Journal of Algebra and related topics
Vol. 1, No 1, (2013), pp 19-30

MODULES WITH NOETHERIAN SECOND SPECTRUM

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ABSTRACT. Let R be a commutative ring and let M be an R-module. In this article, we introduce the concept of the Zariski socles of submodules of M and investigate their properties. Also we study modules with Noetherian second spectrum and obtain some related results.

1. INTRODUCTION

Throughout this paper, R will denote a commutative ring with identity and $\mathbb{Z}$ will denote the ring of integers. We write $N \leq M$ to indicate that N is a submodule of M. Also, $Spec(R)$ will denote the set of all prime ideals of R and $V(I) = \{P \in Spec(R) : I \subseteq P\}$ for any ideal I of R.

Let M be an R-module. A proper submodule P of M is said to be *prime* if for any $r \in R$ and $m \in M$ with $rm \in P$, we have $m \in P$ or $r \in (P :_R M)$. A non-zero submodule S of M is said to be *second* if for each $a \in R$, the endomorphism $S \xrightarrow{a} S$ is either surjective or zero [13]. The *second socle* of M is defined to be the sum of all second submodules of M and denoted by $soc(M)$. If M has no second submodule, then $soc(M)$ is defined to be 0. Also, a submodule N of M is said to be a *socle submodule* of M if $soc(N) = N$ [5] and [6].

Set $X^s := Spec^s(M) = \{S \subseteq M : S \textit{ is a second submodule of } M\}$. We call this set the *second spectrum* of M. For any submodule N of M, $V^{s*}(N)$ is defined to be the set of all second submodules of M contained in N. Of course, $V^{s*}(0)$ is just the empty set and $V^{s*}(M)$ is $Spec^s(M)$.

MSC(2010): 13C13
Keywords: Second submodule, second spectrum, Zariski socle, Noetherian spectrum.
Received: 19 August 2013, Accepted: 10 October 2013.

It is easy to see that for any family of submodules N_i ($i \in I$) of M, $\cap_{i\in I}V^{s*}(N_i) = V^{s*}(\cap_{i\in I}N_i)$. Thus if $\zeta^{s*}(M)$ denotes the collection of all subsets $V^{s*}(N)$ of $Spec^s(M)$, then $\zeta^{s*}(M)$ contains the empty set and $Spec^s(M)$, and $\zeta^{s*}(M)$ is closed under arbitrary intersections. In general $\zeta^{s*}(M)$ is not closed under finite unions. A module M is called a *cotop module* if $\zeta^{s*}(M)$ is closed under finite unions. In this case, $\zeta^{s*}(M)$ is called the *quasi Zariski topology* [4].

Let N be a submodule of M and $V^s(N) = \{S \in Spec^s(M) : Ann_R(N) \subseteq Ann_R(S)\}$. Then

(i) $V^s(M) = Spec^s(M)$ and $V^s(0) = \emptyset$,
(ii) $\cap_{i\in\Lambda}V^s(N_i) = V^s(\cap_{i\in\Lambda}(0 :_M Ann_R(N_i)))$ for any index set Λ,
(iii) $V^s(N) \cup V^s(K) = V^s(N + K)$, where $N, K, N_i \leq M$.

Set $\zeta^s(M) := \{V^s(N) : N \leq M\}$. Then from (i), (ii), and (iii) we see that always there exists a topology, τ^s say, on $Spec^s(M)$ having $\zeta^s(M)$ as the family of all closed sets. We call the topology τ^s the *Zariski topology on* $Spec^s(M)$[4].

There are some interesting results concerning the primeful submodules, Zariski radicals, and modules with Noetherian spectrum (for example see, [11], and [10]). It is natural to ask that to what extent the dual of these results hold. The purpose of this paper is to investigate this question and obtain some related results.

In the rest of this paper, for an R-module M and for an ideal of R, $\overline{R}$ and $\overline{I}$ will denote respectively $R/Ann_R(M)$ and $I/Ann_R(M)$.

2. Secondful R-modules

Definition 2.1. We say that an R-module M is <u>secondful</u> if the natural map $\psi^s : Spec^s(M) \to Spec(\overline{R})$ defined by $S \mapsto \overline{Ann_R(S)}$ is surjective.

Example 2.2. (a) By [4, 3.10], every finite length R-module is secondful.
(b) For each prime integer p the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is not secondful [4, 3.9].
(c) Let R be an integral domain. Then for every Artinian cotorsion R-module, $Hom_R(F, M) = 0$, where F denotes the quotient field of R by [5, 2.9]. Hence by [4, 3.8], every Artinian faithful cotorsion R-module is not secondful.

Part (c) of the following proposition is analogue of dual of Nakayama's Lemma [3, 3.14].

Proposition 2.3. Let M be a secondful R-module. Then we have the following.

(a) If $M \neq 0$ and I is a radical ideal of R, then $Ann_R((0:_M I)) = I$ if and only if $Ann_R(M) \subseteq I$.
(b) If P is a maximal ideal of R such that $(0:_M P) = 0$, then there exists $a \in P$ such that $(1+a)M = 0$.
(c) If I is an ideal of R contained in the Jacobson radical $J(R)$ such that $(0:_M I) = 0$, then $M = 0$.

Proof. (a) The necessity is clear. To prove the sufficiency, we note that $Ann_R(M) \subseteq I = \sqrt{I} = \cap_{P_i \in V(I)} P_i$. Thus for each $P_i \in V(I)$, $Ann_R(M) \subseteq P_i$. As M is secondful, there exist second submodules S_i of M such that $Ann_R(S_i) = P_i$ for each $P_i \in V(I)$. Therefore, $Ann_R((0:_M P_i)) = Ann_R((0:_M Ann_R(S_i))) = P_i$ for each $P_i \in V(I)$. Thus we have

$$Ann_R((0:_M I)) = Ann_R((0:_M \bigcap_{P_i \in V(I)} P_i)) \subseteq \bigcap_{P_i \in V(I)} Ann_R((0:_M P_i))$$

$$= \bigcap_{P_i \in V(I)} P_i = \sqrt{I} = I.$$

Hence $Ann_R((0:_M I)) = I$ because the reverse inclusion is clear.

(b) If $Ann_R(M) \subseteq P$, then by part (a), $Ann_R((0:_M P)) = P$, a contradiction. Therefore, $Ann_R(M) + P = R$. Hence there exists $a \in P$ such that $(1+a)M = 0$.

(c) Suppose that $M \neq 0$. Then $Ann_R(M) \neq R$. Let m be a maximal ideal of R such that $Ann_R(M) \subseteq m$. Now $I \subseteq m$ and $(0:_M m) \subseteq (0:_M I) = 0$. Thus $R = Ann_R((0:_M m))$. But by part (a), $Ann_R((0:_M m)) = m$, a contradiction. □

Let M be an R-module. A proper submodule N of M is said to be *completely irreducible* if $N = \bigcap_{i \in I} N_i$, where $\{N_i\}_{i \in I}$ is a family of submodules of M, implies that $N = N_i$ for some $i \in I$. It is easy to see that every submodule of M is an intersection of completely irreducible submodules of M [9, page 2]. Thus, the intersection of all completely irreducible submodules of M is zero.

Let $P \in Spec(R)$ and N be a submodule of M. *The P-interior of N relative to M* is defined [2, 2.7] as the set

$$I_P^M(N) = \cap\{L \mid L \text{ is a completely irreducible submodule of } M \text{ and } rN \subseteq L \text{ for some } r \in R - P\}.$$

If R is an integral domain, M is said to be *cotorsion* if $I_0^M(M) = 0$ [5].

An R-module L is said to be *cocyclic* if L is a submodule of $E(R/m)$ for some maximal ideal m of R, where $E(R/m)$ is the injective envelope of R/m [15].

The *cosupport* of an R-module M [14] is denoted by $Cosupp(M)$ and it is defined by

$$Cosupp(M) = \{P \in Spec(R) | P \supseteq Ann_R(L) \text{ } for \text{ } some \text{ } cocyclic \\ homomorphic \text{ } image \text{ } L \text{ } of \text{ } M\}.$$

Lemma 2.4. A submodule L of an R-module M is completely irreducible if and only if M/L is a cocyclic module [9, 1.1].

Theorem 2.5. *(cf. [14, 2.3]) Let M be a secondful R-module, then $Cosupp(M) = V(Ann_R(M))$. However, the converse is not true in general.*

Proof. If $M = 0$, then $Cosupp(M) = V(Ann_R(M)) = \emptyset$. So suppose that $M \neq 0$. Then $V(Ann_R(M)) \neq \emptyset$. Let $P \in V(Ann_R(M))$. Then as M is secondful, there exists a second submodule S of M such that $Ann_R(S) = P$. Thus $S = I^M_P(S)$ by [5, 2.10]. Since S is second, $S \neq 0$. Therefore, $0 \neq S = I^M_P(S) \subseteq I^M_P(M)$. Thus there exists a completely irreducible submodule L of M such that $I^M_P(M) \not\subseteq L$ because the intersection of all completely irreducible submodules of M is zero. Hence for each $r \in R - P$, $rM \not\subseteq L$. This implies that $Ann_R(M/L) \subseteq P$ and so $P \in Cosupp(M)$ by Lemma 2.4. Thus $V(Ann_R(M)) \subseteq Cosupp(M)$. The reverse inclusion is always true. As a counterexample to the converse of the first statement take the $\mathbb{Z}$-module $\mathbb{Q}$. Then since $Spec^s(\mathbb{Q}) = \{\mathbb{Q}\}$, $\mathbb{Q}$ is not secondful. But $V(Ann_{\mathbb{Z}}(\mathbb{Q})) = Spec(\mathbb{Z}) = Cosupp(\mathbb{Q})$. □

3. Zariski socles of submodules

Definition 3.1. Let M be an R-module. The *Zariski socle* of a submodule N of M, denoted by $Z.soc(N)$ or Z.socle of N, is the sum of all members of $V^s(N)$, that is,

$$Z.soc(N) = \sum\{S : \ S \in V^s(N)\} =$$

$$\sum\{S : \ Ann_R(N) \subseteq Ann_R(S), S \in Spec^s(M)\}.$$

If $V^s(N) = \emptyset$, then $Z.soc(N) = 0$. We say that a submodule N of M is a *Zariski socle submodule* (or a *Z.socle submodule*) if $N = Z.soc(N)$.

Let Y is a nonempty subset of $Spec^s(M)$. Then we write $T(Y)$ to denote the sum of the members of Y. Clearly, if Y_1 and Y_2 are subsets of $Spec^s(M)$, then $T(Y_1 \cup Y_2) = T(Y_1) + T(Y_2)$.

Lemma 3.2. Let M be an R-module.

(a) For $N \leq M$, $soc(N) = T(V^{s*}(N))$. If $V^{s*}(N) = \emptyset$, then $soc(N) = 0$.

(b) For $N \leq M$, $Z.soc(N) = T(V^s(N))$. If $V^s(N) = \emptyset$, then $Z.soc(N) = 0$.

(c) Let $N \leq M$ and let $Y \subseteq Spec^s(M)$. Then $V^s(T(Y)) = cl(Y)$, the closure of Y. Hence $V^s(T(V^s(N))) = V^s(Z.soc(N)) = V^s(N)$.

Proof. (a) and (b) are straightforward and (c) is proved in [4, 5.1].

Lemma 3.3. Let N be a submodule of an R-module M. Then we have the following.

(a) $V^s((0:_M I)) = V^s((0:_M \sqrt{I})) = V^{s*}((0:_M I)) = V^{s*}((0:_M \sqrt{I}))$.

(b) $V^s(N) = V^s((0:_M Ann_R(N))) = V^s((0:_M \sqrt{Ann_R(N)})) = V^{s*}((0:_M Ann_R(N))) = V^{s*}((0:_M \sqrt{Ann_R(N)}))$.

Proof. (a) Since $(0:_M \sqrt{I}) \subseteq (0:_M I)$, we have $V^{s*}((0:_M \sqrt{I})) \subseteq V^{s*}((0:_M I))$. Thus $V^{s*}((0:_M \sqrt{I})) \subseteq V^{s*}((0:_M I)) \subseteq V^s((0:_M I))$. Now we show that $V^s((0:_M I)) \subseteq V^{s*}((0:_M \sqrt{I}))$. So suppose that $S \in V^s((0:_M I))$.

$$I \subseteq Ann_R((0:_M I)) \subseteq Ann_R(S) \Rightarrow \sqrt{I} \subseteq \sqrt{Ann_R(S)} = Ann_R(S) \Rightarrow$$
$$S \subseteq ((0:_M Ann_R(S)) \subseteq ((0:_M \sqrt{I})) \Rightarrow S \in V^{s*}((0:_M \sqrt{I})).$$

So we have

$$V^{s*}((0:_M \sqrt{I})) \subseteq V^{s*}((0:_M I)) \subseteq V^s((0:_M I)) \subseteq V^{s*}((0:_M \sqrt{I})).$$

Hence

$$V^{s*}((0:_M \sqrt{I})) = V^{s*}((0:_M I)) = V^s((0:_M I)).$$

Now the claim follows from this by replacing I by $\sqrt{I}$.

(b) It is enough to show that $V^s(N) = V^s((0:_M Ann_R(N))$ by part (a). We have

$$S \in V^s(N) \Leftrightarrow Ann_R((0:_M Ann_R(N)) = Ann_R(N) \subseteq Ann_R(S)$$
$$\Leftrightarrow S \in V^s((0:_M Ann_R(N))).$$

□

An R-module M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = (0:_M I)$ [1].

Proposition 3.4. Let N and K be submodules of an R-module M, $S \in Spec^s(M)$, and I be an ideal of R. Then we have the following.

(a) If $S \in V^s(N)$, then $S \subseteq Z.soc(N)$.

(b) $soc(N) \subseteq Z.soc(N)$.

(c) $Z.soc((0:_M I)) = Z.soc((0:_M \sqrt{I})) = soc((0:_M I)) = soc((0:_M \sqrt{I}))$.

(d) $Z.soc(N) = Z.soc((0:_M Ann_R(N)) = Z.soc((0:_M \sqrt{Ann_R(N)}) =$

$$soc((0:_M Ann_R(N)) = soc((0:_M \sqrt{Ann_R(N)}).$$

(e) If M is a comultiplication module, then $Z.soc(N) = soc(N)$.

(f) If $Ann_R(N) \subseteq Ann_R(K)$, then $V^s(K) \subseteq V^s(N) \Leftrightarrow Z.soc(K) \subseteq Z.soc(N)$.

(g) If S is a second submodule of M, then $Ann_R(N) \subseteq Ann_R(S) \Leftrightarrow V^s(S) \subseteq V^s(N)$. Hence $Ann_R(N) = Ann_R(S) \Leftrightarrow V^s(N) = V^s(S) \Leftrightarrow Z.soc(N) = Z.soc(S)$.

Proof. The proof is straightforward by using Lemma 3.2 and Lemma 3.3. □

Theorem 3.5. *Let M ba an R-module and N, K be two submodules of M. Then we have the following.*

(a) $Z.soc(Z.soc(N)) = Z.soc(N$.

(b) $V^s(N + K) = V^s(N) \cup V^s(K)$.

(c) $Z.soc(N + K) = Z.soc(N) + Z.soc(K)$.

(d) *If M is secondful, then $N \neq 0 \Leftrightarrow V^s(N) \neq \emptyset \Leftrightarrow Z.soc(N) \neq 0$.*

(e) $\sqrt{Ann_R(N)} \subseteq Ann_R(Z.soc(N))$ *and hence* $Z.soc(N) \subseteq (0:_M \sqrt{Ann_R(N)})$. *If M is secondful, then we have the equality* $\sqrt{Ann_R(N)} = Ann_R(Z.soc(N))$.

Proof. (a) This follows from the Lemma 3.2.

(b) By [4, 3.2].

(c) This follows from part (b) because

$$Z.soc(N + K) = T(V^s(N + K)) = T(V^s(N) \cup V^s(K)) =$$

$$T(V^s(N)) + T(V^s(K)) = Z.soc(N) + Z.soc(K).$$

(d) Suppose that $N \neq 0$. Then $Ann_R(N) \neq R$ so that there exists a $P \in Spec(R)$ such that $Ann_R(N) \subseteq P$. Since $Ann_R(M) \subseteq P$ and M is secondful, there exists a $S \in Spec^s(M)$ with $P = Ann_R(S) \supseteq Ann_R(N)$. It follows that $S \in V^s(N)$ and hence $V^s(N) \neq \emptyset$. So we assume that $V^s(N) \neq \emptyset$ and that $S \in V^s(N)$. Then $Z.soc(N) = T(V^s(N)) \supseteq S \neq 0$. Also, $Z.soc(N) \neq 0$. This implies that $N \neq 0$ by Definition 3.1.

(e) We may assume that $V^s(N) \neq \emptyset$ (otherwise, $Z.soc(N) = 0$). Then

$$Ann_R(Z.soc(N)) = Ann_R(\sum_{S \in V^s(N)} S) = \bigcap_{S \in V^s(N)} Ann_R(S)$$

$$\supseteq \bigcap_{P\in V(Ann_R(N))} P = \sqrt{Ann_R(N)}.$$

Now let M be a secondful module and let $P \in V(Ann_R(N))$. Then $Ann_R(M) \subseteq Ann_R(N) \subseteq P$ implies that there exists a second submodule S with $Ann_R(N) \subseteq Ann_R(S) = P$. Therefore,

$$\sqrt{Ann_R(N)} = \bigcap_{P\in V(Ann_R(N))} P = \bigcap_{S\in V^s(N)} Ann_R(S) =$$

$$Ann_R(\sum_{S\in V^s(N)} S) = Ann_R(Z.soc(N)).$$

□

Proposition 3.6. Let N be a submodule of an R-module M. Then the following statements are equivalent.

(a) $Z.soc(N) \subseteq N$.
(b) $V^s(N) = V^{s*}(N)$.
(c) $Z.soc(N) = soc(N)$.

Hence we conclude that a Z.socle submodule is a socle submodule.

Proof. $(a) \Rightarrow (b)$. Clearly, $V^{s*}(N) \subseteq V^s(N)$. To see the reverse inclusion, let $S \in V^s(N)$. Then $S \subseteq Z.soc(N)$ so that $S \subseteq N$ by part (a). Hence $V^s(N) \subseteq V^{s*}(N)$ as required.

$(b) \Rightarrow (c)$ and $(c) \Rightarrow (a)$ are clear. To see the last assertion, let K be a Z.socle submodule. Then $K = Z.soc(K)$. Now the claim follows from $(a) \Rightarrow (c)$ and the proof is completed. □

Remark 3.7. Let M be an R-module.

Although the second socles and the Zariski socles of submodules of M have many similar properties, however, they are not the same. The following example shows this.

(a) Let M be a non-zero vector space. Then every nontrivial subspace N of M is second and hence a socle submodule. But N itself is not a *Z.socle* submodule. In fact $Z.soc(N) = M$. This shows that $Z.soc(N) \not\subseteq N$ in general.
(b) By Theorem 3.5,

$$Z.soc(N + K) = Z.soc(N) + Z.soc(K).$$

But in [6], the present authors showed that this relation is true for second socles of submodules under some restrictive conditions. We think this is not true in general and hence we posed as a question in [6].

4. Modules with Noetherian second spectrum

Throughout this section $\mathbb{Z}^+$ will denote the set of positive integer.

A topological space X is *Noetherian* provided that the open (respectively, closed) subset of X satisfy the ascending (respectively, descending) chain condition, or the maximal (respectively, minimal) condition [7] and [8].

Theorem 4.1. *An R-module M has Noetherian second spectrum if and only if the DCC for Zariski socle submodules of M hold.*

Proof. The proof follows from the following facts. (1) if L and K are two submodules of M, then $V^s(L) \supseteq V^s(K)$ if and only if $T(V^s(L)) \supseteq T(V^s(K))$. (2) N is a Zariski socle submodule if and only if $N = T(V^s(N))$ by Lemma 3.2. □

Remark 4.2. (See [4, 3.11].) Let M be an R-module and let $\psi^s : X^s \to Spec(\overline{R})$ be the natural map defined by $S \mapsto \overline{Ann_R(S)}$. If ψ^s is surjective, then ψ^s is both closed and open; more precisely, for every $N \leq M$, $\psi^s(V^s(N)) = V(\overline{Ann_R(N)})$ and $\psi^s(X^s - V^s(N)) = Spec(\overline{R}) - V(\overline{Ann_R(N)})$.

Theorem 4.3. *Let M be a secondful R-module. Then M has a Noetherian second spectrum if and only if the ring $\overline{R}$ has Noetherian spectrum.*

Proof. We may assume $M \neq 0$. Let M be an R-module and let $\psi^s : X^s \to Spec(\overline{R})$ be the natural map defined by $S \mapsto \overline{Ann_R(S)}$. Suppose $\overline{R}$ has Noetherian spectrum and let $V^s(N_1) \supseteq V^s(N_2) \supseteq ...$ be a descending chain of closed sets in X^s, where $N_i \leq M$. Since M is secondful, Then ψ^s is a closed mapping by Remark 4.2 and $\psi^s(V^s(N)) = V(\overline{Ann_R(N)})$ for every $N \leq M$. Hence $\psi^s(V^s(N_1)) \supseteq \psi^s(V^s(N_2)) \supseteq ...$ is a descending chain of closed sets in $\overline{R}$. So there exists $k \in \mathbb{Z}^+$ such that $\psi^s(V^s(N_k)) = \psi^s(V^s(N_{k+i})) \Leftrightarrow V(\overline{Ann_R(N_k)}) = V(\overline{Ann_R(N_{k+i})})$, where $i \geq 1$. Now

$$V^s(N_k) = \{S \in X^s : Ann_R(S) \supseteq Ann_R(N_k)\} =$$
$$\{S \in X^s : Ann_R(S) \in V(Ann_R(N_k))\} =$$
$$\{S \in X^s : Ann_R(S) \in V(Ann_R(N_{k+i}))\} = V^s(N_{k+i}),$$

for each $i \geq 1$.

The reverse implication proved similarly by using Remark 4.2 and surjectivity of ψ^s. □

Corollary 4.4. (a) Every Artinian R-module has Noetherian second spectrum.

(b) If R has Noetherian spectrum, then every secondful R-module has Noetherian second spectrum.
(c) Every Noetherian R-module has Noetherian second spectrum.
(d) Every secondful module over a commutative Noetherian ring has Noetherian second spectrum.

Proof. (a) This follows from Theorem 4.1.
(b) If R has Noetherian spectrum, so does $R/Ann_R(M)$ by [12, p. 632]. Thus the result follows from Theorem 4.3.
(c) Suppose that M is a Noetherian R-module. Then by [5, 2.3], the DCC for socle submodules of M hold. Now since every Zariski socle submodule is a socle submodule by Proposition 3.6, the DCC for Zariski socle submodules of M hold. Thus the result follows from the Theorem 4.1.
(d) This follows from [10, 3.4] and part (b). □

An R-module M is said to be a *weak comultiplication module* if M does not have any second submodule or for every second submodule S of M, $S = (0 :_M I)$, where I is an ideal of R [5].

Theorem 4.5. *If R has Noetherian spectrum and M is a non-zero secondful weak comultiplication R-module, then any Zariski socle submodule N of M is the sum of a finite number of second submodules of M.*

Proof. Since R has Noetherian spectrum, the ideal $I = Ann_R(N)$ has a finite number of minimal prime divisors $P_1, P_2, ..., P_k$, say, so that $\sqrt{I} = \sqrt{Ann_R(N)} = \cap_{i=1}^k P_i$ [10, 5.3]. Since M is secondful, for each i, there is a second submodule S_i of M such that $Ann_R(S_i) = P_i$. As M is a weak comultiplication R-module, we have $(0 :_M P_i) = S_i$ by [5, 3.3]. Thus for each i, $(0 :_M P_i)$ is a second submodule of M. Now by Proposition 3.4, we have

$$N = Z.soc(N) = soc((0 :_M \sqrt{Ann_R(N)})) = soc((0 :_M \cap_{i=1}^k P_i))$$
$$= soc(\sum_{i=1}^{k}(0 :_M P_i)) = \sum_{i=1}^{k}(0 :_M P_i).$$

Thus N is the sum of a finite number of second submodule. □

Corollary 4.6. Let R, M, and N be as in Theorem 4.5. Then N is a Zariski socle submodule of M if and only if N has a finite representation of the form $N = \sum_{i=1}^k (0 :_M P_i)$ for some $P_i \in V(Ann_R(N))$ and $k \in \mathbb{Z}^+$.

Proof. The necessity follows from Theorem 4.5. Conversely, suppose that $N = \sum_{i=1}^{k}(0 :_M P_i)$ for some $P_i \in V(Ann_R(N))$ and $k \in \mathbb{Z}^+$. Then by Theorem 3.5 (c)

$$Z.soc(N) = Z.soc(\sum_{i=1}^{k}(0 :_M P_i)) = \sum_{i=1}^{k} Z.soc((0 :_M P_i)) =$$
$$\sum_{i=1}^{k} soc((0 :_M P_i)) = N,$$

because as mentioned in the proof of Theorem 4.5, each $(0 :_M P_i)$ is a second submodule of M. □

Recall that an ideal I of R is an RFG-ideal if $\sqrt{I} = \sqrt{J}$ for some finitely generated ideal J of R [12].

Definition 4.7. We say that a submodule N of an R-module M is an *RFG^*-submodule* if $Z.soc(N) = Z.soc((0 :_M I))$ for some finitely generated ideal I of R. Also, we say that M has *property (RFG^*)* if every submodule of M is an RFG^*-submodule.

Example 4.8. If R is a Noetherian ring, then every R-module has property (RFG^*) by Proposition 3.4 (d),.

Theorem 4.9. *Let M be a secondful R-module. Then M has property (RFG^*) if and only if M has a Noetherian second spectrum.*

Proof. Let N be a submodule of M. Then the following statements are equivalent.

(a) There exists a finitely generated ideal $I = \sum_{i=1}^{k} Rr_i$ of R such that $Z.soc(N) = Z.soc((0 :_M I))$, where $r_i \in R$ and $k \in \mathbb{Z}^+$.
(b) $V^s(N) = V^s((0 :_M I)) = V^s((0 :_M \sum_{i=1}^{k} Rr_i)) = V^s(\cap_{i=1}^{k}(0 :_M r_i)) = \cap_{i=1}^{k} V^s((0 :_M r_i))$, where $r_i \in R$ and $k \in \mathbb{Z}^+$.
(c) The open set $U = X^s - V^s(N) = X^s - (\cap_{i=1}^{k} V^s((0 :_M r_i)) = \cup_{i=1}^{k}(X^s - V^s((0 :_M r_i))$, where $r_i \in R$ and $k \in \mathbb{Z}^+$.
(d) The open set $U = X^s - V^s(N)$ is quasi-compact as U is a finite union of quasi-compact subsets $X^s - V^s((0 :_M r_i))$ with $r_i \in R$. (We recall that if M is a secondful, then the open sets $X^s - V^s((0 :_M r))$ for every $r \in R$ are quasi-compact and form a base for the Zariski topology of X^s [4, 5.3 and 5.4].)

By using the equivalences above, we deduce that M has property (RFG^*) if and only if every open subset U of form $X^s - V^s(N)$, where $N \leq M$, is quasi-compact if and only if M has Noetherian second spectrum [8, p. 97, Poroposition 9]. □

Lemma 4.10. Let M be a secondful faithful R-module and N be a submodule of M. Then N is an RFG^*-submodule of M if and only $Ann_R(N)$ is an RFG-ideal of R.

Proof. First suppose that N is a submodule of M such that $Ann_R(N)$ is an RFG-ideal of R. Then $\sqrt{Ann_R(N)} = \sqrt{J}$ for some finitely generated ideal J or R. Then by using Proposition 3.4, we have

$$Z.soc(N) = Z.soc((0 :_M \sqrt{Ann_R(N)}) =$$

$$Z.soc((0 :_M \sqrt{J})) = Z.soc((0 :_M J)).$$

Thus N is an RFG^*-submodule of M. Conversely, let I be a finitely generated ideal of R such that $Z.soc(N) = Z.soc((0 :_M I))$. Since M is secondful, by using Theorem 3.5 and Proposition 3.4, we get

$$\sqrt{Ann_R(N)} = Ann_R(Z.soc(N)) = Ann_R((Z.soc((0 :_M I))$$

$$= Ann_R(Z.soc((0 :_M \sqrt{I})) = \sqrt{Ann_R((0 :_M \sqrt{I})}$$

Since M is a secondful faithful R-module, $\sqrt{Ann_R((0 :_M \sqrt{I})} = \sqrt{I}$ by Proposition 2.3 (a). Therefore, $\sqrt{Ann_R(N)} = \sqrt{I}$, where I is a finitely generated ideal of R, as desired. □

We recall that R has Noetherian spectrum if and only if every prime ideal of R is an RFG-ideal [12, p.634, Corollary 2.4].

Theorem 4.11. *Let M be a faithful secondful R-module. Then M has Noetherian second spectrum if and only if every second submodule of M is an RFG^*-submodule.*

Proof. The necessity is clear by Theorem 4.9. Conversely we assume that every second submodule is an RFG^*-submodule. By using Theorem 4.3, it is enough to show that every prime ideal $\overline{P}$ of $\overline{R}$ is an RFG-ideal. Let $P \in Spec(R)$ such that $Ann_R(M) \subseteq P$ and $P/Ann_R(M) = \overline{P}$. As M is secondful, there exists a P-second submodule, say S. By hypothesis, S is an RFG^*-submodule so that $P = Ann_R(S)$ is an RFG-ideal by Lemma 4.10. Hence $\overline{P} = P/Ann_R(M)$ is an RFG-ideal of $\overline{R}$ by [12, p. 633, Poroposition 2.2(iv)]. This implies that every prime ideal of $\overline{R}$ is an RFG-ideal. Therefore, $\overline{R}$ has Noetherian spectrum, as required.

□

Corollary 4.12. Let M ba a faithful secondful R-module. If N and K are RFG^*-submodules of M, then so is $N + K$.

Proof. Let N and K be RFG^*-submodules of M. Then both $Ann_R(N)$ and $Ann_R(K)$ are RFG^*-ideals by Lemma 4.10. Thus $Ann_R(N+K) = Ann_R(N) \cap Ann_R(K)$ is an RFG^*-ideal by [12, p. 633, Proposition 2.2 (i)]. Therefore, $N + K$ is an RFG^*-submodule by Lemma 4.10. □

Acknowledgments

I would like to thank professor H. Ansari-Toroghy for his valuable advice during several helpful conversations on these subjects.

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Journal of Algebra and related topics
Vol. 1, No 1, (2013), pp 80

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چکیده
فرض کنیم R یک حلقه جابجایی و M یک R- مدول باشد. در این مقاله مفهوم سوکل های زاریسکیِ زیر مدول های M را تعریف نموده و خواص آن ها را مورد بررسی قرار می دهیم. بعلاوه مدول های با طیف ثانی نوتری را مطالعه و نتایج جدیدی را درباره آن ها اثبات می کنیم.

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