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TRACES OF PERMUTING n-ADDITIVE MAPPINGS IN $*$-PRIME RINGS

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Abstract. In this paper, we prove that a nonzero square closed $*$-Lie ideal U of a $*$-prime ring $\Re$ of Char $\Re \neq (2^n - 2)$ is central, if one of the following holds: $(i)\delta(x)\delta(y) \mp x \circ y \in Z(\Re)$, $(ii)[x,y] - \delta(xy)\delta(yx) \in Z(\Re)$, $(iii)\delta(x) \circ \delta(y) \mp [x,y] \in Z(\Re)$, $(iv)\delta(x) \circ \delta(y) \mp xy \in Z(\Re)$, $(v)\delta(x)\delta(y) \mp yx \in Z(\Re)$, where δ is the trace of n-additive map $F : \underbrace{\Re \times \Re \times .... \times \Re}_{n-times} \longrightarrow \Re$, for all $x, y \in U$.

1. Introduction

Throughout the paper, $\Re$ will denote an associative ring with centre $Z(\Re)$. A ring $\Re$ is said to be $*$-prime if $a\Re b = a\Re b^* = \{0\}$ implies that either $a = 0$ or $b = 0$. For each pair of elements $x, y \in \Re$ we shall write $[x, y]$ the commutator $xy - yx$; while the symbol $x \circ y$ will stand for the anti-commutator $xy + yx$. An additive map $x \to x^*$ of $\Re$ into itself is called an involution on $\Re$ if it satisfies the conditions: (i) $(x^*)^* = x$, (ii) $(xy)^* = y^*x^*$ for all $x, y \in \Re$. A ring $\Re$ equipped with an involution $*$ is called a $*$-ring or ring with involution. An additive subgroup U of a ring $\Re$ is said to be a Lie ideal of $\Re$ if $[U, \Re] \subseteq U$. A Lie ideal U of a $*$-ring $\Re$ is said to be a $*$-Lie ideal, if $U^* = U$. A Lie ideal U is said to be square closed, if for all $u \in U$, then $u^2 \in U$. An element x in a $*$-ring $\Re$ is said to be hermitian element if $x^* = x$ and skew-hermitian if $x^* = -x$. The set of hermitian and skew-hermitian elements of $\Re$

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will be denoted by $H(\Re)$ and $S(\Re)$, respectively. The involution $*$ is said to be of the first kind if $Z(\Re) \subseteq H(\Re)$, otherwise it is said to be of the second kind. In the later case $S(\Re) \cap Z(\Re) \neq \{0\}$.
An additive map $f : \Re \to \Re$ is said to be a derivation, if $f(xy) = f(x)y + xf(y)$ for all $x, y \in \Re$. An additive map $g : \Re \to \Re$ is said to be a generalized derivation, if there exists a derivation $f : \Re \to \Re$ such that $g(xy) = g(x)y + xf(y)$ for all $x, y \in \Re$. Let $n \geq 2$ be a fixed positive integer. A map $\digamma : \underbrace{\Re \times \Re \times .... \times \Re}_{n-times} \longrightarrow \Re$ is said to be symmetric (permuting), if $\digamma(x_1, x_2, ..., x_n) = \digamma(x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(n)})$ for all $x_i \in \Re$ for every permutation $(\pi(1), \pi(2), ..., \pi(n))$.
A map $\digamma : \underbrace{\Re \times \Re \times .... \times \Re}_{n-times} \longrightarrow \Re$ is said to be n-additive, if $\digamma$ is additive in each variable x_i; $i = 1, 2, ..., n$ i.e., $\digamma(x_1, x_2, ..., x_i + y_i, ..., x_n) = \digamma(x_1, x_2, ..., x_i, ..., x_n) + \digamma(x_1, x_2, ..., y_i, ..., x_n)$ for all $x_i, y_i \in \Re$ and $i = 1, 2, ..., n$. On the other hand, let $\Re = \left\{ \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} \mid a, b \in \mathbb{C} \right\}$, where $\mathbb{C}$ is a complex field. It is clear that, $\Re$ is a noncommutative ring under matrix addition and matrix multiplication.

Define map $\digamma : \underbrace{\Re \times \Re \times .... \times \Re}_{n-times} \longrightarrow \Re$ by

$$\digamma\left[\begin{bmatrix} 0 & 0 \\ a_1 & b_1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ a_2 & b_2 \end{bmatrix}, \cdots, \begin{bmatrix} 0 & 0 \\ a_n & b_n \end{bmatrix}\right] = \begin{bmatrix} 0 & 0 \\ a_1 a_2 a_3 ... a_n & 0 \end{bmatrix}.$$

Then it is easy to see that $\digamma$ is a symmetric n-additive map.

Let $n \geq 2$ be a fixed positive integer and let a map $\delta : \Re \to \Re$ defined by $\delta(x) = \digamma(x, x, ..., x)$ is called the trace of $\digamma$. It is obvious that in case, when $\digamma : \underbrace{\Re \times \Re \times .... \times \Re}_{n-times} \longrightarrow \Re$ is a symmetric n-additive mapping, the trace δ of $\digamma$ satisfies the relation
$\delta(x+y) = \delta(x)+\delta(y)+\sum_{k=1}^{n-1} \binom{n}{k} h_k(x, y)$. Since we have $\digamma(0, x_2, ..., x_n) = \digamma(0 + 0, x_2, ..., x_n) = \digamma(0, x_2, ..., x_n) + \digamma(0, x_2, ..., x_n)$ for all $x, y \in \Re$, we obtain $\digamma(0, x_2, ..., x_n) = 0$ for all $x_i \in \Re$; $i = 1, 2, ..., n$. Hence we get $0 = \digamma(0, x_2, ..., x_n)$, i.e. $0 = \digamma(x_1 - x_1, x_2, ..., x_n)$. This implies that $0 = \digamma(x_1, x_2, ..., x_n) + \digamma(-x_1, x_2, ..., x_n)$. It gives that $\digamma(-x_1, x_2, ..., x_n) = -\digamma(x_1, x_2, ..., x_n)$ for all $x_i \in \Re; i = 1, 2, ..., n$. This tells us that δ is an odd function, if n is odd and δ is an even function, if n is even.
In 1992, Daif and Bell [7, Theorem 1] proved that if a semiprime ring $\Re$ admits a derivation f such that $f([x, y]) - [x, y] \in Z(\Re)$ for all $x, y \in \Re$,

then $\Re$ is commutative. Further, Ashraf at. el. in [6] obtained the commutativity of a prime ring $\Re$ admitting a generalized derivation g on $\Re$ satisfying one of the following : (i) $g(xy) \mp xy \in Z(\Re)$, (ii) $g(xy) \mp yx \in Z(\Re)$, (iii) $g(x)g(y) \mp xy \in Z(\Re)$ for all x, y in some appropriate subset of $\Re$. In 2007, Oukhtite and Salhi [5] proved that let $\Re$ be a $*$-prime ring such that char $\Re \neq 2, 3$. Let L be a nonzero Lie ideal of $\Re$ and d a nonzero derivation of $\Re$ commuting with $*$. If $d^2(L) \subset Z(\Re)$, then $L \subset Z(\Re)$.

Motivated by the aforementioned results, we prove that a nonzero square closed $*$-Lie ideal U of a $*$-prime ring $\Re$ of Char $\Re \neq (2^n - 2)$ is central, if it satisfies one of the following: $(i)\delta(x)\delta(y) \mp x \circ y \in Z(\Re)$, $(ii)[x, y] - \delta(xy)\delta(yx) \in Z(\Re)$, $(iii)\delta(x) \circ \delta(y) \mp [x, y] \in Z(\Re)$, $(iv)\delta(x) \circ \delta(y) \mp xy \in Z(\Re)$, $(v)\delta(x)\delta(y) \mp yx \in Z(\Re)$, where δ is the trace of an n-additive map $\digamma : \underbrace{\Re \times \Re \times .... \times \Re}_{n-times} \longrightarrow \Re$.

2. Main results

We should do a great deal of calculation with commutators and anti-commutator routinely using the following basic identities:

$$x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$$
$$(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$$
$$[x, yz] = (x \circ y)z - y(x \circ z) = y[x, z] + [x, y]z$$

Lemma 2.1. [5, Lemma 1] *Let $\Re$ be a $*$-prime ring and U be a nonzero square closed $*$-Lie ideal of $\Re$. If $[U, U] = 0$, then $U \subseteq Z(\Re)$.*

Theorem 2.2. *Let $\Re$ be a $*$-prime ring of Char $\Re \neq (2^n - 2)$ and U be a nonzero square closed $*$-Lie ideal of $\Re$. Let $\digamma : \underbrace{\Re \times \Re \times .... \times \Re}_{n-times} \longrightarrow \Re$ be a symmetric n-additive map and δ be the trace of $\digamma$. If $\delta(x)\delta(y) \mp x \circ y \in Z(\Re)$ for all $x, y \in U$, then $U \subseteq Z(\Re)$.*

Proof. Suppose that

$$\delta(x)\delta(y) - x \circ y \in Z(\Re) \text{ for all } x, y \in U. \tag{2.1}$$

Replacing y by $y + z$ in (2.1), we get

$$\delta(x)(\delta(y) + \delta(z) + \sum_{k=1}^{n-1} \binom{n}{k} h_k(y, z)) - x \circ y - x \circ z \in Z(\Re). \tag{2.2}$$

Comparing (2.1) and (2.2), we have

$$\delta(x)\sum_{k=1}^{n-1}\binom{n}{k}h_k(y,z)\in Z(\Re).$$

That is

$$\delta(x)\left(\binom{n}{1}h_1(y,z)+\ldots+\binom{n}{n-1}h_{n-1}(y,z)\right)\in Z(\Re). \tag{2.3}$$

Substituting y for z in (2.3), we get

$$\delta(x)\left(\binom{n}{1}h_1(y,y)+\binom{n}{2}h_2(y,y)+\ldots+\binom{n}{n-1}h_{n-1}(y,y)\right)\in Z(\Re).$$

This implies that

$$\delta(x)\Bigg(\binom{n}{1}F(\underbrace{y,y,\ldots,y}_{(n-1)-times},\underbrace{y}_{1-times})+\binom{n}{2}F(\underbrace{y,y,\ldots,y}_{(n-2)-times},\underbrace{y}_{2-times})+\ldots$$
$$+\binom{n}{n-1}F(\underbrace{y}_{1-times},\underbrace{y,y,\ldots,y}_{(n-1)-times})\Bigg)\in Z(\Re)\text{ for all }x,y\in U.$$

This gives

$$\delta(x)\left(\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n-1}\right)F(y,y,\ldots,y)\in Z(\Re)\text{ for all }x,y\in U.$$

That is

$$(2^n-2)\delta(x)F(y,y,\ldots,y)\in Z(\Re)\text{ for all }x,y\in U.$$

Since $\Re$ is (2^n-2) torsion free, we have

$$\delta(x)F(y,y,\ldots,y)\in Z(\Re)\text{ for all }x,y\in U.$$

This implies that

$$\delta(x)\delta(y)\in Z(\Re)\text{ for all }x,y\in U. \tag{2.4}$$

Using (2.1) and (2.4), we obtain

$$x\circ y\in Z(\Re)\text{ for all }x,y\in U. \tag{2.5}$$

Thus $[r,(x\circ y)]=0$ for all $x,y\in U,\ r\in\Re$. Replacing y by $2yx$ and again using the fact that Char $\Re\neq(2^n-2)$, we get

$$(y\circ x)[r,x]=0\text{ for all }x,y\in U,\ r\in\Re. \tag{2.6}$$

Substituting sr for r, we have

$$(y\circ x)\Re[r,x]=\{0\}\text{ for all }x,y\in U,\ r\in\Re. \tag{2.7}$$

For all $x \in U \cap S_*(\Re)$, relation (2.7), yields that $(y \circ x)\Re[r,x] = \{0\} = (y \circ x)\Re([r,x])^*$. Since $\Re$ is $*$-prime ring, we obtain either $(y \circ x) = 0$ or $[r,x] = 0$. Now for any $x \in U$, using the fact $x - x^* \in U \cap S_*(\Re)$, we get $y \circ (x - x^*) = 0$ or $[r, x - x^*] = 0$. If $y \circ (x - x^*) = 0$, then $(y \circ x - y \circ x^*) = 0$, as $y \circ x = 0$; $y \circ x^* = 0$, so we have either $y \circ x = 0$ or $[r,x] = 0$. On the other hand, if $[r, x - x^*] = 0$, then $[r,x] = [r,x^*]$. This implies that, $[r,x^*] = 0$. In conclusion, for all $x, y \in U$, $r \in R$, we have either $(y \circ x) = 0$ or $[r,x] = 0$. Let $A = \{x \in U \mid (y \circ x) = 0\}$, $B = \{x \in U \mid [r,x] = 0\}$, for all $x, y \in U$, $r \in \Re$. Then A and B both are additive subgroups of U and $A \cup B = U$. But a group cannot be union of two its proper subgroups and therefore $A = U$ or $B = U$. If $A = U$, then $(y \circ x) = 0$ for all $x, y \in U$. Replacing x by $[x, rx]$ in the last expression, we get $[x,r][y,x] = 0$ for all $x, y \in U$, $r \in \Re$. Again replacing r by sr, we get

$$[x,s]\Re[y,x] = \{0\} \text{ for all } x, y \in U; \text{ for all } s \in \Re. \tag{2.8}$$

If $x \in U \cap S_*(\Re)$, then $[x,s]\Re[y,x] = ([x,s])^*\Re[y,x] = \{0\}$. Thus $*$-primeness of $\Re$ yields that, either $[x,s] = 0$ or $[y,x] = 0$, but for any $x \in U$, $x - x^*, x + x^* \in U \cap S_*(\Re)$. Then either $[x - x^*, s] = 0$ or $[y, x - x^*] = 0$. If $[x - x^*, s] = 0$, then from (2.8) $[x,s]\Re[y,x] = ([x,s])^*\Re[y,x] = \{0\}$ for all $x, y \in U$ for all $s \in U$. Hence either $[x,s] = 0$ or $[y,x] = 0$. Let $A_1 = \{x \in U \mid [x,s] = 0\}$ and $B_1 = \{x \in U \mid [y,x] = 0\}$. Again A_1 and B_1 are additive subgroups of U such that $A_1 \cup B_1 = U$. But a group can not be union of two its proper subgroups and therefore $A_1 = U$ or $B_1 = U$. If $A_1 = U$, then $[x,s] = 0$ for all $x \in U$ this implies that $U \subseteq Z(\Re)$ on the other hand, if $B_1 = U$, then we have $[y,x] = 0$ for all $x, y \in U$ and hence $U \subseteq Z(\Re)$ by Lemma 2.1. Thus, in both the cases we find that $U \subseteq Z(\Re)$. Now if $B = U$ then $[x,s] = 0$ for all $x \in U$ for all $s \in \Re$ and again $U \subseteq Z(\Re)$ and hence in both the cases we find that $U \subseteq Z(\Re)$.

Similarly, we can prove the result in case $\delta(x)\delta(y) + x \circ y \in Z(\Re)$ for all $x, y \in U$.

□

Theorem 2.3. *Let $\Re$ be a $*$-prime ring of Char $\Re \neq (2^n - 2)$ and U be a nonzero square closed $*$-Lie ideal of $\Re$. Let* $F : \underbrace{\Re \times \Re \times .... \times \Re}_{n-times} \longrightarrow \Re$ *be a symmetric n-additive map and δ be the trace of F. If $\delta(x) \circ \delta(y) \mp x \circ y \in Z(\Re)$ for all $x, y \in U$, then $U \subseteq Z(\Re)$.*

Proof. Suppose

$$\delta(x) \circ \delta(y) - x \circ y \in Z(\Re) \text{ for all } x, y \in U. \tag{2.9}$$

Replacing y by $y+z$ in (2.9), we get

$$\delta(x)\circ\left(\delta(y)+\delta(z)+\sum_{k=1}^{n-1}\binom{n}{k}h_k(y,z)\right)-x\circ y-x\circ z\in Z(\Re). \quad (2.10)$$

Comparing (2.9) and (2.10), we have

$$\delta(x)\circ\sum_{k=1}^{n-1}\binom{n}{k}h_k(y,z)\in Z(\Re).$$

This gives

$$\delta(x)\circ\left(\binom{n}{1}h_1(y,z)+...+\binom{n}{n-1}h_{n-1}(y,z)\right)\in Z(\Re). \quad (2.11)$$

Substituting y for z in (2.11), we get

$$\delta(x)\circ\left(\binom{n}{1}h_1(y,y)+\binom{n}{2}h_2(y,y)+...+\binom{n}{n-1}h_{n-1}(y,y)\right)\in Z(\Re).$$

This implies that

$$\delta(x)\circ\left(\binom{n}{1}F(\underbrace{y,y,...,y}_{(n-1)-times},\underbrace{y}_{1-times})+\binom{n}{2}F(\underbrace{y,y,...,y}_{(n-2)-times},\underbrace{y}_{2-times})+...\right.$$
$$\left.+\binom{n}{n-1}F(\underbrace{y}_{1-times},\underbrace{y,y,...,y}_{(n-1)-times})\right)\in Z(F) \text{ for all } x,y\in U.$$

This gives

$$\delta(x)\circ\left(\binom{n}{1}+\binom{n}{2}+...+\binom{n}{n-1}\right)F(y,y,...,y)\in Z(\Re) \text{ for all } x,y\in U.$$

Therefore, we have

$$(2^n-2)\delta(x)\circ F(y,y,...,y)\in Z(\Re) \text{ for all } x,y\in U.$$

Since $\Re$ is (2^n-2) torsion free. Then

$$\delta(x)\circ F(y,y,...,y)\in Z(\Re) \text{ for all } x,y\in U.$$

This implies that

$$\delta(x)\circ\delta(y)\in Z(\Re) \text{ for all } x,y\in U. \quad (2.12)$$

Using (2.9) and (2.12), we obtain

$$x\circ y\in Z(\Re) \text{ for all } x,y\in U.$$

Using the same arguments, as we have done in the proof of the Theorem 2.2, we get the result.

□

Theorem 2.4. *Let $\Re$ be a $*$-prime ring of Char $\Re \neq (2^n - 2)$ and U be a nonzero square closed $*$-Lie ideal of $\Re$. Let $\digamma : \underbrace{\Re \times \Re \times .... \times \Re}_{n-times} \longrightarrow \Re$ be a symmetric n-additive map and δ be the trace of $\digamma$. If $[x, y] - \delta(xy) + \delta(yx) \in Z(\Re)$ for all $x, y \in U$, then $U \subseteq Z(\Re)$.*

Proof. Suppose that

$$[x, y] - \delta(xy) + \delta(yx) \in Z(\Re) \text{ for all } x, y \in U. \tag{2.13}$$

Replacing y by $y + z$ in (2.13), we get

$$\begin{aligned} [x, y] + [x, z] - \delta(xy) - \delta(xz) - \sum_{k=1}^{n-1} \binom{n}{k} h_k(xy, xz) \\ + \delta(yx) + \delta(zx) + \sum_{k=1}^{n-1} \binom{n}{k} h_k(yx, zx) \in Z(\Re). \end{aligned} \tag{2.14}$$

Comparing (2.13) and (2.14), we have

$$\sum_{k=1}^{n-1} \binom{n}{k} h_k(yx, zx) - \sum_{k=1}^{n-1} \binom{n}{k} h_k(yx, zx) \in Z(\Re). \tag{2.15}$$

Substituting z for y in (2.15), we get

$$\sum_{k=1}^{n-1} \binom{n}{k} h_k(yx, yx) - \sum_{k=1}^{n-1} \binom{n}{k} h_k(xy, xy) \in Z(\Re) \text{ for all } x, y \in U.$$

We obtain

$$\begin{aligned} \binom{n}{1} h_1(yx, yx) + \binom{n}{2} h_2(yx, yx) + ... + \binom{n}{n-1} h_{n-1}(yx, yx) \\ - \binom{n}{1} h_1(xy, xy) - \binom{n}{2} h_2(xy, xy) - ... - \binom{n}{n-1} h_{n-1}(xy, xy) \in Z(\Re). \end{aligned}$$

This implies that

$$\Bigg(\binom{n}{1}F(\underbrace{yx,yx,...,yx}_{(n-1)-times},\underbrace{yx}_{1-times})+\binom{n}{2}F(\underbrace{yx,yx,...,yx}_{(n-2)-times},\underbrace{yx}_{2-times})+...$$

$$+\binom{n}{n-1}F(\underbrace{yx}_{1-times},\underbrace{yx,yx,...,yx}_{(n-1)-times})-\binom{n}{1}F(\underbrace{xy,xy,...,xy}_{(n-1)-times},\underbrace{xy}_{1-times})$$

$$-\binom{n}{2}F(\underbrace{xy,xy,...,xy}_{(n-2)-times},\underbrace{xy}_{2-times})-...$$

$$-\binom{n}{n-1}F(\underbrace{xy}_{1-times},\underbrace{xy,xy,...,xy}_{(n-1)-times})\Bigg)\in Z(\Re).$$

This gives

$$\left(\binom{n}{1}+\binom{n}{2}+...+\binom{n}{n-1}\right)F(yx,yx,...,yx)$$

$$-\left(\binom{n}{1}+...+\binom{n}{n-1}\right)F(xy,xy,...,xy)\in Z(\Re).$$

That is

$$(2^n-2)\delta(yx)-(2^n-2)\delta(xy)\in Z(\Re)\text{ for all }x,y\in U.$$

Therefore, we have

$$(2^n-2)\Bigg(\delta(yx)-\delta(xy)\Bigg)\in Z(\Re)\text{ for all }x,y\in U.$$

Since $\Re$ is (2^n-2) torsion free, then

$$\delta(yx)-\delta(xy)\in Z(\Re)\text{ for all }x,y\in U. \tag{2.16}$$

Comparing (2.13) and (2.16), we get $[x,y]\in Z(\Re)$ Thus $[r,[x,y]]=0$ for all $x,y\in U$, $r\in\Re$. Replacing y by $2yx$ and using the fact that $\Re$ is not of characteristic (2^n-2), we get $[r,[x,yx]]=[r,[x,y]x]=[x,y][r,x]$. Again, replacing r by ry, we have $[y,x]\Re[y,x]=\{0\}$ for all $x,y\in U$. Therefore, $[y,x]\Re[y,x]=[y,x]\Re([y,x])^*=\{0\}$ and hence $*$-primeness of $\Re$ yields that $[y,x]=0$ for all $x,y\in U$, by Lemma 2.1, $U\subseteq Z(\Re)$. □

Theorem 2.5. *Let $\Re$ be a $*$-prime ring of Char $\Re\neq(2^n-2)$ and U be a nonzero square closed $*$-Lie ideal of $\Re$. Let* $F:\underbrace{\Re\times\Re\times....\times\Re}_{n-times}\longrightarrow\Re$

be a symmetric n-additive map and δ be the trace of $\digamma$. If $\delta(x)\circ\delta(y)\mp[x,y]\in Z(\Re)$ for all $x,y\in U$, then $U\subseteq Z(\Re)$.

Proof. Suppose that

$$\delta(x)\circ\delta(y)-[x,y]\in Z(\Re) \text{ for all } x,y\in U. \tag{2.17}$$

Replacing y by $y+z$ in (2.17), we get

$$\delta(x)\circ(\delta(y)+\delta(z)+\sum_{k=1}^{n-1}\binom{n}{k}h_k(y,z))-[x,y]-[x,z]\in Z(\Re). \tag{2.18}$$

Comparing (2.17) and (2.18), we have

$$\delta(x)\circ\sum_{k=1}^{n-1}\binom{n}{k}h_k(y,z)\in Z(\Re).$$

This implies that

$$\delta(x)\circ\left(\binom{n}{1}h_1(y,z)+...+\binom{n}{n-1}h_{n-1}(y,z)\right)\in Z(\Re). \tag{2.19}$$

Substituting z for y in (2.19), we get

$$\delta(x)\circ\left(\binom{n}{1}h_1(y,y)+\binom{n}{2}h_2(y,y)+...+\binom{n}{n-1}h_{n-1}(y,y)\right)\in Z(\Re).$$

Therefore, we have

$$\delta(x)\circ\Bigg(\binom{n}{1}\digamma(\underbrace{y,y,...,y}_{(n-1)-times},\underbrace{y}_{1-times})+\binom{n}{2}\digamma(\underbrace{y,y,...,y}_{(n-2)-times},\underbrace{y}_{2-times})+...$$
$$+\binom{n}{n-1}\digamma(\underbrace{y}_{1-times},\underbrace{y,y,...,y}_{(n-1)-times})\Bigg)\in Z(\Re) \text{ for all } x,y\in U.$$

That is

$$\delta(x)\circ\left(\binom{n}{1}+\binom{n}{2}+...+\binom{n}{n-1}\right)\digamma(y,y,...,y)\in Z(\Re) \text{ for all } x,y\in U,$$

and we have

$$(2^n-2)\delta(x)\circ\digamma(y,y,...,y)\in Z(\Re) \text{ for all } x,y\in U.$$

Since $\Re$ is (2^n-2) torsion free, we get

$$\delta(x)\circ\digamma(y,y,...,y)\in Z(\Re) \text{ for all } x,y\in U.$$

This implies that

$$\delta(x)\circ\delta(y)\in Z(\Re). \tag{2.20}$$

Using (2.17) and (2.20), we obtain $[x,y]\in Z(\Re)$ for all $x,y\in U$. Arguing the similar manner as in the proof of Theorem 2.4, we get the result.

Similarly, we can prove the case if, $\delta(x)\circ\delta(y)+[x,y]\in Z(\Re)$ for all x, $y\in U$.

$\square$

Theorem 2.6. *Let $\Re$ be a $*$-prime ring of Char $\Re\neq(2^n-2)$ and U be a nonzero square closed $*$-Lie ideal of $\Re$. Let $\digamma:\underbrace{\Re\times\Re\times....\times\Re}_{n-times}\longrightarrow\Re$ be a symmetric n-additive map and δ be the trace of $\digamma$. If $\delta(x)\circ\delta(y)\mp xy\in Z(\Re)$ for all $x,y\in U$, then $U\subseteq Z(\Re)$.*

Proof. Assume that

$$\delta(x)\circ\delta(y)-xy\in Z(\Re)\text{ for all }x,y\in U. \tag{2.21}$$

Replacing y by $y+z$ in (2.21), we get

$$\delta(x)\circ(\delta(y)+\delta(z)+\sum_{k=1}^{n-1}\binom{n}{k}h_k(y,z))-xy-xz\in Z(\Re). \tag{2.22}$$

Comparing (2.21) and (2.22), we have

$$\delta(x)\circ\sum_{k=1}^{n-1}\binom{n}{k}h_k(y,z)\in Z(\Re).$$

Thus, we obtain

$$\delta(x)\circ\left(\binom{n}{1}h_1(y,z)+...+\binom{n}{n-1}h_{n-1}(y,z)\right)\in Z(\Re). \tag{2.23}$$

Substituting y for z in (2.23), we get

$$\delta(x)\circ\left(\binom{n}{1}h_1(y,y)+\binom{n}{2}h_2(y,y)+...+\binom{n}{n-1}h_{n-1}(y,y)\right)\in Z(\Re).$$

This implies that

$$\delta(x)\circ\Bigg(\binom{n}{1}\digamma(\underbrace{y,y,...,y}_{(n-1)-times},\underbrace{y}_{1-times})+\binom{n}{2}\digamma(\underbrace{y,y,...,y}_{(n-2)-times},\underbrace{y}_{2-times})+...$$
$$+\binom{n}{n-1}\digamma(\underbrace{y}_{1-times},\underbrace{y,y,...,y}_{(n-1)-times})\Bigg)\in Z(\Re).$$

Therefore, we have

$$\delta(x)\circ\left(\binom{n}{1}+\binom{n}{2}+...+\binom{n}{n-1}\right)F(y,y,...,y)\in Z(\Re).$$

This implies that

$$(2^n-2)\delta(x)\circ F(y,y,...,y)\in Z(\Re) \text{ for all } x,y\in U.$$

Since $\Re$ is (2^n-2) torsion free, we get

$$\delta(x)\circ F(y,y,...,y)\in Z(\Re) \text{ for all } x,y\in U.$$

This implies that

$$\delta(x)\circ\delta(y)\in Z(\Re). \tag{2.24}$$

Using (2.21) and (2.24), we obtain

$$xy\in Z(\Re) \text{ for all } x,y\in U. \tag{2.25}$$

Interchanging the role of x and y in (2.25) and subtracting from (2.25), we find

$$[x,y]\in Z(\Re) \text{ for all } x,y\in U.$$

Arguing in the similar manner as in the Theorem 2.5, we get the result.

The prove is same for the case $\delta(x)\circ\delta(y)+xy\in Z(\Re)$ for all $x,y\in U$. □

Theorem 2.7. *Let $\Re$ be a $*$-prime ring of Char $\Re\neq(2^n-2)$ and U be a nonzero square closed $*$-Lie ideal of $\Re$. Let $F:\underbrace{\Re\times\Re\times....\times\Re}_{n-times}\longrightarrow\Re$ be a symmetric n-additive map and δ be the trace of F. If $\delta(x)\circ\delta(y)\mp yx\in Z(\Re)$ for all $x,y\in U$, then $U\subseteq Z(\Re)$.*

Proof. The proof runs on the same parallel lines as of the Theorem 2.6. □

The following examples illustrates that $\Re$ to be $*$-prime ring and Char $\Re\neq(2^n-2)$ for $n>1$ are essential in the hypothesis of the above theorems.

Example 2.8. Let $\Re=\left\{\begin{bmatrix} a & b\\ 0 & c\end{bmatrix} \mid a,b,c\in\mathbb{S}, \text{ ring of integers}\right\}$ and $U=\left\{\begin{bmatrix} 0 & b\\ 0 & 0\end{bmatrix} \mid b\in\mathbb{S}\right\}$. Then $Z(\Re)=\left\{\begin{bmatrix} a & 0\\ 0 & a\end{bmatrix} \mid a\in\mathbb{S}\right\}$. Define map $\mathcal{F} : \underbrace{\Re \times \Re \times .... \times \Re}_{n-times} \longrightarrow \Re$ by

$$\mathcal{F}\left[\begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix}, \cdots, \begin{bmatrix} a_n & b_n \\ 0 & c_n \end{bmatrix}\right] = \begin{bmatrix} a_1a_2a_3...a_n & 0 \\ 0 & 0 \end{bmatrix}.$$

It can be verified that $\mathcal{F}$ is n-additive with trace δ defined by $\delta : \Re \longrightarrow \Re$ as $\delta\left[\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right] = \mathcal{F}\left[\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}, \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}, \cdots, \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right]$ satisfying hypothesis of the above Theorems. However $U \nsubseteq Z(\Re)$.

Example 2.9. Let $\Re = \left\{ \begin{bmatrix} x & 0 \\ y & z \end{bmatrix} \mid x, y, z \in \mathbb{S}, \text{ring of integers} \right\}$ and $U = \left\{ \begin{bmatrix} 0 & 0 \\ y & 0 \end{bmatrix} \mid y \in \mathbb{S} \right\}$, here $Z(\Re) = \left\{ \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} \mid x \in \mathbb{S} \right\}$. Define a map $\mathcal{F} : \underbrace{\Re \times \Re \times .... \times \Re}_{n-times} \longrightarrow \Re$ by

$$\mathcal{F}\left[\begin{bmatrix} x_1 & 0 \\ y_1 & z_1 \end{bmatrix}, \begin{bmatrix} x_2 & 0 \\ y_2 & z_2 \end{bmatrix}, \cdots, \begin{bmatrix} x_n & 0 \\ y_n & z_n \end{bmatrix}\right] = \begin{bmatrix} 0 & 0 \\ 0 & z_1z_2z_3...z_n \end{bmatrix}.$$

It can be verified that $\mathcal{F}$ is n-additive with trace δ defined by $\delta : \Re \longrightarrow \Re$ as $\delta\left[\begin{bmatrix} x & 0 \\ y & z \end{bmatrix}\right] = \mathcal{F}\left[\begin{bmatrix} x & 0 \\ y & z \end{bmatrix}, \begin{bmatrix} x & 0 \\ y & z \end{bmatrix}, \cdots, \begin{bmatrix} x & 0 \\ y & z \end{bmatrix}\right]$ satisfying hypothesis of the above Theorems. However $U \nsubseteq Z(\Re)$.

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