ON $(\sigma, \delta)$-SKEW MCCOY MODULES

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Abstract. Let $(\sigma, \delta)$ be a quasi derivation of a ring $R$ and $M_R$ a right $R$-module. In this paper, we introduce the notion of $(\sigma, \delta)$-skew McCoy modules which extends the notion of McCoy modules and $\sigma$-skew McCoy modules. This concept can be regarded also as a generalization of $(\sigma, \delta)$-skew Armendariz modules. We study some connections between reduced modules, semicommutative modules, $(\sigma, \delta)$-compatible modules and $(\sigma, \delta)$-skew McCoy modules. Furthermore, we will give some results showing that the property of being an $(\sigma, \delta)$-skew McCoy module transfers well from a module $M_R$ to its skew triangular matrix extensions and vice versa.

1. Introduction

Throughout this paper, $R$ denotes an associative ring with unity and $M_R$ a right $R$-module. For a subset $X$ of a module $M_R$, $r_R(X) = \{a \in R | Xa = 0\}$ and $\ell_R(X) = \{a \in R | aX = 0\}$ will stand for the right and the left annihilator of $X$ in $R$ respectively. An Ore extension of a ring $R$ is denoted by $R[x; \sigma, \delta]$, where $\sigma$ is an endomorphism of $R$ and $\delta$ is a $\sigma$-derivation, i.e., $\delta: R \to R$ is an additive map such that $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$ for all $a, b \in R$ (the pair $(\sigma, \delta)$ is also called a quasi-derivation of $R$). Recall that elements of $R[x; \sigma, \delta]$ are polynomials in $x$ with coefficients written on the left. Multiplication in $R[x; \sigma, \delta]$ is given by the multiplication in $R$ and the condition $xa = \sigma(a)x + \delta(a)$, for all $a \in R$. In the next, $S$ will stand for the Ore extension $R[x; \sigma, \delta]$.
For any $0 \leq i \leq j$ ($i, j \in \mathbb{N}$), $f_i^j \in \text{End}(R, +)$ will denote the map which is the sum of all possible words in $\sigma, \delta$ built with $i$ factors of $\sigma$ and $j - i$ factors of $\delta$ (e. g., $f_n^0 = \sigma^n$ and $f_0^n = \delta^n, n \in \mathbb{N}$). We have $x^j a = \sum_{i=0}^{j} f_i^j(a)x^i$ for all $a \in R$, where $i, j$ are nonnegative integers with $j \geq i$.

Let $M_R$ be a right $R$-module, we can make $M[x]$ into a right $S$-module by allowing polynomials from $S$ to act on polynomials in $M[x]$ in the obvious way, and applying the above “twist” whenever necessary. The verification that this defines a valid $S$-module structure on $M[x]$ is almost identical to the verification that $S$ is a ring, and it is straightforward. An excellent discussion of Ore extension rings may be found in [13].

From now on, we will use the notation of Lee and Zhou [14], for the right $R[x; \sigma, \delta]$-module $M[x]$. Consider

$$M[x; \sigma, \delta] := \left\{ \sum_{i=0}^{n} m_i x^i \mid n \geq 0, m_i \in M \right\};$$

which is an $S$-module under an obvious addition and the action of monomials of $R[x; \sigma, \delta]$ on monomials in $M[x; \sigma, \delta]$ via $(mx^i)(ax^\ell) = m\sum_{i=0}^{j} f_i^j(a)x^{i+\ell}$ for all $a \in R$ and $j, \ell \in \mathbb{N}$. The $R[x; \sigma, \delta]$-module $M[x; \sigma, \delta]$ is called the skew polynomial extension related to the quasi-derivation $(\sigma, \delta)$.

A module $M_R$ is semicommutative, if for any $m \in M$ and $a \in R$, $ma = 0$ implies $mRa = 0$, [6]. Let $\sigma$ an endomorphism of $R$, $M_R$ is called an $\sigma$-semicommutative module, if for any $m \in M$ and $a \in R$, $ma = 0$ implies $mR\sigma(a) = 0$, [7]. For a module $M_R$ and a quasi-derivation $(\sigma, \delta)$ of $R$, we say that $M_R$ is $\sigma$-compatible, if for each $m \in M$ and $a \in R$, we have $ma = 0 \iff m\sigma(a) = 0$. Moreover, we say that $M_R$ is $\delta$-compatible, if for each $m \in M$ and $a \in R$, we have $ma = 0 \Rightarrow m\delta(a)=0$. If $M_R$ is both $\sigma$-compatible and $\delta$-compatible, we say that $M_R$ is $(\sigma, \delta)$-compatible (see [3]). In [7], a module $M_R$ is called $\sigma$-skew Armendariz, if $m(x)f(x) = 0$ where $m(x) = \sum_{i=0}^{n} m_ix^i \in M[x; \sigma]$ and $f(x) = \sum_{j=0}^{m} a_j x^j \in R[x; \sigma]$ implies $m_i \sigma^j(a_j) = 0$ for all $i, j$. According to Lee and Zhou [14], $M_R$ is called $\sigma$-Armendariz, if it is $\sigma$-compatible and $\sigma$-skew Armendariz. Chen and Cui [8, 9], introduced both concepts of McCoy modules and $\sigma$-skew McCoy modules. A module $M_R$ is called McCoy if $m(x)g(x) = 0$, where $m(x) = \sum_{i=0}^{n} m_ix^i \in M[x]$ and $g(x) = \sum_{j=0}^{q} b_j x^j \in R[x] \setminus \{0\}$ implies that there exists $a \in R \setminus \{0\}$ such that $m(x)a = 0$. A module $M_R$ is called $\sigma$-skew McCoy if $m(x)g(x) = 0$, where $m(x) = \sum_{i=0}^{n} m_ix^i \in M[x; \sigma]$ and $g(x) = \sum_{j=0}^{q} b_j x^j \in R[x; \sigma] \setminus \{0\}$ implies that there exists $a \in R \setminus \{0\}$ such
that $m(x)a = 0$. Following Alhevas and Moussavi [1], a module $M_R$ is called $(\sigma, \delta)$-skew Armendariz, if whenever $m(x)g(x) = 0$ where $m(x) = \sum_{i=0}^{p} m_i x^i \in M[x; \sigma, \delta]$ and $g(x) = \sum_{j=0}^{q} b_j x^j \in R[x; \sigma, \delta]$, we have $m_i x^i b_j x^j = 0$ for all $i, j$. According to Lee and Zhou [14], a module $M_R$ is called $\sigma$-reduced, if for any $m \in M$ and $a \in R$. We have: $ma = 0$ implies $mR \cap Ma = 0$, and $ma = 0$ if and only if $m\sigma(a) = 0$. The module $M_R$ is called reduced if $M_R$ is id$_R$-reduced. Moreover, $M_R$ is reduced if and only if it is semicommutative with $ma^2 = 0$ implies $ma = 0$ for any $m \in M$ and $a \in R$ (see [14, Lemma 1.2]).

In this paper, we introduce the concept of $(\sigma, \delta)$-skew McCoy modules which is a generalization of McCoy modules and $\sigma$-skew McCoy modules. This concept can be regarded also as a generalization of $(\sigma, \delta)$-skew Armendariz modules and $(\sigma, \delta)$-skew Armendariz rings. We show that, $(\sigma, \delta)$-compatible reduced modules are $(\sigma, \delta)$-skew McCoy. In particular, $\sigma$-reduced modules are $\sigma$-skew McCoy. Also, many connections between reduced modules, semicommutative modules, $(\sigma, \delta)$-compatible modules and $(\sigma, \delta)$-skew McCoy modules are studied. Furthermore, we show that $(\sigma, \delta)$-skew McCoyness passes from a module $M_R$ to its skew triangular matrix extension $V_n(M, \sigma)$. In this sens, we complete the definition of skew triangular matrix rings $V_n(R, \sigma)$ given by Isfahani [19], by introducing the notion of skew triangular matrix modules, and we will give some results on $(\sigma, \delta)$-skew McCoy triangular matrix modules.

2. $(\sigma, \delta)$-skew McCoy modules

In this section, we introduce the concept of $(\sigma, \delta)$-skew McCoy modules which is a generalization of McCoy modules, $\sigma$-skew McCoy modules and $(\sigma, \delta)$-skew Armendariz modules.

**Definition 2.1.** Let $M_R$ be a module and $M[x; \sigma, \delta]$ the corresponding $(\sigma, \delta)$-skew polynomial module over $R[x; \sigma, \delta]$.

- (1) $M_R$ is called $(\sigma, \delta)$-skew McCoy if $m(x)g(x) = 0$, where $m(x) = \sum_{i=0}^{p} m_i x^i \in M[x; \sigma, \delta]$ and $g(x) = \sum_{j=0}^{q} b_j x^j \in R[x; \sigma, \delta] \setminus \{0\}$, implies that there exists $a \in R \setminus \{0\}$ such that $m(x)a = 0$.

- (2) $R$ is called $(\sigma, \delta)$-skew McCoy if $R$ is $(\sigma, \delta)$-skew McCoy as a right $R$-module.

**Remark 2.2.** Let $m(x) = \sum_{i=0}^{p} m_i x^i \in M[x; \sigma, \delta]$ and $a \in R$. Then

$$m(x)a = 0 \iff \sum_{i=\ell}^{p} m_i f_i^\ell(a) = 0 \quad \forall \ell = 0, 1, \ldots, p.$$
If \( \sigma = id_R \) and \( \delta = 0 \), we get the concept of McCoy module, and if only \( \delta = 0 \), we get the concept of \( \sigma \)-skew McCoy module. An ideal \( I \) of a ring \( R \) is called \((\sigma, \delta)\)-stable, if \( \sigma(I) \subseteq I \) and \( \delta(I) \subseteq I \).

**Proposition 2.3.** (1) Let \( I \) be any nonzero ideal of \( R \). If \( I \) is \((\sigma, \delta)\)-stable, then \( R/I \) is a \((\sigma, \delta)\)-skew McCoy module as an \( R \)-module.

(2) For any index set \( I \), if \( M_i \) is a \((\sigma_i, \delta_i)\)-skew McCoy module as an \( R_i \)-module for each \( i \in I \), then \( \prod_{i \in I} M_i \) is a \((\sigma, \delta)\)-skew McCoy as an \( \prod_{i \in I} R_i \)-module, where \( (\sigma, \delta) = (\sigma_i, \delta_i)_{i \in I} \).

(3) Every submodule of a \((\sigma, \delta)\)-skew McCoy module is \((\sigma, \delta)\)-skew McCoy. In particular, if \( I \) is a right ideal of a \((\sigma, \delta)\)-skew McCoy ring, then \( I_R \) is a \((\sigma, \delta)\)-skew McCoy module.

(4) A module \( M_R \) is \((\sigma, \delta)\)-skew McCoy if and only if every finitely generated submodule of \( M_R \) is \((\sigma, \delta)\)-skew McCoy.

Proof. (1) Let \( m(x) = \sum_{i=0}^{p} \overline{m}_i x^i \in (R/I)[x; \sigma, \delta] \), where \( \overline{m}_i = r_i + I \in R/I \) for all \( i = 0, 1, \ldots, p \) and \( r \) an arbitrary nonzero element of \( I \). We have \( m(x)r) = \sum_{i=0}^{p} (r_i + I) \sum_{\ell=0}^{i} f_{\ell}(r)x^{\ell} \in I[x; \sigma, \delta] \), because \( f_{\ell}(r) \in I \) for all \( \ell = 0, 1, \ldots, i \). Hence \( m(x)r = 0 \).

(2) Let \( M = \prod_{i \in I} M_i \) and \( R = \prod_{i \in I} R_i \) such that each \( M_i \) is an \((\sigma_i, \delta_i)\)-skew McCoy as \( R_i \)-module for all \( i \in I \). Take \( m(x) = (m_i(x))_{i \in I} \in M[x; \sigma, \delta] \) and \( f(x) = (f_i(x))_{i \in I} \in R[x; \sigma, \delta] \setminus \{0\} \), where \( m_i(x) = \sum_{s=0}^{p} m_i(s)x^s \in M_i[x; \sigma_i, \delta_i] \) and \( f_i(x) = \sum_{t=0}^{q} a_i(t)x^t \in R_i[x; \sigma_i, \delta_i] \) for each \( i \in I \). If \( m(x)f(x) = 0 \), then \( m_i(x)f_i(x) = 0 \) for each \( i \in I \). Since \( M_i \) is \((\sigma_i, \delta_i)\)-skew McCoy, there exists \( 0 \neq r_i \in R_i \) such that \( m_i(x)r_i = 0 \) for each \( i \in I \). Thus \( m(x)r = 0 \) where \( 0 \neq r = (r_i)_{i \in I} \in R \).

(3) and (4) are obvious. \( \square \)

If \( M_R \) is an \((\sigma, \delta)\)-compatible module, then \( ma = 0 \Rightarrow m f_i^j(a) = 0 \) for any nonnegative integers \( i, j \) such that \( i \geq j \), where \( m \in M_R \) and \( a \in R \). For a subset \( U \) of \( M_R \) and \((\sigma, \delta)\) a quasi-derivation of \( R \), the set of all skew polynomials with coefficients in \( U \) will be denoted by \( U[x; \sigma, \delta] \).

**Lemma 2.4.** Let \( M_R \) be a module and \((\sigma, \delta)\) a quasi-derivation of \( R \). The following are equivalent:

1. For any \( U \subseteq M[x; \sigma, \delta] \), \( (r_R[x; \sigma, \delta](U) \cap R)[x; \sigma, \delta] = r_{R[x; \sigma, \delta]}(U) \).

2. For any \( m(x) = \sum_{i=0}^{p} m_i x^i \in M[x; \sigma, \delta] \) and \( f(x) = \sum_{j=0}^{q} a_j x^j \in R[x; \sigma, \delta] \). If \( m(x)f(x) = 0 \), then \( \sum_{i=0}^{p} m_i f_i^j(a_j) = 0 \) for all \( i, j \).

Proof. (1) \( \Rightarrow \) (2). Let \( m(x) = \sum_{i=0}^{p} m_i x^i \in M[x; \sigma, \delta] \) and \( f(x) = \sum_{j=0}^{q} a_j x^j \in R[x; \sigma, \delta] \). If \( m(x)f(x) = 0 \), we have \( f(x) \in r_{R[x; \sigma, \delta]}(m(x)) = (r_{R[x; \sigma, \delta]}(m(x)) \cap R)[x; \sigma, \delta] \). Then \( a_j \in r_{R[x; \sigma, \delta]}(m(x)) \) for all \( j \), so that
m(x)a_j = 0 for all j. But m(x)a_j = 0 ⇔ \sum_{i=0}^{p} m_i f_i^\ell(a_j) = 0 for all 0 \leq i \leq p. Thus \sum_{i=1}^{p} m_i f_i^\ell(a_j) = 0 for all i, j.

(2) ⇒ (1). Let U \subseteq M[x; \sigma, \delta], we have always \( r_R(x; \sigma, \delta)(U) \cap R[x; \sigma, \delta] \subseteq r_R(x; \sigma, \delta)(U) \). Conversely, let \( f(x) \in r_R(x; \sigma, \delta)(U) \) then by (2), we have \( Ua_j = 0 \) for all j and so \( a_j \in r_R(x; \sigma, \delta)(U) \cap R \). Therefore \( f(x) \in (r_R(x; \sigma, \delta)(U) \cap R)[x; \sigma, \delta] \). □

**Theorem 2.5.** Let \( M_R \) be a module and \( N \) a nonzero submodule of \( M[x; \sigma, \delta] \) such that \( r_R[x; \sigma, \delta](N) = (r_R[x; \sigma, \delta](N) \cap R)[x; \sigma, \delta] \). Then

\[
r_R[x; \sigma, \delta](N) \neq 0 \implies r_R(N) \neq 0.
\]

**Proof.** If \( r_R[x; \sigma, \delta](N) \neq 0 \), then there exists \( 0 \neq f(x) = \sum_{i=0}^{p} a_i x^i \in r_R[x; \sigma, \delta](N) \). But \( r_R[x; \sigma, \delta](N) = (r_R[x; \sigma, \delta](N) \cap R)[x; \sigma, \delta] \) by Lemma 2.4. Therefore all \( a_i \) are in \( r_R[x; \sigma, \delta](N) \), so \( a_i \in r_R(N) \) for all \( i \). Since \( f(x) \neq 0 \), there exists \( i_0 \in \{0, 1, \cdots p\} \) such that \( 0 \neq a_{i_0} \in r_R(N) \). So that \( r_R(N) \neq 0 \). □

**Proposition 2.6.** If \( M_R \) is an \((\sigma, \delta)\)-skew Armendariz module, then it is \((\sigma, \delta)\)-skew McCoy.

**Proof.** Let \( m(x) = \sum_{i=0}^{p} m_i x^i \in M[x; \sigma, \delta] \) and \( g(x) = \sum_{j=0}^{q} b_j x^j \in R[x; \sigma, \delta] \setminus \{0\} \). If \( m(x)g(x) = 0 \), then \( m_i x^i b_j x^j = 0 \) for all \( i, j \). Since \( g(x) \neq 0 \), we have \( b_{j_0} \neq 0 \) for some \( j_0 \in \{0, 1, \cdots, p\} \). Therefore \( m_i x^i b_{j_0} x^{j_0} = 0 \) for all \( i \). On the other hand, we have

\[
m_i x^i b_{j_0} x^{j_0} = \sum_{\ell=0}^{p} \sum_{i=\ell}^{p} m_i f_{i}^\ell(b_{j_0}) x^{\ell+j_0} = 0,
\]

and so \( \sum_{i=\ell}^{p} m_i f_{i}^\ell(b_{j_0}) = 0 \) for all \( \ell = 0, 1, \cdots, p \). Then \( m(x)b_{j_0} = 0 \), hence \( M_R \) is \((\sigma, \delta)\)-skew McCoy. □

Note that, the converse of Proposition 2.6 does not hold by the next example.

**Example 2.7.** Let \( \mathbb{Z}_4 \) be the ring of integers modulo 4, and consider the ring

\[
R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z}_4 \right\}.
\]

Let \( \sigma: R \to R \) be an endomorphism defined by \( \sigma\left( \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix} \). Clearly, \( \sigma \) in an automorphism of \( R \).

(1) \( R_R \) is \( \sigma \)-skew McCoy, by [5, Theorem 9]. In fact, \( R \) is commutative, \( \sigma \)-reversible and \( \sigma \) is an automorphism of \( R \) (see [4, Example 2.7(i)]).
(2) $R_R$ is not $\sigma$-skew Armendariz, by [11, Example 7]. Indeed, for $p = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} x \in R[x; \sigma]$, we have $p^2 = 0$. However

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \sigma \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \neq 0$$

**Corollary 2.8.** If $M_R$ is a reduced $(\sigma, \delta)$-compatible module, then it is $(\sigma, \delta)$-skew McCoy.

**Proof.** Clearly from [1, Theorem 2.19] and Proposition 2.6. □

A module $(\sigma, \delta)$-skew McCoy need not to be McCoy by [9, Example 2.3(2)]. The next example shows that, there exists a module which is McCoy but not $(\sigma, \delta)$-skew McCoy for some quasi-derivation $(\sigma, \delta)$.

**Example 2.9.** Let $\mathbb{Z}_2$ be the ring of integers modulo 2, and consider the ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with the usual addition and multiplication. Let $\sigma$ be an endomorphism of $R$ defined by $\sigma((a, b)) = (b, a)$ and $\delta$ an $\sigma$-derivation of $R$ defined by $\delta((a, b)) = (a, b) - \sigma((a, b))$. Then $R$ is a commutative reduced ring, and so it is McCoy. However, for $p(x) = (1, 0)x$ and $q(x) = (1, 1) + (1, 0)x \in R[x; \sigma, \delta]$. We have $p(x)q(x) = 0$, but $p(x)(a, b) \neq 0$ for any $0 \neq (a, b) \in R$. Therefore, $R$ is not $(\sigma, \delta)$-skew McCoy. Note that, $R$ is not $(\sigma, \delta)$-compatible, because $(0, 1)(1, 0) = (0, 0)$, but $(0, 1)\sigma((1, 0)) = (0, 1)^2 \neq (0, 0)$ and $(0, 1)\delta((1, 0)) = (0, 1)(1, 1) = (0, 1) \neq (0, 0)$.

With the help of Examples 2.9 and 2.10, we see that “$(\sigma, \delta)$-compatibility” and “reducibility” of $M_R$ in Corollary 2.8 are not superfluous.

**Example 2.10.** Let $\mathbb{Z}_2$ be the ring of integers modulo 2. Consider the ring $R = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$ and the endomorphism $\sigma: R \rightarrow R$ defined by

$$\sigma \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}.$$

(1) $R$ is $\sigma$-compatible. Indeed, let $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, $B = \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} \in R$. Then

$$AB = 0 \iff \begin{cases} aa' = 0 \\ bb' = 0 \\ ab' + bc' = 0 \end{cases}$$
\[
\begin{align*}
\sigma, \delta
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
 aa' = 0 \\
 bb' = 0 \\
 a(-b') + bc' = 0
\end{cases}
\end{align*}
\]

(2) \( R \) is not reduced, because \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0 \) and \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \). Let \( m = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). Then, \( R \) is not \( \sigma \)-skew McCoy. To see this, simply observe that for \( m = \left( \begin{array}{ll}
1 & 0 \\
0 & 0
\end{array} \right) + \left( \begin{array}{ll}
1 & 1 \\
0 & 0
\end{array} \right) x \), \( f = \left( \begin{array}{ll}
0 & 0 \\
0 & 1
\end{array} \right) + \left( \begin{array}{ll}
0 & 1 \\
0 & 0
\end{array} \right) x \in R[x; \sigma] \), we have \( mf = 0 \). But \( mr \neq 0 \) for any nonzero \( r \in R \).

**Corollary 2.11.** \( \sigma \)-reduced modules are \( \sigma \)-skew McCoy.

**Proof.** Clearly from the fact that \( \sigma \)-reduced modules are \( \sigma \)-compatible and reduced. \( \square \)

From Example 2.12, we see that the converse of Corollary 2.11 need not be true.

**Example 2.12.** Consider a ring of polynomials over \( \mathbb{Z}_2 \), \( R = \mathbb{Z}_2[x] \). Let \( \sigma: R \to R \) be an endomorphism defined by \( \sigma(f(x)) = f(0) \). Then, \( R \) is \( \sigma \)-skew McCoy, because it is \( \sigma \)-skew Armendariz by [11, Example 5]. Moreover, \( R \) is not \( \sigma \)-compatible. In fact, let \( f = 1 + x \), \( g = x \in R \), we have \( fg = (1 + x)x \neq 0 \), however \( f\sigma(g) = (1 + x)\sigma(x) = 0 \).

**Definition 2.13.** Let \( M_R \) be a module and \( \sigma \) an endomorphism of \( R \). We say that \( M_R \) satisfies the condition \((C_\sigma)\) if whenever \( m\sigma(a) = 0 \) with \( m \in M \) and \( a \in R \), then \( ma = 0 \).

**Lemma 2.14.** If \( M[x; \sigma, \delta]_{R[x; \sigma, \delta]} \) is semicommutative with the condition \((C_\sigma)\), then \( M_R \) is \((\sigma, \delta)\)-compatible.

**Proof.** Let \( m \in M \) and \( a \in R \). It suffices to verify that \( ma = 0 \) implies \( ma = 0 \) and \( ma = 0 \). Indeed, if \( ma = 0 \) then \( m\sigma(a)x + m\delta(a) = 0 \), which gives \( m\sigma(a) = 0 \) and \( m\delta(a) = 0 \). \( \square \)

**Lemma 2.15.** Let \( M_R \) be an \((\sigma, \delta)\)-compatible module, if \( ma^2 = 0 \) implies \( ma = 0 \) for any \( m \in M \) and \( a \in R \). Then

\[ m\sigma(a)a = 0 \implies ma = 0 \]

\[ ma = 0 \implies ma = 0. \]

**Proof.** The proof is straightforward. \( \square \)

**Proposition 2.16.** Let \( M_R \) be a reduced \((\sigma, \delta)\)-compatible module. For \( m(x) = \sum_{i=0}^{p} m_i x^i \in M[x; \sigma, \delta] \) and \( f(x) = \sum_{j=0}^{q} a_j x^j \in R[x; \sigma, \delta] \). If \( m(x)f(x) = 0 \), then \( m_i a_j = 0 \) for all \( i \) and \( j \).
Proof. We will use freely the fact that, if $ma = 0$ then $m\sigma^i(a) = m\delta^j(a) = 0$ for any nonnegative integers $i, j$ with $j \geq i$. From $m(x)f(x) = 0$, we have the following system of equations:

(0) \[ m_p\sigma^p(a_q) = 0, \]

(1) \[ m_p\sigma^p(a_{q-1}) + m_{p-1}\sigma^{p-1}(a_q) + m_pf_{p-1}^p(a_q) = 0, \]

(2) \[ m_p\sigma^p(a_{q-2}) + m_{p-1}\sigma^{p-1}(a_{q-1}) + m_pf_{p-2}^p(a_{q-1}) + m_{p-2}\sigma^{p-2}(a_q) + m_{p-1}f_{p-2}^{p-1}(a_q) + m_pf_{p-2}^p(a_q) = 0, \]

(3) \[ m_p\sigma^p(a_{q-3}) + m_{p-1}\sigma^{p-1}(a_{q-2}) + m_pf_{p-3}^p(a_{q-2}) + m_{p-2}\sigma^{p-2}(a_{q-1}) + m_{p-1}f_{p-2}^{p-1}(a_{q-1}) + m_{p-3}\sigma^{p-3}(a_q) + m_{p-2}f_{p-3}^{p-2}(a_q) + m_{p-1}f_{p-3}^{p-1}(a_q) + m_pf_{p-3}^p(a_q) = 0, \]

\[ \vdots \]

(\ell) \[ \sum_{j+k=\ell} \sum_{i=0}^p \sum_{k=0}^q (m_i \sum_{j=0}^i f_j^i(a_k)) = 0, \]

\[ \vdots \]

(p + q) \[ \sum_{i=0}^p m_i\delta^i(a_0) = 0. \]

From equation (0), we have $m_p a_q = 0$ by $\sigma$-compatibility. Multiplying equation (1) on the right side by $a_q$, we get

(1') \[ m_p\sigma^p(a_{q-1})a_q + m_{p-1}\sigma^{p-1}(a_q)a_q + m_pf_{p-1}^p(a_q)a_q = 0, \]

Since $M_R$ is semicommutative, we have

\[ m_p a_q = 0 \Rightarrow m_p\sigma^p(a_{q-1})a_q = m_pf_{p-1}^p(a_q)a_q = 0. \]

By Lemma 2.15, equation (1') gives $m_{p-1}a_q = 0$. Also, by $(\sigma, \delta)$-compatibility, equation (1) implies $m_p\sigma^p(a_{q-1}) = 0$, because $m_p a_q = m_{p-1}a_q = 0$. Thus $m_p a_{q-1} = 0$.

Summarizing at this point, we have

(\alpha) \[ m_p a_q = m_{p-1}a_q = m_p a_{q-1} = 0 \]

Now, multiplying equation (2) on the right side by $a_q$, we get

(2') \[ m_p\sigma^p(a_{q-2})a_q + m_{p-1}\sigma^{p-1}(a_{q-1})a_q + m_pf_{p-1}^p(a_{q-1})a_q + m_{p-2}\sigma^{p-2}(a_q)a_q + m_{p-1}f_{p-2}^{p-1}(a_q)a_q + m_pf_{p-2}^p(a_q)a_q = 0, \]
With the same manner as above, equation (2') gives 
\[ m_{p-2}\sigma^{p-2}(a_q)a_q = 0 \]
and thus \( m_{p-2}a_q = 0 \) \((\beta)\). Also, multiplying equation (2) on the right side by \( a_{q-1} \), we get
\begin{align*}
(2'') \quad m_p\sigma^p(a_q-2)a_{q-1} + m_{p-1}\sigma^{p-1}(a_q-1)a_{q-1} + m_p f_{p-1}^p(a_q-1)a_{q-1} \\
+ m_{p-2}\sigma^{p-2}(a_q)a_{q-1} + m_{p-1} f_{p-2}^p(a_q)a_{q-1} + m_p f_{p-2}^p(a_q)a_{q-1} = 0
\end{align*}
Equations (\(\alpha\)) and (\(\beta\)) implies
\[ 0 = m_p\sigma^p(a_q-2)a_{q-1} = m_p f_{p-1}^p(a_q-1)a_{q-1} = m_{p-2}\sigma^{p-2}(a_q)a_{q-1} \\
= m_{p-1} f_{p-2}^p(a_q)a_{q-1} = m_p f_{p-2}^p(a_q)a_{q-1}
\]
Hence, equation (2'') gives \( m_{p-1}\sigma^{p-1}(a_q-1)a_{q-1} = 0 \) and by Lemma 2.15, we get \( m_{p-1}a_{q-1} = 0 \) \((\gamma)\). Now, from (\(\alpha\)), (\(\beta\)) and (\(\gamma\)), we get
\[ m_{p-1}\sigma^{p-1}(a_q-1) = m_p f_{p-1}^p(a_q-1) = m_{p-2}\sigma^{p-2}(a_q) = m_{p-1} f_{p-2}^p(a_q) = m_p f_{p-2}^p(a_q) = 0.
\]
Hence, equation (2) implies \( m_p\sigma^p(a_{q-2}) = 0 \), so that \( m_p^2a_{q-2} = 0 \). Summarizing at this point, we have \( m_ia_j = 0 \) with \( i + j \in \{p + q, p + q - 1, p + q - 2\} \).

Continuing this procedure yields \( m_ia_j = 0 \) for all \( i, j \).

**Corollary 2.17** ([1, Theorem 2.19]). If \( M_R \) is a reduced \((\sigma, \delta)\)-compatible module, then it is \((\sigma, \delta)\)-skew Armendariz.

**Proof.** Clearly from Proposition 2.16.

**Remark 2.18.** Note that, Corollary 2.8 can be obtained as a corollary of Proposition 2.16. Indeed, for \( m(x) = \sum_{i=0}^{p} m_i x^i \in M[x; \sigma, \delta] \) and \( f(x) = \sum_{j=0}^{q} a_j x^j \in R[x; \sigma, \delta] \setminus \{0\} \) such that \( m(x)f(x) = 0 \), from Proposition 2.16 and [3, Lemma 2.1], we have \( m(x)a_j = 0 \) for all \( j \).

Since \( f(x) \neq 0 \), there exists \( j_0 \in \{0, 1, \cdots, q\} \) such that \( a_{j_0} \neq 0 \), and hence \( M_R \) is \((\sigma, \delta)\)-skew McCoy.

Let \( M_R \) be a module and \((\sigma, \delta)\) a quasi derivation of \( R \). We say that \( M_R \) satisfies the condition \((*)\), if for any \( m(x) \in M[x; \sigma, \delta] \) and \( f(x) \in R[x; \sigma, \delta], m(x)f(x) = 0 \) implies \( m(x)f(x) = 0 \). If \( M[x; \sigma, \delta] \) is semicommutative as a right \( R[x; \sigma, \delta] \)-module, then we have the property \((*)\). A module \( M_R \) which satisfies the condition \((*)\) is semicommutative. But the converse need not be true, by the next example.

**Example 2.19.** Take the ring \( R = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \), with \((\sigma, \delta)\) as considered in Example 2.9. Since \( R \) is commutative, the module \( R_R \) is semicommutative. However, it does not satisfy the condition \((*)\). For \( p(x) = (1, 0)x \) and \( q(x) = (1, 1) + (1, 0)x \in R[x; \sigma, \delta] \). We have \( p(x)q(x) = 0 \), but \( p(x)(1, 0)q(x) = (1, 0) + (1, 0)x \neq 0 \). Thus \( p(x)Rq(x) \neq 0 \).
Proposition 2.20. Let $M_R$ be an $(\sigma, \delta)$-compatible module which satisfies $(\ast)$. If for any $m(x) = \sum_{i=0}^{p} m_i x^i \in M[x; \sigma, \delta]$ and $f(x) = \sum_{j=0}^{q} a_j x^j \in R[x; \sigma, \delta] \setminus \{0\}$, $m(x)f(x) = 0$. Then $m_i a_q^{p+1} = 0$ for all $i = 0, 1, \cdots, p$.

Proof. Let $m(x) = \sum_{i=0}^{p} m_i x^i \in M[x; \sigma, \delta]$ and $f(x) = \sum_{j=0}^{q} a_j x^j \in R[x; \sigma, \delta] \setminus \{0\}$, such that $m(x)f(x) = 0$. We can suppose $a_q \neq 0$. From $m(x)f(x) = 0$, we get $m_p \sigma^p(a_q) = 0$. Since $M_R$ is $(\sigma, \delta)$-compatible, we have $m_p a_q = 0$ which implies $m_p x^p a_q = 0$. Since $m(x)f(x) = 0$ implies $m(x)a_q f(x) = 0$. We have

$$0 = (m_p x^p + m_{p-1} x^{p-1} + \cdots + m_1 x + m_0)(a_q x^q + a_q a_{q-1} x^{q-1} + \cdots + a_q a_1 x + a_q a_0)$$

$$= (m_{p-1} x^{p-1} + \cdots + m_1 x + m_0)(a_q x^q + a_q a_{q-1} x^{q-1} + \cdots + a_q a_1 x + a_q a_0).$$

If we put $f'(x) = a_q f(x)$ and $m'(x) = \sum_{i=0}^{p-1} m_i x^i$, then we get $m_{p-1} a_q^2 = 0$. Continuing this procedure yields $m_i a_q^{p+1-i} = 0$ for all $i = 0, 1, \cdots, p$. Consequently $m_i a_q^{p+1} = 0$ for all $i = 0, 1, \cdots, p$. \qed

Corollary 2.21. Let $R$ be a reduced ring. Then

1. If $M_R$ is $(\sigma, \delta)$-compatible satisfying $(\ast)$, then $M_R$ is $(\sigma, \delta)$-skew McCoy.

2. If $M[x; \sigma, \delta]_{R[x; \sigma, \delta]}$ is semicommutative satisfying $(C_\sigma)$, then $M_R$ is $(\sigma, \delta)$-skew McCoy.

Proof. (1) Let $m(x) = \sum_{i=0}^{p} m_i x^i \in M[x; \sigma, \delta]$ and $f(x) = \sum_{j=0}^{q} a_j x^j \in R[x; \sigma, \delta] \setminus \{0\}$, such that $m(x)f(x) = 0$. We can suppose $a_q \neq 0$. By Proposition 2.20, we have $m_i a_q^{p+1} = 0$ for all $i = 0, 1, \cdots, p$. Since $M_R$ is $(\sigma, \delta)$-compatible, we get $m_i x^i a_q^{p+1} = m_i \sum_{\ell=0}^{p} f_\ell(a_q^{p+1}) x^\ell = 0$ for all $i$. Hence $m(x) a_q^{p+1} = 0$ where $a_q^{p+1} \neq 0$, because $R$ is reduced. Consequently $M_R$ is $(\sigma, \delta)$-skew McCoy.

(2) Obvious from (1) and Lemma 2.14. \qed

3. $(\sigma, \delta)$-skew McCoyness of some matrix extensions

This section is devoted to presenting many results on $(\sigma, \delta)$-skew McCoyness of some matrix extensions. At first, we define skew triangular matrix modules $V_n(M, \sigma)$, based on the definition of skew triangular
matrix rings $V_n(R, \sigma)$ given by Isfahani [19]. Let $\sigma$ be an endomorphism of a ring $R$ and $M_R$ a right $R$-module. For $n \geq 2$. Consider

$$V_n(R, \sigma) := \left\{ \begin{pmatrix} a_0 & a_1 & a_2 & \ldots & a_{n-1} \\ 0 & a_0 & a_1 & \ldots & a_{n-2} \\ 0 & 0 & a_0 & a_1 & \ldots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & a_1 \\ 0 & 0 & 0 & 0 & \ldots & a_0 \end{pmatrix} \right| a_0, a_2, \ldots, a_{n-1} \in R \right\}$$

and

$$V_n(M, \sigma) := \left\{ \begin{pmatrix} m_0 & m_1 & m_2 & m_3 & \ldots & m_{n-1} \\ 0 & m_0 & m_1 & m_2 & \ldots & m_{n-2} \\ 0 & 0 & m_0 & m_1 & \ldots & m_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & m_1 \\ 0 & 0 & 0 & 0 & \ldots & m_0 \end{pmatrix} \right| m_0, m_2, \ldots, m_{n-1} \in M \right\}$$

Clearly $V_n(M, \sigma)$ is a right $V_n(R, \sigma)$-module under the usual matrix addition operation and the following scalar product operation.

$$\begin{pmatrix} m_0 & m_1 & m_2 & m_3 & \ldots & m_{n-1} \\ 0 & m_0 & m_1 & m_2 & \ldots & m_{n-2} \\ 0 & 0 & m_0 & m_1 & \ldots & m_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & m_1 \\ 0 & 0 & 0 & 0 & \ldots & m_0 \end{pmatrix} \cdot \begin{pmatrix} a_0 & a_1 & a_2 & \ldots & a_{n-1} \\ 0 & a_0 & a_1 & \ldots & a_{n-2} \\ 0 & 0 & a_0 & a_1 & \ldots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & a_1 \\ 0 & 0 & 0 & 0 & \ldots & a_0 \end{pmatrix} = \begin{pmatrix} c_0 & c_1 & c_2 & c_3 & \ldots & c_{n-1} \\ 0 & c_0 & c_1 & c_2 & \ldots & c_{n-2} \\ 0 & 0 & c_0 & c_1 & \ldots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & c_1 \\ 0 & 0 & 0 & 0 & \ldots & c_0 \end{pmatrix}$$

where

$$c_i = m_0\sigma^0(a_i) + m_1\sigma^1(a_{i-1}) + m_2\sigma^2(a_{i-2}) + \cdots + m_i\sigma^i(a_0)$$

for each $0 \leq i \leq n-1$.

We denote elements of $V_n(R, \sigma)$ by $(a_0, a_1, \ldots, a_{n-1})$, and elements of $V_n(M, \sigma)$ by $(m_0, m_1, \ldots, m_{n-1})$. There is a ring isomorphism

$$\varphi: R[x, \sigma]/(x^n) \to V_n(R, \sigma)$$
given by $\varphi(a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + (x^n)) = (a_0, a_1, a_2, \ldots, a_{n-1})$, and an abelian group isomorphism

$$\phi: M[x, \sigma]/M[x, \sigma](x^n) \to V_n(M, \sigma)$$
Then $mV$ if and only if $V_n(M, \sigma)$ is $\sigma$-skew McCoy as an $V_n(R, \sigma)$-module for any nonnegative integer $n \geq 2$.

Proof. Note that

$$V_n(R, \sigma)[x, \sigma] \cong V_n(R[x, \sigma], \sigma) \text{ and } V_n(M, \sigma)[x, \sigma] \cong V_n(M[x, \sigma], \sigma)$$

. We only prove when $n = 2$, because other cases can be proved with the same manner. Suppose $M_R$ is $\sigma$-skew McCoy. Let $0 \neq m(x) \in V_2(M, \sigma)[x, \sigma]$ and $0 \neq f(x) \in V_2(R, \sigma)[x, \sigma]$ such that $m(x)f(x) = 0$, where

$$m(x) = \sum_{i=0}^{p} \left( \begin{array}{cc} m_{11}^{(i)} & m_{12}^{(i)} \\ 0 & m_{11} \end{array} \right) x^i = \left( \begin{array}{c} \sum_{i=0}^{p} m_{11}^{(i)} x^i \\ \sum_{i=0}^{p} m_{12}^{(i)} x^i \end{array} \right)$$

$$f(x) = \sum_{j=0}^{q} \left( \begin{array}{cc} a_{11}^{(j)} & a_{12}^{(j)} \\ 0 & a_{11} \end{array} \right) x^j = \left( \begin{array}{c} \sum_{j=0}^{q} a_{11}^{(j)} x^j \\ \sum_{j=0}^{q} a_{12}^{(j)} x^j \end{array} \right)$$

Then

$$\left( \begin{array}{cc} \alpha_{11} & \alpha_{12} \\ 0 & \alpha_{11} \end{array} \right) \left( \begin{array}{cc} \beta_{11} & \beta_{12} \\ 0 & \beta_{11} \end{array} \right) = 0,$$

which gives $\alpha_{11}\beta_{11} = 0$ and $\alpha_{11}\beta_{12} + \alpha_{12}\beta_{11} = 0$ in $M[x; \sigma]$. If $\alpha_{11} \neq 0$, then there exists $0 \neq \beta \in \{\beta_{11}, \beta_{12}\}$ such that $\alpha_{11}\beta = 0$. Since $M_R$ is $\sigma$-skew McCoy, there exists $0 \neq c \in R$ which satisfies $\alpha_{11}c = 0$, thus

$$\left( \begin{array}{cc} \alpha_{11} & \alpha_{12} \\ 0 & \alpha_{11} \end{array} \right) \left( \begin{array}{cc} 0 & c \\ 0 & 0 \end{array} \right) = \left( \begin{array}{cc} 0 & \alpha_{11}c \\ 0 & 0 \end{array} \right) = 0.$$

If $\alpha_{11} = 0$ then

$$\left( \begin{array}{cc} 0 & \alpha_{12} \\ 0 & 0 \end{array} \right) \left( \begin{array}{cc} 0 & c \\ 0 & 0 \end{array} \right) = 0, \text{ for any } 0 \neq c \in R. \text{ Therefore,}$$

$V_2(M, \sigma)$ is $\sigma$-skew McCoy.

Conversely, suppose $V_2(M, \sigma)$ is a $\sigma$-skew McCoy module. Let $0 \neq m(x) = m_0 + m_1 x + \cdots + m_p x^p \in M[x; \sigma]$ and $0 \neq f(x) = a_0 + a_1 x + \cdots + a_q x^q \in R[x; \sigma]$, such that $m(x)f(x) = 0$. Then

$$\left( \begin{array}{cc} m(x) & 0 \\ 0 & m(x) \end{array} \right) \left( \begin{array}{cc} f(x) & 0 \\ 0 & f(x) \end{array} \right) = \left( \begin{array}{cc} m(x)f(x) & 0 \\ 0 & m(x)f(x) \end{array} \right) = 0.$$
So there exists $0 \neq \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in V_2(R, \sigma)$ such that
\[
\begin{pmatrix} m(x) & 0 \\ 0 & m(x) \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = 0
\]
because $V_2(M, \sigma)$ is $\overline{\sigma}$-skew McCoy. Thus $m(x)a = m(x)b = 0$, where $a \neq 0$ or $b \neq 0$. Therefore, $M_R$ is $\sigma$-skew McCoy.

\begin{corollary}
For a nonnegative integer $n \geq 2$, we have:
\begin{enumerate}
\item $M_R$ is $\sigma$-skew McCoy if and only if $M[x; \sigma]/M[x; \sigma](x^n)$ is $\overline{\sigma}$-skew McCoy.
\item $R$ is $\sigma$-skew McCoy if and only if $R[x; \sigma]/(x^n)$ is $\overline{\sigma}$-skew McCoy.
\item $M_R$ is McCoy if and only if $M[x]/M[x](x^n)$ is McCoy.
\item $R$ is McCoy if and only if $R[x]/(x^n)$ is McCoy.
\end{enumerate}
\end{corollary}

For a nonnegative integer $n \geq 2$, let $R$ be a ring and $M$ a right $R$-module.

Consider
\[
S_n(R) := \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \ldots & a_{1n} \\ 0 & a & a_{23} & \ldots & a_{2n} \\ 0 & 0 & a & \ldots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & a \end{pmatrix} \mid a, a_{ij} \in R \right\}
\]
and
\[
S_n(M) := \left\{ \begin{pmatrix} m & m_{12} & m_{13} & \ldots & m_{1n} \\ 0 & m & m_{23} & \ldots & m_{2n} \\ 0 & 0 & m & \ldots & m_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & m \end{pmatrix} \mid m, m_{ij} \in M \right\}
\]

Clearly, $S_n(M)$ is a right $S_n(R)$-module under the usual matrix addition operation and the following scalar product operation. For $U = (u_{ij}) \in S_n(M)$ and $A = (a_{ij}) \in S_n(R)$, $UA = (m_{ij}) \in S_n(M)$ with $m_{ij} = \sum_{k=1}^{n} u_{ik}a_{kj}$ for all $i, j$. A quasi derivation $(\sigma, \delta)$ of $R$ can be extended to a quasi derivation $(\overline{\sigma}, \overline{\delta})$ of $S_n(R)$ as follows: $\overline{\sigma}((a_{ij})) = (\sigma(a_{ij}))$ and $\overline{\delta}((a_{ij})) = (\delta(a_{ij}))$. We can easily verify that $\overline{\delta}$ is a $\overline{\sigma}$-derivation of $S_n(R)$.

\begin{theorem}
A module $M_R$ is $(\sigma, \delta)$-skew McCoy if and only if $S_n(M)$ is $(\overline{\sigma}, \overline{\delta})$-skew McCoy as an $S_n(R)$-module for any nonnegative integer $n \geq 2$.
\end{theorem}

\begin{proof}
The proof is similar to [5, Theorem 14].
\end{proof}

Now, for $n \geq 2$. Consider
\[ V_n(R) := \begin{cases} 
\left( \begin{array}{cccc}
 a_0 & a_1 & a_2 & a_3 & \cdots & a_{n-1} \\
 0 & a_0 & a_1 & a_2 & \cdots & a_{n-2} \\
 0 & 0 & a_0 & a_1 & \cdots & a_{n-3} \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & 0 & 0 & \cdots & a_1 \\
 0 & 0 & 0 & 0 & \cdots & a_0 
\end{array} \right) 
| a_0, a_1, a_2, \cdots, a_{n-1} \in R 
\end{cases} \]

and

\[ V_n(M) := \begin{cases} 
\left( \begin{array}{cccc}
 m_0 & m_1 & m_2 & m_3 & \cdots & m_{n-1} \\
 0 & m_0 & m_1 & m_2 & \cdots & m_{n-2} \\
 0 & 0 & m_0 & m_1 & \cdots & m_{n-3} \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & 0 & 0 & \cdots & m_1 \\
 0 & 0 & 0 & 0 & \cdots & m_0 
\end{array} \right) 
| m_0, m_1, m_2, \cdots, m_{n-1} \in M 
\end{cases} \]

With the same method as above, \( V_n(M) \) is a right \( V_n(R) \)-module, and a quasi derivation \((\sigma, \delta)\) of \( R \) can be extended to a quasi derivation \((\overline{\sigma}, \overline{\delta})\) of \( V_n(R) \). Note that \( V_n(M) \cong M[x]/M[x](x^n) \) where \( M[x](x^n) \) is a submodule of \( M[x] \) generated by \( x^n \) and \( V_n(R) \cong R[x]/(x^n) \) where \( (x^n) \) is an ideal of \( R[x] \) generated by \( x^n \).

**Proposition 3.4.** A module \( M_R \) is \((\sigma, \delta)\)-skew McCoy if and only if \( V_n(M) \) is \((\overline{\sigma}, \overline{\delta})\)-skew McCoy as an \( V_n(R) \)-module for any nonnegative integer \( n \geq 2 \).

**Proof.** The proof is similar to that of [5, Theorem 14] or [9, Proposition 2.27]. \( \square \)

**Corollary 3.5.** For a nonnegative integer \( n \geq 2 \), we have:

1. \( M_R \) is \((\sigma, \delta)\)-skew McCoy if and only if \( M[x]/M[x](x^n) \) is \((\overline{\sigma}, \overline{\delta})\)-skew McCoy.
2. \( R \) is \((\sigma, \delta)\)-skew McCoy if and only if \( R[x]/(x^n) \) is \((\overline{\sigma}, \overline{\delta})\)-skew McCoy.
3. \( R \) is McCoy if and only if \( R[x]/(x^n) \) is McCoy.

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