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Introduction of the numerical methods in quantum calculus with uncertainty

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Abstract. The aim of this study is the introduction of the numerical methods for solving the fuzzy q-differential equations that many real life problems can be modelized in the form of these equations. q-Taylor's expansion method is among important and famous methods for solving these problems. In this paper, applications of the fuzzy q-Taylor's expansion, the fuzzy local q-Taylor's expansion and the fuzzy q-Euler's method, based on the generalized Hukuhara q-differentiability are illustrated which are two numerical methods for finding approximate solution of the fuzzy initial value q-problems (for short FIVq-Ps).

Keywords: Generalized Hukuhara q-derivative, fuzzy q-Taylor's theorem, fuzzy local q-Taylor's expansion, fuzzy q-Euler's method.
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1 Introduction

Mathematics and Physics are two main topics and applicable fields of sciences. For the first time, relations between these topics especially quantum calculus (q-calculus) and q-differential operators were studied by Jackson in [19]. It has a lot of applications in different mathematical areas such as: number theory, combinatorics, basic hyper-geometric functions, orthogonal polynomials and other sciences: quantum theory, mechanics, theory of relativity, capacitor theory, electrical circuits, particle physics, viscoelastic, electro-analytical chemistry, neurology, diffusion systems, control theory and statistics; for instance see the articles [2,7,26]. Recently, there have

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appeared many papers which study for more details on q-calculus, we refer the reader to the references [10,12–15,17,21–24].

In recent decades, the research on fuzzy calculus is becoming an interesting and useful topic among applied scientists [11, 28]. Among the applications in this field there are conversion problems into fuzzy differential equations and find their approximate solution using numerical methods [1,3,4,29]. One of the numerical methods that plays an important role in studying the local behavior of a suitable fuzzy-valued function is fuzzy Taylor's expansion. This expansion is a powerful tool in approximation theory and other fuzzy numerical methods. Anastassiou in [5] proved the fuzzy Taylor's expansion for Hukuhara differentiable functions.

The quantum calculus is usually referred in mathematics as the calculus without limit and replaces the classical derivative with a difference operator. As a result, q-Taylor's theorem will also be important in quantum calculus. The concept of the q-Taylor's series was formulated by Jackson [18] in 1910. A new notation of q-Taylor's formula was presented by Jing et al. and Ernst in [10,20]. After that, Ismail and Stanton in [16] introduced the q-type interpolation series. In [25] López et al. used the approach of [27] to give the sufficient conditions for convergence of Ismail-Stanton q-Taylor's series. In 2008, Annaby et al. [6] proved q-Taylor's series for Jackson q-difference operators. So that, the fuzzy q-derivative and fuzzy q-fractional derivative in Caputo sense based on the generalized Hukuhara difference, to show the importance of the q-derivative by establishing a connection between quantum calculus and fuzzy calculus was introduced in [30] by Noeiaghdam et al. In this work, we will investigate one of the basic applications of fuzzy q-Taylor's expansion that is in solving the FIVq-Ps are described.

There are many numerical, analytical and semi-analytical methods for solving q-differential equations. But we know that, solving the fuzzy form of these problems is difficult. Thus we prefer to apply the numerical method for fuzzy q-differential equations. Studying novel numerical methods to convert the fuzzy q-problems to the crisp form, solving the obtained crisp problems and finding the approximate solutions are among challenging and interesting problems that in recent years many authors have been focused on these topics. But cases, we can not transform the fuzzy problems to the crisp form easily. This paper aims to provide the numerical methods for solving fuzzy q-differential equations, without converting them to crisp forms it mean that we the full fuzzy methods are applied to solve the fuzzy q-differential equations.

In the most simple problems of the q-differential equations, the need to use numerical and approximate methods such as q-Euler's method is less felt, so it has not been introduced so far. Due to the explanations given for the use of fuzzy numerical methods in solving fuzzy q-differential equations and that there are fuzzy q-differential equations that cannot be solved by analytical methods, it will certainly always be important to introduce methods for finding approximate solution to such problems. Accordingly, in this paper, after introducing local q-Taylor's expansion and the q-Euler's method in crisp form, we have introduced the fuzzy local q-Taylor's expansion and fuzzy q-Euler's method to solve the FIVq-Ps.

This paper is organized as follows: Section 2, presents some preliminaries, including the basic definitions of the fuzzy calculus and quantum calculus. Principal properties for fuzzy q-derivative and also fuzzy q-Taylor's theorem are expressed in Section 3. In the main section, local q-Taylor's expansion and q-Euler's methods and also fuzzy form of these methods based on the concept of generalized Hukuhara q-differentiability for finding approximation solution of FIVq-Ps are introduced.

2 Notations and Preliminary Results

Fuzzy calculus: In this section, we review some definitions and notations of fuzzy logic and the fuzzy calculus which can be found in the articles [9, 28–30]. Basically, an arbitrary fuzzy number u in parametric form is an ordered pair of functions $[\underline{u}(r), \overline{u}(r)]$, $0 \leq r \leq 1$ which satisfies in the following requirements:

I. $\underline{u}(r)$ is a bounded monotonic increasing left continuous function in (0,1],

II. $\overline{u}(r)$ is a bounded monotonic decreasing left continuous function in (0,1],

III. $\underline{u}(r) \leq \overline{u}(r), \ 0 \leq r \leq 1$.

The set of all these fuzzy numbers is denoted by $\mathbb{R}_{\mathcal{F}}$. A crisp number k is simply represented by $\overline{u}(r) = \underline{u}(r) = k, \ \ 0 \leq r \leq 1$, and called singleton. For arbitrary numbers $u, v \in \mathbb{R}_{\mathcal{F}}$ and scalar k, we define

$$\textit{addition}: \ [u \oplus v]_r = [\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r)],$$

$$\textit{scalar multiplication}: \ \begin{array}{l} k \geq 0 \Rightarrow [k \odot u]_r = [k\underline{u}(r), k\overline{u}(r)]. \\ k < 0 \Rightarrow [k \odot u]_r = [k\overline{u}(r), k\underline{u}(r)]. \end{array}$$

The meaning of fuzzy-valued function is a function $y : A \to \mathbb{R}_{\mathcal{F}}, \ A \in \mathbb{R}$ where $[y(t)]_r = [\underline{y}(t;r), \overline{y}(t;r)]$ so called the r-cut or parametric form of the fuzzy-valued function y.

Let $u, v \in \mathbb{R}_{\mathcal{F}}$, if there exists $w \in \mathbb{R}_{\mathcal{F}}$, such that $u = v \oplus w$, then w is called the Hukuhara difference (H-difference for short) of u and v, and it is denoted by $u \ominus v$ and for $w \in \mathbb{R}_{\mathcal{F}}$ existential, it is the generalized Hukuhara difference (gH-difference for short) defined as

$$u \ominus_{gH} v = w \Leftrightarrow \begin{cases} (i) \ \ u = v \oplus w, \\ or \ (ii) \ \ v = u \oplus (-1)w. \end{cases}$$

It is easy to show that (i) and (ii) are both valid if and only if w is a crisp number.

Now, the Hausdorff distance between fuzzy numbers is given by $\mathbf{d} : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}^+ \cup \{0\}$ as $\mathbf{d}(u, v) = \sup_{0 \leq r \leq 1} \max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\}$, and it is easy to see that following properties of the metric $\mathbf{d}$ for $\lambda \in \mathbb{R}$ and $\forall u, \ v, \ w, \ z \ \in \mathbb{R}_{\mathcal{F}}$ are valid:

I. $\mathbf{d}(u \oplus w, v \oplus w) = \mathbf{d}(u, v)$,

II. $\mathbf{d}(\lambda u, \lambda v) = \mid \lambda \mid \mathbf{d}(u, v)$.

III. $\mathbf{d}(u \oplus v, w \oplus z) \leq \mathbf{d}(u, w) + \mathbf{d}(v, z)$,

In the following, we summarize the fundamental definitions and results without proving the relevant properties on q-calculus. For more details refer to [8, 22, 23, 31], and the references given therein.

Quantum calculus: In expression $\dfrac{y(t) - y(t_0)}{t - t_0}$ as t approaches t_0, the limit if it exists, gives the familiar definition of the derivative $\mathcal{D}y(t)$ (normal derivative) of a function $y(t)$ at $t = t_0$. However, if we take $t = qt_0$ or $t = t_0 + h$, where q is a fixed number different from 1, and h is a fixed number different from 0 and do not take the limit, with these changes we obtain the so-called quantum calculus. To make this statement precisely, we need the following definitions: Let $\mu \in \mathbb{R}$. The time-scale $\mathbb{T}_\mu$, for $0 < q < 1$ is defined as

$$\mathbb{T}_\mu = \{t : t = \mu q^n,\ n \in \mathbb{Z}^+\} \cup \{0\},$$

where $\mathbb{Z}^+$ is the set of nonnegative integer numbers. The q-analogues of a positive integer number a is defined as follows

$$[a]_q = \frac{1-q^a}{1-q}.$$

Consider an arbitrary function $y : \mathbb{T}_\mu \to \mathbb{R}$. The q-differential and q-derivative of y are respectively,

$$d_q y(t) = y(t) - y(qt),$$
$$\mathcal{D}_q y(t) = \frac{d_q y(t)}{d_q t} = \frac{y(t) - y(qt)}{(1-q)t}, \quad t \in \mathbb{T}_\mu - \{0\}.$$

It is obvious that if $t = 0$ then $\mathcal{D}_q y(0) = \mathcal{D}y(0)$, where $\mathcal{D}y$ is normal derivative. Further we define

$$\mathcal{D}_q^0 y = y, \quad \mathcal{D}_q^n y = \mathcal{D}_q(\mathcal{D}_q^{n-1} y), \quad n = 1, 2, 3, \ldots.$$

If the function y is differentiable at t, we have $\lim_{q\to 1} \mathcal{D}_q y(t) = \mathcal{D}y(t)$. The q-derivative operator is a special case of Hahn's q-operator, which is defined in [23]. The q-antiderivative ${}_qI_a$ for y is defined in the following form

$$\int_a^b y(s) d_q s = b(1-q)\sum_{i=0}^{\infty} q^i y(bq^i) - a(1-q)\sum_{i=0}^{\infty} q^i y(aq^i),$$

that is called the Jackson integral of $y(x)$. As for q-derivative, we can define an operator q-antiderivative ${}_qI_a^n$ as follows

$${}_qI_a^0 y(t) = y(t), \quad {}_qI_a^n y(t) = {}_qI_a({}_qI_a^{n-1} y(t)), \quad n = 1, 2, 3, \ldots.$$

Also, for the operators ${}_qI_a$ and $\mathcal{D}_q$ we have

$$\mathcal{D}_q . {}_qI_a y(t) = y(t), \quad {}_qI_a . \mathcal{D}_q y(t) = y(t) - y(a).$$

The q-fractional function is defined by

$$(t-s)_q^n = \begin{cases} \prod_{i=0}^{n-1}(t - q^i s), & n \in \mathbb{N}, \\ t^n \prod_{i=0}^{\infty} \dfrac{1 - \frac{s}{t} q^i}{1 - \frac{s}{t} q^{i+n}}, & (t \neq 0)\ O.W. \end{cases}$$

The q-gamma function denoted by $\Gamma_q(.)$ is defined as follows

$$\Gamma_q(\alpha) = \frac{(1-q)_q^{\alpha-1}}{(1-q)^{\alpha-1}}, \quad \alpha \in \mathbb{R}/\{0\} \cup \mathbb{Z}_-, \quad 0 < q < 1,$$

$$\Gamma_q(\alpha+1) = \frac{1-q^\alpha}{1-q}\Gamma_q(\alpha) = [\alpha]_q \Gamma_q(\alpha), \quad \Gamma_q(1) = 1, \quad \alpha > 0.$$

3 Definitions and properties of fuzzy q-differentiability

Before considering methods for approximating the solutions of the FIVq-Ps, we need some definitions and results from the fuzzy q-differentiability. In the sequel, fuzzy q-derivative based on the concept of generalized Hukuhara q-differentiability is introduced. Consequently, the obtained results and properties of fuzzy q-differentiability are useful and can be proved.

Definition 1. *For arbitrary fuzzy-valued function $y : \mathbb{T}_\mu \to \mathbb{R}_\mathcal{F}$, q-differential by gH-difference is*

$$^{gH}\mathcal{D}_q y(t) = y(t) \ominus_{gH} y(qt).$$

Therefore, the generalized Hukuhara q-derivative (for short ${}^F_q[gH]$-derivative) of y is defined by

$${}^F_q\mathcal{D}y(t) = \frac{^{gH}\mathcal{D}_q y(t)}{\mathcal{D}_q t} = \frac{y(t) \ominus_{gH} y(qt)}{(1-q)t}, \quad t \in \mathbb{T}_\mu - \{0\}.$$

On the other hand, for the fuzzy-valued functions $y(t)$ and $y(qt)$, we have

$$[y(t) \ominus_{gH} y(qt)]_r = \Big[\min\{\underline{y}(t;r) - \underline{y}(qt;r), \bar{y}(t;r) - \bar{y}(qt;r)\},$$
$$\max\{\underline{y}(t;r) - \underline{y}(qt;r), \bar{y}(t;r) - \bar{y}(qt;r)\}\Big],$$

If the gH-difference exists, then $[y(t) \ominus y(qt)]_r = [y(qt) \ominus_{gH} y(t)]_r$. The conditions for the existence of $^{gH}\mathcal{D}_q y(t) = y(t) \ominus_{gH} y(qt) \in \mathbb{R}_\mathcal{F}$ with respect to r are

$$case(i)\begin{cases} ^{gH}\mathcal{D}_q\underline{y}(t;r) = \underline{y}(t;r) - \underline{y}(qt;r), \text{ is increasing,} \\ ^{gH}\mathcal{D}_q\overline{y}(t;r) = \overline{y}(t;r) - \overline{y}(qt;r), \text{ is decreasing,} \\ ^{gH}\mathcal{D}_q\underline{y}(t;r) \leq^{gH} \mathcal{D}_q\overline{y}(t;r), \end{cases}$$

$$case(ii)\begin{cases} ^{gH}\mathcal{D}_q\underline{y}(t;r) = \overline{y}(t;r) - \overline{y}(qt;r), \text{ is increasing,} \\ ^{gH}\mathcal{D}_q\overline{y}(t;r) = \underline{y}(t;r) - \underline{y}(qt;r), \text{ is decreasing,} \\ ^{gH}\mathcal{D}_q\underline{y}(t;r) \leq^{gH} \mathcal{D}_q\overline{y}(t;r). \end{cases}$$

Definition 2. *Let $y : \mathbb{T}_\mu \to \mathbb{R}_\mathcal{F}$ and $y(t) \ominus_{gH} y(qt)$ exists. If*

$$[{}^F_q\mathcal{D}y_{i-gH}(t)]_r = [\mathcal{D}_q\underline{y}(t;r),\ \mathcal{D}_q\overline{y}(t;r)], \quad 0 \leq r \leq 1,$$

we say that y is ${}^F_q[(i) - gH]$-differentiable and if

$$[{}^F_q\mathcal{D}y_{ii-gH}(t)]_r = [\mathcal{D}_q\overline{y}(t;r),\ \mathcal{D}_q\underline{y}(t;r)], \quad 0 \leq r \leq 1,$$

y is ${}^F_q[(ii) - gH]$-differentiable where

$$\mathcal{D}_q\underline{y}(t;r) = \frac{\underline{y}(t;r) - \underline{y}(qt;r)}{(1-q)t}, \quad \mathcal{D}_q\overline{y}(t;r) = \frac{\overline{y}(t;r) - \overline{y}(qt;r)}{(1-q)t}.$$

Definition 3. *Let $y : \mathbb{T}_\mu \to \mathbb{R}_\mathcal{F}$ be a fuzzy-valued function on $\mathbb{T}_\mu$. A point $\xi \in \mathbb{T}_\mu - \{0\}$ is said to be a switching point for the ${}_q^F[gH]$-differentiably of y, if in any neighborhood V of ξ there exist points $t_1 < \xi < t_2$ such that*
(type I) *y is ${}_q^F[(i) - gH]-$differentiable at t_1 while y is not ${}_q^F[(ii) - gH]-$differentiable at t_1, and y is ${}_q^F[(ii) - gH]-$differentiable at t_2 while y is not ${}_q^F[(i) - gH]-$differentiable at t_2, or*
(type II) *y is ${}_q^F[(ii) - gH]-$differentiable at t_1 while y is not ${}_q^F[(i) - gH]-$differentiable at t_1, and y is ${}_q^F[(i) - gH]-$differentiable at t_2 while y is not ${}_q^F[(ii) - gH]-$differentiable at t_2.*

Definition 4. *Let $y : \mathbb{T}_\mu \to \mathbb{R}_\mathcal{F}$. We say that y is ${}_q^F[gH]$-differentiable of the n^{th}-order at t_0 whenever the function y is ${}_q^F[gH]$-differentiable of the order m $(m = 1, 2, \ldots, n-1)$ at t_0, moreover the ${}_q^F[gH]$-differentiable type has no change (there is not any switching point on $\mathbb{T}_\mu$). Then there exists ${}_q^F\mathcal{D}^{(n)}y(t_0) \in \mathbb{R}_\mathcal{F}$ such that*

$$ {}_q^F\mathcal{D}^{(n)}y(t_0) = \frac{{}_q^F\mathcal{D}^{(n-1)}y(t_0) \ominus_{gH} \ {}_q^F\mathcal{D}^{(n-1)}y(qt_0)}{(1-q)t_0}. $$

Theorem 1. *[30] Suppose that $[y(t)]_r = [\underline{y}(t;r), \bar{y}(t;r)]$. The function y is ${}_q^F[gH]$-differentiable if and only if $\underline{y}(t;r)$ and $\bar{y}(t;r)$ are q-differentiable with respect to t for all $r \in [0,1]$ and*

$$ [{}_q^F\mathcal{D}y(t)]_r = [\min\{\mathcal{D}_q\underline{y}(t;r),\ \mathcal{D}_q\bar{y}(t;r)\}, \max\{\mathcal{D}_q\underline{y}(t;r),\ \mathcal{D}_q\bar{y}(t;r)\}]. $$

Definition 5. *Let y be a fuzzy-valued function and $F(t) \ominus_{gH} F(qt)$ exists. Then the fuzzy Jackson integral is defined by*

$$ F(t) = (1-q)t \odot \sum_{j=0}^{\infty} q^j \odot y(q^j t) $$

Throughout the rest of this paper, the notation $\mathcal{C}_f(\mathbb{T}_\mu, \mathbb{R}_\mathcal{F})$ is called the set of fuzzy-valued continuous functions defined on $\mathbb{T}_\mu$ and $\mathcal{C}_f^n(\mathbb{T}_\mu, \mathbb{R}_\mathcal{F})$, $n \in \mathbb{N}$ for the space of fuzzy-valued functions y on $\mathbb{T}_\mu$ such that itself and its first n, q-derivatives are all in $\mathcal{C}_f(\mathbb{T}_\mu, \mathbb{R}_\mathcal{F})$.

Theorem 2. *[30] Let $y : \mathbb{T}_\mu \to \mathbb{R}_\mathcal{F}$, be ${}_q^F[gH]$-differentiable function such that the type of ${}_q^F[gH]$-differentiability of y in $\mathbb{T}_\mu$ does not change. Then for $t \in \mathbb{T}_\mu$,*

I. *If $y(s)$ is ${}_q^F[(i) - gH]$-differentiable then ${}_q^F\mathcal{D}y_{i-gH}(s)$ is q-integrable over $\mathbb{T}_\mu$ and*

$$ y(t) = y(a) \oplus \int_a^t {}_q^F\mathcal{D}y_{i-gH}(s)d_qs. $$

II. *If $y(s)$ is ${}_q^F[(ii) - gH]$-differentiable then ${}_q^F\mathcal{D}y_{ii-gH}(s)$ is q-integrable over $\mathbb{T}_\mu$ and*

$$ y(t) = y(a) \ominus (-1) \int_a^t {}_q^F\mathcal{D}y_{ii-gH}(s)d_qs. $$

Theorem 3. *Let ${}_q^F\mathcal{D}^{(j)}y : \mathbb{T}_\mu \to \mathbb{R}_\mathcal{F}$ and ${}_q^F\mathcal{D}^{(j)}y \in \mathcal{C}_f(\mathbb{T}_\mu, \mathbb{R}_\mathcal{F})$, $j = 1, \ldots, n$. For all $t \in \mathbb{T}_\mu$*

I. *Assume that* ${}^F_q\mathcal{D}^{(j)}y,\ j=1,\dots,n$ *are* ${}^F_q[(i)-gH]$*-differentiable and there is no change in the type of* ${}^F_q[gH]$*-differentiability on* $\mathbb{T}_\mu$*. Then*

$$ {}^F_q\mathcal{D}^{(j-1)}y_{i-gH}(t) = {}^F_q\mathcal{D}^{(j-1)}y_{i-gH}(a) \oplus \int_a^t {}^F_q\mathcal{D}^{(j)}y_{i-gH}(s)d_qs. $$

II. *If* ${}^F_q\mathcal{D}^{(j)}y,\ j=1,\dots,n$ *are* ${}^F_q[(ii)-gH]$*-differentiable and the type of* ${}^F_q[gH]$*-differentiability does not change on* $\mathbb{T}_\mu$*, then*

$$ {}^F_q\mathcal{D}^{(j-1)}y_{ii-gH}(t) = {}^F_q\mathcal{D}^{(j-1)}y_{ii-gH}(a) \oplus \int_a^t {}^F_q\mathcal{D}^{(j)}y_{ii-gH}(s)d_qs. $$

III. *Assume that* ${}^F_q\mathcal{D}^{(j)}y$ *for* $j=1,\dots,n$ *exist and in each order of differentiation, the type of* ${}^F_q[gH]$*-differentiability changes on* $\mathbb{T}_\mu$*(i.e.* ${}^F_q[gH]$*-differentiability changes from* ${}^F_q[(i)-gH]$ *to* ${}^F_q[(ii)-gH]$ *and vice versa), then*

$$ \begin{cases} {}^F_q\mathcal{D}^{(j-1)}y_{i-gH}(t) = {}^F_q\mathcal{D}^{(j-1)}y_{i-gH}(a) \ominus (-1)\int_a^t {}^F_q\mathcal{D}^{(j)}y_{ii-gH}(s)d_qs, & j \text{ is even number}, \\ {}^F_q\mathcal{D}^{(j-1)}_{ii-gH}y(t) = {}^F_q\mathcal{D}^{(j-1)}y_{ii-gH}(a) \ominus (-1)\int_a^t {}^F_q\mathcal{D}^{(j)}y_{i-gH}(s)d_qs, & j \text{ is odd number}. \end{cases} $$

Proof. By assuming ${}^F_q\mathcal{D}^{(j)}y \in \mathcal{C}_f(\mathbb{T}_q, \mathbb{R}_\mathcal{F}),\ i=0,\dots,n$, we give the proof only for parts **II** and **III**, the part **I** is left because the proof for it is similar.

II. Our proof starts with the observation that ${}^F_q\mathcal{D}^{(j)}y,\ j=1,\dots,n$ are ${}^F_q[(ii)-gH]$-differentiable. Hence, using properties of ${}^F_q[gH]$-derivative and Theorem 2, we have

$$ \begin{aligned} \left[\int_a^t {}^F_q\mathcal{D}^{(j)}y_{ii-gH}(s)d_qs\right]_r &= \left[\int_a^t \mathcal{D}_q^{(j)}\bar{y}(s;r)d_qs, \int_a^t \mathcal{D}_q^{(j)}\underline{y}(s;r)d_qs\right] \\ &= \left[\int_a^t \mathcal{D}_q(\mathcal{D}_q^{(j-1)}\bar{y})(s;r)d_qs, \int_a^t \mathcal{D}_q(\mathcal{D}_q^{(j-1)}\underline{y})(s;r)d_qs\right] \\ &= [\mathcal{D}_q^{(j-1)}\bar{y}(t;r) - \mathcal{D}_q^{(j-1)}\bar{y}(a;r), \mathcal{D}_q^{(j-1)}\underline{y}(t;r) - \mathcal{D}_q^{(j-1)}\underline{y}(a;r)] \\ &= [\mathcal{D}_q^{(j-1)}\bar{y}(t;r), \mathcal{D}_q^{(j-1)}\underline{y}(t;r)] - [\mathcal{D}_q^{(j-1)}\bar{y}(a;r), \mathcal{D}_q^{(j-1)}\underline{y}(a;r)] \\ &= [{}^F_q\mathcal{D}^{(j-1)}y_{ii-gH}(t) \ominus {}^F_q\mathcal{D}^{(j-1)}y_{ii-gH}(a)]_r, \end{aligned} $$

leads to

$$ {}^F_q\mathcal{D}^{(j-1)}y_{ii-gH}(t) = {}^F_q\mathcal{D}^{(j-1)}y_{ii-gH}(a) \oplus \int_a^t {}^F_q\mathcal{D}^{(j)}y_{ii-gH}(s)d_qs. $$

In the following, taking into account that the next part can be proved in much the same way, the only difference being in the analysis of assumptions.

III. Under the conditions stated in the part III, by Definition 1 and Theorem 2 and assumming that j is an even number, we get

$$\begin{aligned}
&\left[{}_{q}^{F}\mathcal{D}^{(j-1)}y_{i-gH}(t) \ominus (-1)\odot \int_a^t {}_{q}^{F}\mathcal{D}^{(j)}y_{ii-gH}(s)d_qs\right]_r\\
&= [\mathcal{D}_q^{(j-1)}\underline{y}(t;r), \mathcal{D}_q^{(j-1)}\bar{y}(t;r)] + \left[-\int_a^t \mathcal{D}_q^{(j)}\underline{y}(s;r)d_qs, -\int_a^t \mathcal{D}_q^{(j)}\bar{y}(s;r)d_qs\right]\\
&= [\mathcal{D}_q^{(j-1)}\underline{y}(t;r), \mathcal{D}_q^{(j-1)}\bar{y}(t;r)]\\
&+ [\mathcal{D}_q^{(j-1)}\underline{y}(a;r) - \mathcal{D}_q^{(j-1)}\underline{y}(t;r), \mathcal{D}_q^{(i-1)}\bar{y}(a;r) - \mathcal{D}_q^{(j-1)}\bar{y}(t;r)]\\
&= [\mathcal{D}_q^{(j-1)}\underline{y}(a;r),\ \mathcal{D}_q^{(j-1)}\bar{y}(a;r)] = [\ {}_{q}^{F}\mathcal{D}^{(j-1)}y_{i-gH}(a)]_r,
\end{aligned}$$

that leads to

$${}_{q}^{F}\mathcal{D}^{(j-1)}y_{i-gH}(t) = \ {}_{q}^{F}\mathcal{D}_{i-gH}^{(j-1)}y(a) \ominus (-1)\int_a^t \ {}_{q}^{F}\mathcal{D}^{(j)}y_{ii-gH}(s)d_qs.$$

The proof for odd numbers by a similar process is obvious and yields

$${}_{q}^{F}\mathcal{D}^{(j-1)}y_{ii-gH}(t) = \ {}_{q}^{F}\mathcal{D}^{(j-1)}y_{ii-gH}(a) \ominus (-1)\int_a^t \ {}_{q}^{F}\mathcal{D}^{(j)}y_{i-gH}(s)d_qs,$$

which completes the proof. □

Theorem 4. ***(Fuzzy q-Taylor's Theorem)*** *Let ${}_{q}^{F}\mathcal{D}^{(n)}y \in \mathcal{C}_f(\mathbb{T}_\mu, \mathbb{R}_{\mathcal{F}})$ and $t \in \mathbb{T}_\mu$. For finding the fuzzy q-Taylor's expansion of y around $a \in \mathbb{T}_\mu$, we have*

I. *If ${}_{q}^{F}\mathcal{D}^{(j)}y,\ j = 0, 1, \ldots, n-1$ are ${}_{q}^{F}[(i) - gH]$-differentiable, provided that type of ${}_{q}^{F}[gH]$-differentiability has no change on $\mathbb{T}_\mu$. Then there exists $\eta \in \mathbb{T}_\mu$ such that*

$$y(t) = y(a) \oplus \sum_{j=1}^{n-1}\left[\frac{(t-a)_q^j}{\Gamma_q(j+1)} \odot {}_{q}^{F}\mathcal{D}^{(j)}y_{i-gH}(a)\right] \oplus \frac{(t-a)_q^n}{\Gamma_q(n+1)} \odot {}_{q}^{F}\mathcal{D}^{(n)}y_{i-gH}(\eta).$$

II. *If ${}_{q}^{F}\mathcal{D}^{(j)}y,\ j = 0, 1, \ldots, n-1$ are ${}_{q}^{F}[(ii) - gH]$-differentiable, provided that type of ${}_{q}^{F}[gH]$-differentiability has no change on $\mathbb{T}_\mu$. Then there exists $\eta \in \mathbb{T}_\mu$ such that*

$$y(t) = y(a) \ominus (-1)\sum_{j=1}^{n-1}\left[\frac{(t-a)_q^j}{\Gamma_q(j+1)} \odot {}_{q}^{F}\mathcal{D}^{(j)}y_{ii-gH}(a)\right] \ominus (-1)\frac{(t-a)_q^n}{\Gamma_q(n+1)} \odot {}_{q}^{F}\mathcal{D}^{(n)}y_{ii-gH}(\eta).$$

III. *Suppose that ${}_{q}^{F}\mathcal{D}^{(j)}y, j = 1, \ldots, n$ exist and in each order the type of ${}_{q}^{F}[gH]$-differentiability changes on $\mathbb{T}_\mu$, then there exists $\eta \in \mathbb{T}_\mu$ such that*

$$\begin{aligned}
y(t) = y(a) \ominus (-1) &\sum_{\substack{j=1\\ j \text{ is odd}}}^{n-1}\left[\frac{(t-a)_q^j}{\Gamma_q(j+1)} \odot {}_{q}^{F}\mathcal{D}^{(j)}y_{ii-gH}(a)\right] \oplus \sum_{\substack{j=1\\ j \text{ is even}}}^{n-1}\left[\frac{(t-a)_q^j}{\Gamma_q(j+1)}\right.\\
&\left.\odot\, {}_{q}^{F}\mathcal{D}^{(j)}y_{i-gH}(a)\right] \ominus (-1)\frac{(t-a)_q^n}{\Gamma_q(n+1)} \odot {}_{q}^{F}\mathcal{D}^{(n)}y_{ii-gH}(\eta).
\end{aligned}$$

IV. *If y has a switching point at $\xi \in \mathbb{T}_\mu$ of type II (i.e. ${}^F_q[gH]$-differentiability changes from ${}^F_q[(ii)-gH]$ to ${}^F_q[(i)-gH]$) and suppose that the type of differentiability for ${}^F_q\mathcal{D}^{(j)}y,\ j = 1,\ldots,n-1$ are ${}^F_q[(i)-gH]$-differentiable. Then there exists $\eta \in \mathbb{T}_\mu$ such that*

$$y(t) = \begin{cases} y(a) \ominus (-1)(t-a) \odot {}^F_q\mathcal{D}y_{ii-gH}(a) \\ \oplus \sum_{j=2}^{n-1}\left[\dfrac{(t-a)_q^j}{\Gamma_q(j+1)} \odot {}^F_q\mathcal{D}^{(j)}y_{i-gH}(a)\right] \\ \oplus\dfrac{(t-a)_q^n}{\Gamma_q(n+1)} \odot {}^F_q\mathcal{D}^{(n)}y_{i-gH}(\eta), & 0 < t \le \xi, \\ \\ y(a) \oplus \sum_{j=1}^{n-1}\left[\dfrac{(t-a)_q^j}{\Gamma_q(j+1)} \odot {}^F_q\mathcal{D}^{(j)}y_{i-gH}(a)\right] \\ \oplus\dfrac{(t-a)_q^n}{\Gamma_q(n+1)} \odot {}^F_q\mathcal{D}^{(n)}y_{i-gH}(\eta), & \xi \le t. \end{cases}$$

V. *If y has a switching point at $\xi \in \mathbb{T}_\mu$ of type I (i.e. ${}^F_q[gH]$-differentiability changes from ${}^F_q[(i)-gH]$ to ${}^F_q[(ii)-gH]$) and suppose that the type of differentiability for ${}^F_q\mathcal{D}^{(j)}y,\ j = 1,\ldots,n-1$ are ${}^F_q[(ii)-gH]$-differentiable. Then there exists $\eta \in \mathbb{T}_\mu$ such that*

$$y(t) = \begin{cases} y(a) \oplus (t-a) \odot {}^F_q\mathcal{D}y_{i-gH}(a) \\ \ominus(-1)\sum_{j=2}^{n-1}\left[\dfrac{(t-a)_q^j}{\Gamma_q(j+1)} \odot {}^F_q\mathcal{D}^{(j)}y_{ii-gH}(a)\right] \\ \ominus(-1)\dfrac{(t-a)_q^n}{\Gamma_q(n+1)} \odot {}^F_q\mathcal{D}^{(n)}y_{ii-gH}(\eta), & 0 < t \le \xi, \\ \\ y(a) \ominus (-1)\sum_{j=1}^{n-1}\left[\dfrac{(t-a)_q^j}{\Gamma_q(j+1)} \odot {}^F_q\mathcal{D}^{(k)}y_{ii-gH}(a)\right] \\ \ominus(-1)\dfrac{(t-a)_q^n}{\Gamma_q(n+1)} \odot {}^F_q\mathcal{D}^{(n)}y_{ii-gH}(\eta), & \xi \le t. \end{cases}$$

Proof. Since ${}^F_q\mathcal{D}^{(n)}y \in \mathcal{C}_f(\mathbb{T}_\mu, \mathbb{R}_\mathcal{F})$, by Theorem 3, we obtain:

I. Suppose ${}^F_q\mathcal{D}^{(j)}y$, for $j = 0,1,\ldots,n$ are fuzzy q-differentiable as in Definition 2 (i) then

$$\begin{aligned} &{}^F_qI_a^{(n)}{}^F_q\mathcal{D}^{(n)}y_{i-gH}(t) \\ &= {}^F_qI_a^{(n-1)}{}^F_q\mathcal{D}^{(n-1)}y_{i-gH}(t) \ominus {}^F_qI_a^{(n-1)}{}^F_q\mathcal{D}^{(n-1)}y_{i-gH}(a) \\ &= {}^F_qI_a^{(n-2)}{}^F_q\mathcal{D}^{(n-2)}y_{i-gH}(t) \ominus {}^F_qI_a^{(n-2)}{}^F_q\mathcal{D}^{(n-2)}y_{i-gH}(a) \ominus {}^F_qI_a^{(n-1)}{}^F_q\mathcal{D}^{(n-1)}y_{i-gH}(a) \\ &\ \vdots \\ &= {}^F_qI_a\, {}^F_q\mathcal{D}y_{i-gH}(t) \ominus {}^F_qI_a^{(2)}{}^F_q\mathcal{D}^{(2)}y_{i-gH}(a) \ominus \cdots \ominus {}^F_qI_a^{(n-1)}{}^F_q\mathcal{D}^{(n-1)}y_{i-gH}(a) \\ &= y(t) \ominus y(a) \ominus {}^F_qI_a\, {}^F_q\mathcal{D}y_{i-gH}(a) \ominus {}^F_qI_a^{(2)}{}^F_q\mathcal{D}^{(2)}y_{i-gH}(a) \ominus \cdots \ominus {}^F_qI_a^{(n-1)}{}^F_q\mathcal{D}^{(n-1)}y_{i-gH}(a). \end{aligned}$$

Since

$${}^F_qI_a^{(j)} = \int_a^t\int_a^t\cdots\int_a^t \underbrace{d_qsd_qs\cdots d_qs}_{j-times} = \frac{(t-a)_q^j}{\Gamma_q(j+1)},$$

so, we have

$$
\begin{aligned}
{}_q^F I_a^{(n)} {}_q^F \mathcal{D}^{(n)} y_{i-gH}(t) = y(t) \ominus y(a) \ominus \frac{(t-a)_q}{\Gamma_q(2)} {}_q^F \mathcal{D} y_{i-gH}(a) \ominus \frac{(t-a)_q^2}{\Gamma_q(3)} {}_q^F \mathcal{D}^{(2)} y_{i-gH}(a) \\
\ominus \cdots \ominus \frac{(t-a)_q^j}{\Gamma_q(j+1)} {}_q^F \mathcal{D}^{(j-1)} y_{i-gH}(a).
\end{aligned}
$$

Also, q-mean-value theorem for $\eta \in (a,t)$, gives

$$
y(t) = y(a) \oplus \sum_{j=1}^{n-1} \left[\frac{(t-a)_q^j}{\Gamma_q(j+1)} \odot {}_q^F \mathcal{D}^{(j)} y_{i-gH}(a) \right] \oplus \frac{(t-a)_q^n}{\Gamma_q(n+1)} \odot {}_q^F \mathcal{D}^{(n)} y_{i-gH}(\eta).
$$

II. Suppose that ${}_q^F \mathcal{D}^{(j)} y$, for $j = 0, 1, \ldots, n$ are fuzzy q-differentiable as in Definition 2(ii), then

$$
\begin{aligned}
&{}_q^F I_a^{(n)} {}_q^F \mathcal{D}^{(n)} y_{ii-gH}(t) \\
&\quad = {}_q^F I_a^{(n-1)} {}_q^F \mathcal{D}^{(n-1)} y_{ii-gH}(t) \ominus {}_q^F I_a^{(n-1)} {}_q^F \mathcal{D}^{(n-1)} y_{ii-gH}(a) \\
&\quad = {}_q^F I_a^{(n-2)} {}_q^F \mathcal{D}^{(n-2)} y_{ii-gH}(t) \ominus {}_q^F I_a^{(j-2)} {}_q^F \mathcal{D}^{(n-2)} y_{ii-gH}(a) \ominus {}_q^F I_a^{(n-1)} {}_q^F \mathcal{D}^{(n-1)} y_{ii-gH}(a) \\
&\quad\ \vdots \\
&\quad = {}_q^F I_a \, {}_q^F \mathcal{D} y_{ii-gH}(t) \ominus {}_q^F I_a^{(2)} {}_q^F \mathcal{D}^{(2)} y_{ii-gH}(a) \ominus \cdots \ominus {}_q^F I_a^{(n-1)} {}_q^F \mathcal{D}^{(n-1)} y_{i-gH}(a) \\
&\quad = (-1) y(t) \ominus (-1) y(a) \ominus {}_q^F I_a {}_q^F \mathcal{D} y_{ii-gH}(a) \ominus {}_q^F I_a^{(2)} {}_q^F \mathcal{D}^{(2)} y_{ii-gH}(a) \\
&\qquad \ominus \cdots \ominus {}_q^F I_a^{(n-1)} {}_q^F \mathcal{D}^{(n-1)} y_{ii-gH}(a).
\end{aligned}
$$

Similar to proof of **I**, we get

$$
y(t) = y(a) \ominus (-1) \sum_{j=1}^{n-1} \left[\frac{(t-a)_q^j}{\Gamma_q(j+1)} \odot {}_q^F \mathcal{D}^{(j)} y_{ii-gH}(a) \right] \ominus (-1) \frac{(t-a)_q^n}{\Gamma_q(n+1)} \odot {}_q^F \mathcal{D}^{(n)} y_{ii-gH}(\eta).
$$

III. Assuming that ${}_q^F[gH]$-differentiability of ${}_q^F \mathcal{D}^{(j)} y$ for $j = 1, \ldots, n$ changes on $\mathbb{T}_\mu$ and ${}_q^F \mathcal{D}^{(n-1)} y$ is fuzzy q-differentiable as in Definition 2(ii) then we have

$$
\begin{aligned}
&{}_q^F I_a^{(n)} {}_q^F \mathcal{D}^{(n)} y_{ii-gH}(t) \\
&\quad = -{}_q^F I_a^{(n-1)} {}_q^F \mathcal{D}^{(n-1)} y_{i-gH}(t) \ominus (-1) {}_q^F I_a^{(n-1)} {}_q^F \mathcal{D}^{(n-1)} y_{i-gH}(a) \\
&\quad = -{}_q^F I_a^{(n-1)} {}_q^F \mathcal{D}^{(n-1)} y_{i-gH}(t) \ominus (-1) [-{}_q^F I_a^{(n-2)} {}_q^F \mathcal{D}^{(n-2)} y_{ii-gH}(a) \\
&\qquad \ominus (-1) {}_q^F I_a^{(n-2)} {}_q^F \mathcal{D}^{(n-2)} y_{ii-gH}(a)] \\
&\quad\ \vdots \\
&\quad = -{}_q^F I_a^{(n-1)} \, {}_q^F \mathcal{D} y_{i-gH}^{(n-1)}(t) \ominus {}_q^F I_a^{(n-2)} {}_q^F \mathcal{D}^{(n-2)} y_{ii-gH}(a) \oplus \cdots \oplus y(x) \ominus y(a) \\
&\quad = \frac{-(t-a)_q^{n-1}}{\Gamma_q(n)} {}_q^F \mathcal{D} y_{i-gH}^{(n-1)}(t) \ominus \frac{-(t-a)_q^{n-2}}{\Gamma_q(n-1)} {}_q^F \mathcal{D}^{(n-2)} y_{ii-gH}(a) \oplus \cdots \oplus y(x) \ominus y(a).
\end{aligned}
$$

Therefore

$$y(t) = y(a) \ominus (-1) \sum_{\substack{j=1 \\ j \text{ is odd}}}^{n-1} \left[\frac{(t-a)_q^j}{\Gamma_q(j+1)} \odot {}_q^F\mathcal{D}^{(j)} y_{ii-gH}(a) \right]$$

$$\oplus \sum_{\substack{j=1 \\ j \text{ is even}}}^{n-1} \left[\frac{(t-a)_q^j}{\Gamma_q(j+1)} \odot {}_q^F\mathcal{D}^{(j)} y_{i-gH}(a) \right] \ominus (-1) \frac{(t-a)_q^n}{\Gamma_q(n+1)} \odot {}_q^F\mathcal{D}^{(n)} y_{ii-gH}(\eta).$$

IV. Suppose ${}_q^F\mathcal{D}^{(j)}y$, for $j = 0, 1, \ldots, n$ are q-differentiable as in Definition 2(i).

$$\begin{aligned} {}_q^F I_a^{(n)} {}_q^F\mathcal{D}^{(n)} y_{i-gH}(t) &= {}_q^F I_a^{(n-1)} {}_q^F\mathcal{D}^{(n-1)} y_{i-gH}(t) \ominus {}_q^F I_a^{(n-1)} {}_q^F\mathcal{D}^{(n-1)} y_{i-gH}(a) \\ &= {}_q^F I_a^{(2)} {}_q^F\mathcal{D}^{(2)} y_{i-gH}(t) \ominus {}_q^F I_a^{(2)} {}_q^F\mathcal{D}^{(2)} y_{i-gH}(a) \\ &\quad \ominus \cdots \ominus {}_q^F I_a^{(n-1)} {}_q^F\mathcal{D}^{(n-1)} y_{i-gH}(a). \end{aligned} \tag{1}$$

Since ${}_q^F\mathcal{D}y$ is q-differentiable on $[a, \xi]$ as in Definition 2.4 (i) and on $[\xi, b]$ as in Definition 2.4 (ii), thus gives

$${}_q^F I_a^{(2)} {}_q^F\mathcal{D}^{(2)} y_{i-gH}(t) = \begin{cases} {}_q^F I_a \, {}_q^F\mathcal{D} y_{i-gH}(t) \ominus {}_q^F I_a \, {}_q^F\mathcal{D} y_{i-gH}(a), & 0 < t \leq \xi, \\ -{}_q^F I_a \, {}_q^F\mathcal{D} y_{i-gH}(a) \ominus (-1) {}_q^F I_a \, {}_q^F\mathcal{D} y_{i-gH}(t), & \xi \leq t. \end{cases} \tag{2}$$

Now, if we substitute Eq. (2) into Eq. (1), the proof of **IV** can be obtained, and the proof of **V** is similar to **IV**, which proves the theorem. □

4 Numerical methods

In this section, first is introduced crisp numerical methods by using the q-Taylor's expansion. Finally, fuzzy local q-Taylor's expansion and fuzzy q-Euler's method for solving the FIVq-Ps based on the generalized Hukuhara q-derivative are presented.

Local q-Taylor's expansion: Consider the following initial value q-problem (for short IVq-P)

$$\begin{cases} \mathcal{D}_q y(t) = f(t, y(t)), \\ y(0) = y_0 \in \mathbb{R}, \end{cases} \tag{3}$$

where $f : [0, b] \to \mathbb{R}$ is continuous and $y(t)$ is an unknown function of variable t. Furthermore, $\mathcal{D}_q y(t)$ is the first order q-derivative of $y(t)$.

Now, by dividing the interval $[0, b]$ with the step length of $\mathfrak{h}_k$, we have the partition $\widehat{I}_N = \{t_0 = 0 < t_1 = q^{N-1}b < t_2 = q^{N-2}b < \ldots < t_{N-2} = q^2 b < t_{N-1} = qb < t_N = b\}$ where $t_k = q^{N-k}b = qt_{k+1}$ are called the mesh points, and $\mathfrak{h}_k = t_k - t_{k-1} = t_k(1-q) = q^{N-k}b(1-q)$ for $k = 1, 2, \ldots, N$ are called the step sizes. Consider the q-Taylor's expansion introduced in

Section 2, for $t_1 = t_0 + \mathfrak{h}_1$. Then if y and it's high order are q-differentiable, we have

$$y(t_1) = y(t_0 + \mathfrak{h}_1) = y(t_0) + (t_1 - t_0)\mathcal{D}_q y(t_0) + \frac{(t_1 - t_0)_q^2}{\Gamma_q(3)}\mathcal{D}_q^{(2)} y(t_0) + ...$$
$$= y_0 + q^{N-1}b\mathcal{D}_q y_0 + \frac{(q^{N-1}b)_q^2}{\Gamma_q(3)}\mathcal{D}_q^{(2)} y_0 + \cdots + \frac{(q^{N-1}b)_q^p}{\Gamma_q(p+1)}\mathcal{D}_q^{(p)} y_0 \cdots .$$

To find an approximation of y_1, we cut above expression to the q-derivative of p^{th}-order and assume that $\mathcal{D}_q^{(j)} y(t_0) = \mathcal{D}_q^{(j)} y_0,\ j = 0, 1, \ldots, p$. Therefore

$$y(t_1) \simeq y_1 = y_0 + q^{N-1}b\mathcal{D}_q y_0 + \frac{[q^{N-1}b]^2}{\Gamma_q(3)}\mathcal{D}_q^{(2)} y_0 + \cdots + \frac{[q^{N-1}b]^p}{\Gamma_q(p+1)}\mathcal{D}_q^{(p)} y_0,$$

that its accuracy depends on the smallness of the $\mathfrak{h}_0$ and the bigness of the p. Using y_1 and $t_2 = t_1 + \mathfrak{h}_2$, we get

$$y(t_2) = y(t_1 + \mathfrak{h}_2) = y(t_1) + (t_2 - t_1)\mathcal{D}_q y(t_1) + \frac{(t_2 - t_1)_q^2}{\Gamma_q(3)}\mathcal{D}_q^{(2)} y(t_1) + \cdots$$
$$= y_1 + (q^{N-2}b - q^{N-1}b)\mathcal{D}_q y_1 + \cdots + \frac{(q^{N-2}b - q^{N-1}b)_q^p}{\Gamma_q(p+1)}\mathcal{D}_q^{(p)} y_1 + \cdots .$$

In this step, by cutting the expression obtained above to p^{th}-order, clearly we get

$$y(t_2) \simeq y_2 = y_1 + q^{N-2}b(1-q)\ \mathcal{D}_q y_1 + [q^{N-2}b(1-q)]^2\ \mathcal{D}_q^{(2)} y_1 + \cdots + [q^{N-2}b(1-q)]^p\ \mathcal{D}_q^{(p)} y_1.$$

As the process continues, we derive

$$y_{k+1} = y_k + q^{N-(k+1)}b(1-q)\ \mathcal{D}_q y_k + [q^{N-(k+1)}b(1-q)]^2\ \mathcal{D}_q^{(2)} y_k$$
$$+ \cdots + [q^{N-(k+1)}b(1-q)]^p\ \mathcal{D}_q^{(p)} y_k,$$

and finally for $\mathfrak{h}_{k+1} = t_{k+1} - t_k = q^{N-(k+1)}b(1-q)$ and $k = 1, 2, \ldots, N-1$, we obtain

$$y_{k+1} = y_k + \mathfrak{h}_{k+1}\mathcal{D}_q y_k + \mathfrak{h}_{k+1}^2\mathcal{D}_q^{(2)} y_k + \cdots + \mathfrak{h}_{k+1}^p\mathcal{D}_q^{(p)} y_k.$$

So, local q-Taylor's expansion is the following form:

$$y_1 = y_0 + \sum_{j=1}^{p}\frac{[q^{N-1}b]^j}{\Gamma_q(j+1)}\mathcal{D}_q^{(j)} y_0 = y_0 + \sum_{j=1}^{p}\frac{\mathfrak{h}_1^j}{\Gamma_q(j+1)}\mathcal{D}_q^{(j)} y_0,$$
$$y_{k+1} = y_k + \sum_{j=1}^{p}[q^{N-(k+1)}b(1-q)]^j\mathcal{D}_q^{(j)} y_k = y_k + \sum_{j=1}^{p}\mathfrak{h}_{k+1}^j\mathcal{D}_q^{(j)} y_k, \quad k = 1, 2, \ldots, N-1.$$

q-Euler's method: Consider the q-Taylor's series expansion of the unique solution of the IVq-P (3) evaluated at the point t_0 which is q-differentiable and continuous, as follows

$$y(t_1) = y(t_0) + q^{N-1}b\ {}_q^F\mathcal{D}y(t_0) + \frac{[q^{N-1}b]^2}{\Gamma_q(3)}\ \mathcal{D}_q^{(2)} y(\eta_0),$$

and at the point t_k

$$y(t_{k+1}) = y(t_k) + \frac{(t_{k+1}-t_k)_q}{\Gamma_q(2)}\mathcal{D}_q y_0(t_k) + \frac{(t_{k+1}-t_k)_q^2}{\Gamma_q(3)}\mathcal{D}_q^2 y(\eta_k),$$

for each $k = 1, \dots, N-1$ and some points $\eta_k \in (t_{k+1}, t_k)$. Let us assume $\mathfrak{h}_{k+1} = t_{k+1} - t_k = q^{N-(k+1)}b(1-q)$. By using problem (3), we have

$$\begin{cases} y(t_1) = y(t_0) + \mathfrak{h}_1 f(t_0, y(t_0)) + \dfrac{\mathfrak{h}_1^2}{q+1}\mathcal{D}_q^{(2)} y(\eta_0), \\ y(t_{k+1}) = y(t_k) + \mathfrak{h}_{k+1}\ f(t_k, y(t_k)) + \mathfrak{h}_{k+1}^2 \mathcal{D}_q^{(2)} y(\eta_k)), \quad k \neq 0. \end{cases}$$

On the other hand, for each $k = 1, \dots, N-1$ we conclude

$$\begin{aligned} &\mathbf{d}(y(t_{k+1}), y(t_k) + \mathfrak{h}_{k+1} f(t_k, y(t_k)) + \mathfrak{h}_{k+1}^2 \mathcal{D}_q^{(2)} y(\eta_k)) \\ &\qquad \leq \mathbf{d}(y(t_{k+1}), y(t_k) + \mathfrak{h}_{k+1} f(t_k, y(t_k))) + \mathbf{d}(0, \mathfrak{h}_{k+1}^2 \mathcal{D}_q^{(2)} y(\eta_k)) \to 0. \end{aligned}$$

It is obvious that when $\mathfrak{h}_{k+1} \to 0$ then $\mathbf{d}(y(t_{k+1}), y(t_k) + \mathfrak{h}_{k+1} f(t_k, y(t_k))) \to 0$ and $\mathbf{d}(0, \mathfrak{h}_{k+1}^2 \mathcal{D}_q^{(2)} y(\eta_k)) \to 0$. So, when $\mathfrak{h}_1 \to 0$, we have

$$\mathbf{d}(y(t_1), y(t_0) + \mathfrak{h}_1 f(t_0, y(t_0)) + \frac{\mathfrak{h}_1^2}{q+1}\mathcal{D}_q^{(2)} y(\eta_0)) \to 0.$$

Then, for sufficiently small $\mathfrak{h}_{k+1}$ ($k = 0, 1, 2, \dots, N-1$), we have $y(t_{k+1}) \approx y(t_k) + \mathfrak{h}_{k+1}\ f(t_k, y(t_k))$. If y_{k+1} is the approximate value of $y(t_{k+1})$, then q-Euler's method is given by the recurrence relation

$$\begin{cases} y_0 = y_0, \\ y_{k+1} = y_k + \mathfrak{h}_{k+1} f(t_k, y_k), \quad k = 0, \dots, N-1. \end{cases} \tag{4}$$

4.1 Fuzzy local q-Taylor's expansion

Consider the following FIVq-P

$$\begin{cases} {}_q^F\mathcal{D} y(t) = f(t, y(t)), \\ y(0) = y_0 \in \mathbb{R}_{\mathcal{F}}, \end{cases} \tag{5}$$

where $f : [0, b] \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ is continuous and $y(t)$ is an unknown fuzzy function of crisp variable t. Furthermore, ${}_q^F\mathcal{D} y(t)$ is the first order ${}_q^F[gH]$-derivative of $y(t)$, with the finite set of switching points. Now, if we consider the mesh points $t_k = q^{N-k}b = qt_{k+1}$ and step sizes $\mathfrak{h}_k = q^{N-k}b(1-q)$ for $k = 1, 2, \dots, N$, then

Case I. If y and it's high order q-derivatives are ${}_q^F[(i) - gH]$-differentiable and type of ${}_q^F[gH]$-differentiability has no change on $\mathbb{T}_\mu$, for $t_1 = t_0 + \mathfrak{h}_1$, then we have

$$y(t_1) = y(t_0 + \mathfrak{h}_1) = y(t_0) \oplus (t_1 - t_0) \odot {}_q^F\mathcal{D} y_{i-gH}(t_0) \oplus \frac{(t_1-t_0)_q^2}{\Gamma_q(3)} \odot {}_q^F\mathcal{D}^{(2)} y_{i-gH}(t_0) \oplus \cdots \tag{6}$$

$$\begin{aligned} &= y_0 \oplus q^{N-1}b \odot {}_q^F\mathcal{D} y_{0,i-gH} \oplus \frac{(q^{N-1}b)_q^2}{\Gamma_q(3)} \odot {}_q^F\mathcal{D}^{(2)} y_{0,i-gH} \\ &\quad \oplus \cdots \oplus \frac{(q^{N-1}b)_q^p}{\Gamma_q(p+1)} \odot {}_q^F\mathcal{D}^{(p)} y_{0,i-gH} \cdots. \end{aligned} \tag{7}$$

To find an approximation of y_1, we cut the expression (6) to the q-derivative of p^{th}-order and assume ${}_q^F\mathcal{D}^{(j)}y_{gH}(t_0) = {}_q^F\mathcal{D}^{(j)}y_{0,gH},\ j = 0, 1, \dots, p$. Therefore

$$\begin{aligned} y(t_1) \simeq y_1 = y_0 &\oplus q^{N-1}b \odot {}_q^F\mathcal{D}y_{0,i-gH} \oplus \frac{[q^{N-1}b]^2}{\Gamma_q(3)} \odot {}_q^F\mathcal{D}^{(2)}y_{0,i-gH} \\ &\oplus \cdots \oplus \frac{[q^{N-1}b]^p}{\Gamma_q(p+1)} \odot {}_q^F\mathcal{D}^{(p)}y_{0,i-gH}, \end{aligned}$$

that its accuracy depends on the smallness of the $\mathfrak{h}_0$ and the bigness of the p. Using y_1 and $t_2 = t_1 + \mathfrak{h}_2$, we get

$$\begin{aligned} y(t_2) = y(t_1 + \mathfrak{h}_2) &= y(t_1) \oplus (t_2 - t_1) \odot {}_q^F\mathcal{D}y_{i-gH}(t_1) \oplus \frac{(t_2 - t_1)_q^2}{\Gamma_q(3)} \odot {}_q^F\mathcal{D}^{(2)}y_{i-gH}(t_1) \oplus \cdots \\ &= y_1 \oplus (q^{N-2}b - q^{N-1}b) \odot {}_q^F\mathcal{D}y_{1,i-gH} \oplus \frac{(q^{N-2}b - q^{N-1}b)_q^2}{\Gamma_q(3)} \odot {}_q^F\mathcal{D}^{(2)}y_{1,i-gH} \\ &\oplus \cdots \oplus \frac{(q^{N-2}b - q^{N-1}b)_q^p}{\Gamma_q(p+1)} \odot {}_q^F\mathcal{D}^{(p)}y_{1,i-gH} \oplus \cdots . \end{aligned}$$

Now, by assuming that $y(t_2) \simeq y_2$ and cut the expression obtained above to p^{th}-order, clearly we obtain

$$\begin{aligned} y_2 = y_1 &\oplus q^{N-2}b(1-q) \odot {}_q^F\mathcal{D}y_{1,i-gH} \oplus [q^{N-2}b(1-q)]^2 \odot {}_q^F\mathcal{D}^{(2)}y_{1,i-gH} \\ &\oplus \cdots \oplus [q^{N-2}b(1-q)]^p \odot {}_q^F\mathcal{D}^{(p)}y_{1,i-gH}. \end{aligned}$$

As the process continues, we derive

$$\begin{aligned} y_{k+1} = y_k &\oplus q^{N-(k+1)}b(1-q) \odot {}_q^F\mathcal{D}y_{k,i-gH} \oplus [q^{N-(k+1)}b(1-q)]^2 \odot {}_q^F\mathcal{D}^{(2)}y_{k,i-gH} \\ &\oplus \cdots \oplus [q^{N-(k+1)}b(1-q)]^p \odot {}_q^F\mathcal{D}^{(p)}y_{k,i-gH}, \end{aligned}$$

and generally for step $\mathfrak{h}_{k+1} = t_{k+1} - t_k = q^{N-(k+1)}b(1-q)$ and $k = 1, 2, \dots, N-1$, it results in

$$y_{k+1} = y_k \oplus \mathfrak{h}_{k+1} \odot {}_q^F\mathcal{D}y_{k,i-gH} \oplus \mathfrak{h}_{k+1}^2 \odot {}_q^F\mathcal{D}^{(2)}y_{k,i-gH} \cdots \oplus \mathfrak{h}_{k+1}^p \odot {}_q^F\mathcal{D}^{(p)}y_{k,i-gH}.$$

Finally, we conclude that

$$\begin{aligned} y_1 &= y_0 \oplus \sum_{j=1}^{p} \frac{[q^{N-1}b]^j}{\Gamma_q(j+1)} \odot {}_q^F\mathcal{D}^{(j)}y_{0,i-gH} = y_0 \oplus \sum_{j=1}^{p} \frac{\mathfrak{h}_1^j}{\Gamma_q(j+1)} \odot {}_q^F\mathcal{D}^{(j)}y_{0,i-gH}, \\ y_{k+1} &= y_k \oplus \sum_{j=1}^{p} [q^{N-(k+1)}b(1-q)]^j \odot {}_q^F\mathcal{D}^{(j)}y_{k,i-gH} \\ &= y_k \oplus \sum_{j=1}^{p} \mathfrak{h}_{k+1}^j \odot {}_q^F\mathcal{D}^{(j)}y_{k,i-gH}, \quad k = 1, 2, \dots, N-1. \end{aligned}$$

Case II. If y and it's high order q-derivatives are ${}_q^F[(ii)-gH]$-differentiable and type of ${}_q^F[gH]$-differentiability is unchangeable on $\mathbb{T}_\mu$, then

$$y_1 = y_0 \ominus (-1)\sum_{j=1}^{p} \frac{\mathfrak{h}_1^j}{\Gamma_q(j+1)} \odot {}_q^F\mathcal{D}^{(j)} y_{0,ii-gH},$$

$$y_{k+1} = y_k \ominus (-1)\sum_{j=1}^{p} \mathfrak{h}_{k+1}^j \odot {}_q^F\mathcal{D}^{(j)} y_{k,ii-gH}, \quad k=1,2,\dots,N-1.$$

Case III. If y is ${}_q^F[(i)-gH]$-differentiable, and it's high order derivatives, change from ${}_q^F[(i)-gH]$ to ${}_q^F[(ii)-gH]$-differentiability according to Definition 3(type I), then

$$y_1 = y_0 \ominus (-1) \sum_{\substack{j=1\\ j\ is\ odd}}^{p} \frac{\mathfrak{h}_1^j}{\Gamma_q(j+1)} \odot {}_q^F\mathcal{D}^{(j)} y_{0,ii-gH} \oplus \sum_{\substack{j=2\\ j\ is\ even}}^{p} \frac{\mathfrak{h}_1^j}{\Gamma_q(j+1)} \odot {}_q^F\mathcal{D}^{(j)} y_{0,i-gH},$$

$$y_{k+1} = y_k \ominus (-1) \sum_{\substack{j=1\\ j\ is\ odd}}^{p} \mathfrak{h}_{k+1}^j \odot {}_q^F\mathcal{D}^{(j)} y_{k,ii-gH} \oplus \sum_{\substack{j=2\\ j\ is\ even}}^{p} \mathfrak{h}_{k+1}^j \odot {}_q^F\mathcal{D}^{(j)} y_{k,i-gH}, \ k=1,\dots,N-1.$$

Case IV. If y has a switching point at $\xi \in \mathbb{T}_\mu$ of type II (i.e. ${}_q^F[gH]$-differentiability changes from ${}_q^F[(ii)-gH]$ to ${}_q^F[(i)-gH]$) and the type of differentiability for ${}_q^F\mathcal{D}^{(j)}y,\ j=1,\dots,n-1$ are ${}_q^F[(i)-gH]$-differentiable, then for $k=0$

$$y_1 = \begin{cases} y_0 \ominus (-1)\mathfrak{h}_1 \odot {}_q^F\mathcal{D} y_{0,ii-gH} \oplus \sum_{j=2}^{p} \dfrac{\mathfrak{h}_1^j}{\Gamma_q(j+1)} \odot {}_q^F\mathcal{D}^{(j)} y_{0,i-gH}, & 0<t\le\xi, \\ y_0 \oplus \sum_{j=1}^{p} \dfrac{\mathfrak{h}_1^j}{\Gamma_q(j+1)} \odot {}_q^F\mathcal{D}^{(j)} y_{0,i-gH}, & \xi \le t. \end{cases}$$

and for $k=1,2,\dots,N-1$,

$$y_{k+1} = \begin{cases} y_k \ominus (-1)\mathfrak{h}_{k+1} \odot {}_q^F\mathcal{D} y_{k,ii-gH} \oplus \sum_{j=2}^{p} \mathfrak{h}_{k+1}^j \odot {}_q^F\mathcal{D}^{(j)} y_{k,i-gH}, & 0<t\le\xi, \\ y_k \oplus \sum_{j=1}^{p} \mathfrak{h}_{k+1}^j \odot {}_q^F\mathcal{D}^{(j)} y_{k,i-gH}, & \xi \le t. \end{cases}$$

Case V. If y has a switching point at $\xi \in \mathbb{T}_\mu$ of type I (i.e. ${}_q^F[gH]$-differentiability changes from ${}_q^F[(i)-gH]$ to ${}_q^F[(ii)-gH]$) and the type of differentiability for ${}_q^F\mathcal{D}^{(j)}y,\ j=1,\dots,n-1$ are ${}_q^F[(ii)-gH]$-differentiable, then for $k=0$

$$y_1 = \begin{cases} y_0 \oplus \mathfrak{h}_1 \odot {}_q^F\mathcal{D} y_{0,i-gH} \ominus (-1)\sum_{j=2}^{p} \dfrac{\mathfrak{h}_1^j}{\Gamma_q(j+1)} \odot {}_q^F\mathcal{D}^{(j)} y_{0,ii-gH}, & 0<t\le\xi, \\ y_0 \ominus (-1)\sum_{j=1}^{p} \dfrac{\mathfrak{h}_1^j}{\Gamma_q(j+1)} \odot {}_q^F\mathcal{D}^{(j)} y_{0,ii-gH}, & \xi \le t. \end{cases}$$

and for $k = 1, 2, \ldots, N-1$,

$$y_{k+1} = \begin{cases} y_k \oplus \mathfrak{h}_{k+1} \odot {}_q^F\mathcal{D}y_{k,i-gH} \ominus (-1) \sum_{j=2}^{p} \mathfrak{h}_{k+1}^j \odot {}_q^F\mathcal{D}^{(j)} y_{k,ii-gH}, & 0 < t \leq \xi, \\ y_k \ominus (-1) \sum_{j=1}^{p} \mathfrak{h}_{k+1}^j \odot {}_q^F\mathcal{D}^{(j)} y_{k,ii-gH}, & \xi \leq t. \end{cases}$$

This method is called the fuzzy local q-Taylor's expansion. One way to avoid calculating the higher order q-derivatives is to put $p = 1$ in the fuzzy local q-Taylor's expansion which the approximated results are obtained by fuzzy q-Euler's method in the following section.

4.2 Fuzzy q-Euler's method

The q-Euler's method which often serves as the basis to construct more complex methods, is a first-order numerical method. Fuzzy q-Euler's method is the natural starting point for any discussion of numerical methods for solving the fuzzy q-differential equation. We know that the fuzzy q-Euler's method can be derived in several ways. By considering the mesh points $t_k = q^{N-k}b = qt_{k+1}$ and the step sizes $\mathfrak{h}_k = q^{N-k}b(1-q)$ for $k = 1, 2, \ldots, N$, the method that is considered here is as follows:

Case I. Consider the fuzzy q-Taylor's series expansion of the unique solution of the FIVq-P (5) about t_k which is ${}_q^F[(i)-gH]$-differentiable and belongs to $\mathcal{C}_f^2(\mathbb{T}_\mu, \mathbb{R}_\mathcal{F})$ such that the type of ${}_q^F[gH]$-differentiability is unchangeable on $[0, b] \in \mathbb{T}_\mu$, as follows

$$y(t_1) = y(t_0) \oplus q^{N-1}b \odot {}_q^F\mathcal{D}y(t_0) \oplus \frac{[q^{N-1}b]^2}{\Gamma_q(3)} \odot {}_q^F\mathcal{D}^{(2)} y(\eta_0),$$

and

$$y(t_{k+1}) = y(t_k) \oplus \frac{(t_{k+1}-t_k)_q}{\Gamma_q(2)} \odot \; {}_q^F\mathcal{D}y_{i-gH}(t_k) \oplus \frac{(t_{k+1}-t_k)_q^2}{\Gamma_q(3)} \odot \; {}_q^F\mathcal{D}^{(2)} y_{i-gH}(\eta_k),$$

for each $k = 1, \ldots, N-1$ and some points $\eta_k \in (t_{k+1}, t_k)$. Let us assume $\mathfrak{h}_{k+1} = t_{k+1} - t_k = q^{N-(k+1)}b(1-q)$ and by using problem (5), we have

$$\begin{cases} y(t_1) = y(t_0) \oplus \mathfrak{h}_1 \odot f(t_0, y(t_0)) \oplus \dfrac{\mathfrak{h}_1^2}{q+1} \odot {}_q^F\mathcal{D}^{(2)} y_{i-gH}(\eta_0), \\ y(t_{k+1}) = y(t_k) \oplus \mathfrak{h}_{k+1} \odot \; f(t_k, y(t_k)) \oplus \mathfrak{h}_{k+1}^2 \odot \; {}_q^F\mathcal{D}^{(2)} y_{i-gH}(\eta_k)), \quad k \neq 0. \end{cases}$$

On the other hand, for each $k = 1, \ldots, N-1$ we conclude

$$\begin{aligned} &\mathbf{d}(y(t_{k+1}), y(t_k) \oplus \mathfrak{h}_{k+1} \odot f(t_k, y(t_k)) \oplus \mathfrak{h}_{k+1}^2 \odot {}_q^F\mathcal{D}^{(2)} y_{i-gH}(\eta_k)) \\ &\quad \leq \mathbf{d}(y(t_{k+1}), y(t_k) \oplus \mathfrak{h}_{k+1} \odot f(t_k, y(t_k))) + \mathbf{d}(0, \mathfrak{h}_{k+1}^2 \odot \; {}_q^F\mathcal{D}^{(2)} y_{i-gH}(\eta_k)) \to 0, \end{aligned}$$

as $\mathfrak{h}_{k+1} \to 0$ since $\mathbf{d}(y(t_{k+1}), y(t_k) \oplus \mathfrak{h}_{k+1} \odot f(t_k, y(t_k))) \to 0$ and $\mathbf{d}(0, \mathfrak{h}_{k+1}^2 \odot {}_q^F\mathcal{D}^{(2)} y_{i-gH}(\eta_k)) \to 0$. So, when $\mathfrak{h}_1 \to 0$, it is obvious that

$$\mathbf{d}(y(t_1), y(t_0) \oplus \mathfrak{h}_1 \odot f(t_0, y(t_0)) \oplus \frac{\mathfrak{h}_1^2}{q+1} \odot {}_q^F\mathcal{D}^{(2)} y_{i-gH}(\eta_0)) \to 0.$$

Table 1: Numerical results of Example 1 for $q = 0.7$ and the node points $t_i = q^i b$.

$q = 0.7$			
$t_i = q^i b$	$\mathfrak{h}_i = t_i - t_{i-1}$	$y_i = y_{i-1} \oplus \mathfrak{h}_i(qt_{i-1}y_{i-1})$	$e_i = y(t_i) - y_{t_i}$
0.2401	0.07203	(0.82,1.,1.2)	(0.0197394,0.0240725,0.028887)
0.343	0.1029	(0.834181,1.01729,1.22075)	(0.0266999,0.0325609,0.0390731)
0.49	0.147	(0.863624,1.0532,1.26384)	(0.0423815,0.0516848,0.0620218)
0.7	0.21	(0.92583,1.12906,1.35487)	(0.0812172,0.0990454,0.118854)
1.	0.3	(1.06193,1.29503,1.55404)	(0.19581,0.238793,0.286552)

Then, for suffciently small $\mathfrak{h}_{k+1}$ $(k = 0, 1, 2, \ldots, N-1)$, we have $y(t_{k+1}) \approx y(t_k) \oplus \mathfrak{h}_{k+1} \odot f(t_k, y(t_k))$. If y_{k+1} is the approximate value of $y(t_{k+1})$, then the fuzzy q-Euler's method can be constructed in the following form

$$\begin{cases} y_0 = y_0, \\ y_{k+1} = y_k \oplus \mathfrak{h}_{k+1} \odot f(t_k, y_k), \quad k = 0, \ldots, N-1. \end{cases} \tag{8}$$

Case II. Let us suppose that $y(t) \in \mathcal{C}^2_f(\mathbb{T}_\mu, \mathbb{R}_\mathcal{F})$ is ${}^F_q[(ii) - gH]$-differentiable such that the type of ${}^F_q[gH]$-differentiability do not change on $\mathbb{T}_\mu$. According to the process described above, in this case the fuzzy q-Euler's is presented as

$$\begin{cases} y_0 = y_0, \\ y_{k+1} = y_k \ominus (-1)\mathfrak{h}_{k+1} \odot f(t_k, y_k), \quad k = 0, \ldots, N-1. \end{cases} \tag{9}$$

Case III. Assume that $y(t)$ has a switching point type I at $\xi \in [0, b]$, such that $t_0, t_1, \ldots, t_j, \xi, t_{j+1}, \ldots, t_N$. So according to Eq. (8) and (9), the fuzzy q-Euler's method is obtained as follows

$$\begin{cases} y_0 = y_0, \\ y_{k+1} = y_k \oplus \mathfrak{h}_{k+1} \odot f(t_k, y_k), & k = 0, 1, \ldots, j, \\ y_{k+1} = y_k \ominus (-1)\mathfrak{h}_{k+1} \odot f(t_k, y_k), & k = j+1, j+2, \ldots, N-1. \end{cases}$$

Case IV. If ξ is a switching point type II, the fuzzy q-Euler's method takes the form

$$\begin{cases} y_0 = y_0, \\ y_{k+1} = y_k \ominus (-1)\mathfrak{h}_{k+1} \odot f(t_k, y_k), & k = 0, 1, \ldots, j, \\ y_{k+1} = y_k \oplus \mathfrak{h}_{k+1} \odot f(t_k, y_k), & k = j+1, j+2, \ldots, N-1. \end{cases}$$

Example 1. Let us consider the following problem

$$\begin{cases} {}^F_q\mathcal{D}y(t) = qt \odot y(t), & t \in \mathbb{T}_\mu, \\ y_0 = (0.82, 1, 1.2) \in \mathbb{R}_f, \end{cases} \tag{10}$$

where the solution is $y(t) = (0.82, 1, 1.2) \odot \sum_{n=0}^{\infty} \dfrac{q^n t^{2n}}{[2n]_q!!}$, in which $[2n]_q!! = [n]_q!(2)^n_q$. Numerical results are demonstrated in Table 1 for $q = 0.7$.

Remark 1. *It is clear that the accuracy of y_1 in fuzzy local q-Taylor's expansion depends on small values of h and large values of p. But we should have the p^{th}-order derivative of y. We know that in the fuzzy local q-Taylor's expansion for $p > 1$, finding the high order derivative of y is difficult. Thus we should use other methods, that they do not apply the high order derivatives of ${}_q^F\mathcal{D}y$. By substituting $p = 1$ in the Taylor expansion, we can approximate the value of y_{i+1} that we call the fuzzy q-Euler's method.*

5 Conclusions

As we know, there are many real life problems in the various fields of science and engineering that can be modelized as differential equations. Thus, solving differential equations have been always one of the concerns of various sciences. Most of the time it is not possible to find the exact solution to the problem using direct methods, so we need to use approximate and numerical methods to solve differential equations.The q-Taylor's expansion method is one of the elementary and original methods for solving differential equations especially initial value q-problems. In this article we have tried to introduce the fuzzy local q-Taylor's expansion and the fuzzy q-Euler's method for solving the FIVq-P. These numerical methods were defined based on the fuzzy q-Taylor's expansion. Accordingly, we will need to introduce local q-Taylor's expansion and q-Euler's method in crisp form then the fuzzy mode. In the future works, we will examine the details of these applications that no attempt has been made here to develop.

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