{M}{pM} \cong \frac{M_j}{pM_j}$. Now since M_j is a Max-injective R-module, $pM_j \in Max(M_j)$ by Theorem 3.3. It follows that $pM \in Max(M)$ and therefore $P = pM \in Max(M)$. Thus M is Max-injective by ((a) $\Leftrightarrow$ (e)) of Theorem 3.3.

(b) Set $T = \{pM_j \oplus (\bigoplus_{j\neq i\in\Lambda} M_i) | p \in V_R^m(Ann(M)), pM_j \neq M_j\}$. Then by ((a) $\Leftrightarrow$ (g)) of Theorem 3.3, $T \subseteq Max(M)$. To see the reverse inclusion, we assume that $Q \in Max(M)$. Then there exists $p \in V_R^m(Ann(M))$ such that $Q = pM$ by ((a) $\Leftrightarrow$ (g)) of Theorem 3.3. By part (a), M_i's are Max-compatible. It follows that $pM_j \neq M_j$ for a unique element $j \in \Lambda$. This implies that $Q = pM = pM_j \oplus (\bigoplus_{j\neq i\in\Lambda} pM_i)$ as desired. Hence the proof is completed.

□

Definition 3.8. [5, Definition. 3.17] Topological space W is a Max-spectral space if W homeomorphic with the maximal ideal space of some ring S (with the topology inherited from $Spec(S)$). Max-spectral spaces have been characterized by Hochster [15, p.57, Proposition. 11] as the topological spaces W which satisfy the following conditions:

(a) W is a T_1-space;
(b) W is quasi-compact.

Notation 3.9. Let M be an R-module and W be a subset of $Max_R(M)$. We will denote the closure of W in $Max_R(M)$ by $cl^m(W)$.

Theorem 3.10. *Consider the following statements for an R module M.*

(a) *$Max(M)$ is a Max-spectral space.*
(b) *M is Max-injective and $Im(\phi)$ is compact, where ϕ is natural map of $Max_R(M)$.*
(c) *M is Max-injective and $Im(\phi)$ is closed, where ϕ is natural map of $Max_R(M)$.*
(d) *M is Max-injective and for any $q \in V_R^m(Ann(M))$, we have*

$$\bigcap_{p \in Im(\phi)} p \subseteq q \Rightarrow qM \neq M.$$

Then $(c) \Leftrightarrow (d) \Rightarrow (a) \Leftrightarrow (b)$. Moreover, if $Max(\bar{R}) = Spec(\bar{R})$ then all statements are equivalent.

Proof. $(a) \Rightarrow (b)$. Let $Max(M)$ be a Max-spectral space. Then $Max(M)$ is T_1-space by [5, Corollary. 3.20] and so M is Max-injective by Remark 3.1. By [5, Lemma. 3.1], the mapping $\phi : Max_R(M) \to Max(\bar{R})$ is continuous and $Max(M)$ is compact by Definition 3.8. This implies that $Im(\phi)$ is compact.

$(b) \Rightarrow (a)$. By [5, Theorem. 3.24].

$(c) \Leftrightarrow (d)$. $Im(\phi)$ is closed iff $Im(\phi) \supseteq cl^m(Im(\phi)) = V_R^m(\bigcap_{p \in Im(\phi)} p)$ by [5, Lemma. 3.4]. But $Im(\phi) = \{p \mid p \in V_R^m(Ann(M)), pM \neq M\}$ by Theorem 3.3. Therefor $Im(\phi)$ is closed iff for any $q \in V_R^m(Ann(M))$, we have $\bigcap_{p \in Im(\phi)} p \subseteq q \Rightarrow qM \neq M$. Now since $Max(\bar{R})$ is compact and every closed subset of a compact space is compact, $(c) \Rightarrow (b)$. But we proved that $(c) \Leftrightarrow (d)$ and $(a) \Leftrightarrow (b)$. Therefore we have $(d) \Rightarrow (a)$.

To see the next assertion let $Max(\bar{R}) = Spec(\bar{R})$. Then $Max(\bar{R})$ is T_2- space by [1, Remark. 4.2]. Hence every compact space of $Max(\bar{R})$ is closed. This implies that $(b) \Rightarrow (c)$ and the proof is completed. □

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تحقیقی بر مدول های Max – انژکتیو

رضا اولیایی سرمزده[*] و سپیده ملکی
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چکیده
فرض کنید R یک حلقه جا بجایی و M یک R- مدول باشد. در این مقاله، خواص بیشتری از مدول های Max- انژکتیو را مورد جستجو قرار داده و شرایطی را مطالعه می کنیم که تحت آنها، طیف ماکزیمال M با توپولوژی زاریسکی یک Max- اسپکترال است.

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