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A GENERALIZATION OF PURE SUBMODULES

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Abstract. Let R be a commutative ring with identity, S a multiplicatively closed subset of R, and M be an R-module. The goal of this work is to introduce the notion of S-pure submodules of M as a generalization of pure submodules of M and prove a number of results concerning of this class of modules. We say that a submodule N of M is *S-pure* if there exists an $s \in S$ such that $s(N \cap IM) \subseteq IN$ for every ideal I of R. Also, We say that M is *fully S-pure* if every submodule of M is S-pure.

1. Introduction

Throughout this paper R will denote a commutative ring with identity and S will denote a multiplicatively closed subset of R.

An R-module M is said to be a *multiplication module* if for every submodule N of M, there exists an ideal I of R such that $N = IM$ [7].

In [9], Cohn defined a submodule N of an R-module M a *pure submodule* if the sequence $0 \to N \otimes F \to M \otimes F$ is exact for every R-module F. In [3], Anderson and Fuller defined the submodule N of an R-module M a *pure submodule* if $IN = N \cap IM$ for every ideal I of R. In [14], Ribenboim called a submodule N of an R-module M a *pure submodule* if $rM \cap N = rN$ for each $r \in R$. Although the first condition implies the second [12, p.158], and the second implies the third, these definitions are not equivalent in general [12, p.158]. The

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three definitions of purity given above are equivalent if M is flat. In particular, if M is a faithful multiplication module [2].

In this paper, our definition of purity will be that of Anderson and Fuller [3].

Let S be a multiplicatively closed subset of R and M be an R-module. Recently the notions of S-Noetherian rings, S-Noetherian modules, S-prime submodules, S-multiplication modules, S-2-absorbing submodules, S-comultiplication modules, S-second submodules, and classical S-2-absorbing submodules introduced and investigated in [5, 1, 8, 15, 4, 16, 17, 10, 13]. The aim of this paper is to introduce the notion of S-pure submodules of M as a generalisation of pure submodules. We provide some information concerning this new class of modules.

2. Main results

Definition 2.1. We say that a submodule N of an R-module M is *S-pure* if there exists an $s \in S$ such that $s(N \cap IM) \subseteq IN$ for every ideal I of R.

Definition 2.2. We say that an R-module M is *fully S-pure* if every submodule of M is S-pure.

Example 2.3. Let M be an R-module such that $Ann_R(M) \cap S \neq \emptyset$. Then M is a fully S-pure R-module.

Proposition 2.4. *Every fully pure R-module is a fully S-pure R-module. The converse is true if $S \subseteq U(R)$, where $U(R)$ is the set of units in R.*

Proof. This is clear. □

The following example shows that the converse of Proposition 2.4 is not true in general.

Example 2.5. Let $\mathbb{Z}$ be the ring of integers. For a prime number p, one can see that the submodule $p\mathbb{Z}$ of the $\mathbb{Z}$-module $\mathbb{Z}$ is not pure. Take the multiplicatively closed subset $S = \{p^n : n \in \mathbb{N} \cup \{0\}\}$ of $\mathbb{Z}$. Then for each $k \in \mathbb{N}$, $p(p\mathbb{Z} \cap (k\mathbb{Z})) \subseteq (p\mathbb{Z})(k\mathbb{Z})$ implies that $p\mathbb{Z}$ is an S-pure submodule of $\mathbb{Z}$.

Theorem 2.6. *Let N and K be two submodules of an R-module M such that $N \subseteq K \subseteq M$. Then we have the following.*

(a) *If N is an S-pure submodule of K and K is an S-pure submodule of M, then N is an S-pure submodule of M.*

(b) *If N is an S-pure submodule of M, then N is an S-pure submodule of K.*

(c) *If K is an S-pure submodule of M, then K/N is an S-pure submodule of M/N.*

(d) *If N is an S-pure submodule of M and K/N is an S-pure submodule of M/N, then K is an S-pure submodule of M.*

(e) *If N is an S-pure submodule of M, then there is a bijection between the S-pure submodules of M containing N and the S-pure submodules of M/N.*

Proof. (a) Let I be an ideal of R. Then since K is an S-pure submodule of M, there exists an $s \in S$ such that $s(K \cap IM) \subseteq IK$. Since N is an S-pure submodule of K, there exists an $t \in S$ such that $t(N \cap IK) \subseteq IN$. Therefore,

$$\begin{aligned} ts(IM \cap N) = st(IM \cap N \cap K) &\subseteq ts(IM \cap K) \cap stN \\ &\subseteq t(IK \cap N) \subseteq IN. \end{aligned}$$

(b) Let I be an ideal of R. Then as N is an S-pure submodule of M, there exists an $s \in S$ such that $s(N \cap IM) \subseteq IN$. Hence

$$s(N \cap IK) \subseteq s(N \cap IM) \subseteq IN.$$

(c) Let I be an ideal of R. Then there exists an $s \in S$ such that $s(K \cap IM) \subseteq IK$. Thus

$$\begin{aligned} s(K/N \cap I(M/N)) = s((K/N) \cap ((IM+N)/N)) &= s((K \cap (IM+N))/N) \\ = (s(K \cap N) + s(K \cap IM))/N &\subseteq (sN + IK)/N \\ &\subseteq (N + IK)/N = I(K/N). \end{aligned}$$

(d) Let I be an ideal of R. Since N is an S-pure submodule of M, there exists an $t \in S$ such that $t(N \cap IM) \subseteq IN$. Since K/N is an S-pure submodule of M/N, there exists an $t \in S$ such that $s(K/N \cap I(M/N)) \subseteq I(K/N)$. This implies that $s(K \cap (IM+N)) + N \subseteq IK + N$. Hence $IM \cap (s(K \cap IM) + N) \subseteq (IK + N) \cap IM$. Therefore,

$$ts(K \cap IM) + t(N \cap IM) \subseteq tIK + t(N \cap IM) \subseteq tIK + IN \subseteq IK.$$

Thus $ts(K \cap IM) \subseteq IK$.

(e) This follows from parts (c) and (d). □

Recall that the saturation S^* of S is defined as $S^* = \{x \in R : x/1 \textit{ is a unit of } S^{-1}R\}$. Clearly, S^* is a multiplicatively closed subset of R containing S [11].

A multiplicatively closed subset S of R is said to satisfy the *maximal multiple condition* if there exists an $s \in S$ such that $t \mid s$ for each $t \in S$.

Proposition 2.7. *Let M be an R-module. Then we have the following.*

(a) *If $S_1 \subseteq S_2$ are multiplicatively closed subsets of R and M is a fully S_1-pure R-module, then M is a fully S_2-pure R-module.*

(b) *M is a fully S-pure R-module if and only if M is a fully S^*-pure R-module.*
(c) *If N is an S-pure submodule of M, then sN is an S-pure submodule of M for each $s \in S$.*
(d) *If N and K are submodules of M such that $N + K$ and $N \cap K$ are S-pure submodules of M. Then N is an S-pure submodule of M.*
(e) *If S is satisfying the maximal multiple condition (e.g., S is finite or $S \subseteq U(R)$) and $\{M_\lambda\}_\Lambda$ is a family of submodules of M with S-pure submodules $N_\lambda \subseteq M_\lambda$, then $\cap_{\lambda\in\Lambda} N_\lambda$ is an S-pure submodule of $\cap_{\lambda\in\Lambda} M_\lambda$.*
(f) *If $f : M \to M$ is an endomorphism and there exists an $s \in S$ such that $sf(x) = f^2(x)$ for each $x \in M$, then $f(M)$ is an S-pure submodule of M.*

Proof. (a) This is clear.

(b) Let M be a fully S-pure R-module. Since $S \subseteq S^*$, by part (a), M is a fully S^*-pure R-module. For the converse, assume that M is a fully S^*-pure module, N is a submodule of M, and I is an ideal of R. Then there exists an $x \in S^*$ such that $x(N \cap IM) \subseteq IN$. As $x \in S^*$, $x/1$ is a unit of $S^{-1}R$ and so $(x/1)(a/s) = 1$ for some $a \in R$ and $s \in S$. This implies that $us = uxa$ for some $u \in S$. Hence,

$$us(N \cap IM) = uxa(N \cap IM) \subseteq x(N \cap IM) \subseteq IN.$$

(c) Let $s \in S$. As N is S-pure, there is an $t \in S$ such that $t(IM \cap N) \subseteq IN$ for each ideal I of R. Therefore,

$$ts(IM \cap sN) \subseteq ts(IM \cap N) \subseteq sIN = I(sN).$$

(d) Let I be an ideal of R. Then there exist $s, t \in S$ such that $s((N + K) \cap IM) \subseteq I(N + K)$ and $t((N \cap K) \cap IM) \subseteq I(N \cap K)$. Let $n \in IM \cap N$. Then $sn \in s(N + K) \cap IM \subseteq I(N + K)$. Hence $sn = \sum_{i=1}^n a_i(n_i + k_i)$ for some $a_i \in I$, $n_i \in N$, and $k_i \in K$. It follows that $\sum_{i=1}^n a_ik_i \in N \cap K$. Hence $t\sum_{i=1}^n a_ik_i \in I(N \cap K) \subseteq IN$. Thus $t\sum_{i=1}^n a_ik_i = \sum_{i=1}^n b_ix_i$ for some $b_i \in I$ and $x_i \in N$. Therefore,

$$stn = st\sum_{i=1}^n a_in_i + s\sum_{i=1}^n b_ix_i \in IN.$$

Hence, $st(N \cap IM) \subseteq N$.

(e) Let I be an ideal of R. Then there exists an $s \in S$ such that $s(N_\lambda \cap IM) \subseteq IN_\lambda$ for each $\lambda \in \Lambda$. This implies that

$$s((\bigcap_{\lambda\in\Lambda} N_\lambda) \bigcap IM) = \bigcap_{\lambda\in\Lambda} s(N_\lambda \bigcap IM) \subseteq I(\bigcap_{\lambda\in\Lambda} N_\lambda).$$

(f) Let I be an ideal of R and $f(x) = \sum_{i=1}^{n} a_i m_i \in f(M) \cap IM$ for some $x, m_i \in M$ and $a_i \in I$. Then $sf(x) = f(f(x)) = \sum_{i=1}^{n} a_i f(m_i) \in If(M)$. This implies that $s(f(M) \cap IM) \subseteq If(M)$. □

Proposition 2.8. *Let M be an R-module. Then we have the following.*

(a) *If N is an S-pure submodule of M, then for each prime ideal $\mathfrak{p}$ of R, $N_{\mathfrak{p}}$ is an $S_{\mathfrak{p}}$-pure submodule of $M_{\mathfrak{p}}$ as an $R_{\mathfrak{p}}$-module.*

(b) *If $N_{\mathfrak{m}}$ is an $S_{\mathfrak{m}}$-pure submodule of an $R_{\mathfrak{m}}$-module $M_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m}$ of R, then N is an S-pure submodule of M.*

Proof. (a) This is straightforward.

(b) Suppose that I is an ideal of R. Then by assumption, for any maximal ideal $\mathfrak{m}$ of R, there is an $h/a \in S_{\mathfrak{m}}$ such that

$$(h(N \cap IM))_{\mathfrak{m}} = h/a(N_{\mathfrak{m}} \cap I_{\mathfrak{m}} M_{\mathfrak{m}}) \subseteq I_{\mathfrak{m}} N_{\mathfrak{m}} = (IN)_{\mathfrak{m}}.$$

This yields that

$$h(N \cap IM) \subseteq IN.$$

□

Theorem 2.9. *Let S satisfying the maximal multiple condition and M be an R-module. Then we have the following.*

(a) *If $\{N_\lambda\}_{\lambda \in \Lambda}$ is a chain of S-pure submodules of M, then $\sum_{\lambda \in \Lambda} N_\lambda$ is an S-pure submodule of M.*

(b) *If N is a submodule of M, then there is a submodule K of M maximal with respect to $K \subseteq N$ and K is an S-pure submodule of M.*

Proof. (a) Let I be an ideal of R. Then there exists an $s \in S$ such that $s(N_\lambda \cap IM) \subseteq IN_\lambda$ for each $\lambda \in \Lambda$. This implies that

$$s(\sum_{\lambda \in \Lambda} N_\lambda \cap IM) = s \sum_{\lambda \in \Lambda} (N_\lambda \cap IM) \subseteq I \sum_{\lambda \in \Lambda} N_\lambda,$$

Since $\{N_\lambda\}_{\lambda \in \Lambda}$ is a chain.

(b) Let

$$\Sigma = \{H \leq N | H \ is \ a \ S - pure \ submodule \ of \ M\}.$$

Then $0 \in \Sigma \neq \emptyset$. Let $\{N_\lambda\}_{\lambda \in \Lambda}$ be a totally ordered subset of Σ. Then $\sum_{\lambda \in \Lambda} N_\lambda \leq N$ and by part (a), $\sum_{\lambda \in \Lambda} N_\lambda$ is an S-pure submodule of M. Thus by using Zorn's Lemma, one can see that Σ has a maximal element, K say, as needed. □

Let R_i be a commutative ring with identity and M_i be an R_i-module, for $i = 1, 2$. Let $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is an R-module. Clearly, every submodule of M is in the form of $N = N_1 \times N_2$ for some submodules N_1 of M_1 and N_2 of M_2. Also, if S_i is a multiplicatively closed subset of R_i for each $i = 1, 2$, then $S = S_1 \times S_2$ is a multiplicatively closed subset of R.

Theorem 2.10. *Let M_i be an R_i-module and $S_i \subseteq R_i$ be a multiplicatively closed subset for $i = 1, 2$. Assume that $M = M_1 \times M_2$, $R = R_1 \times R_2$, and $S = S_1 \times S_2$. Then M is a fully S-pure R-module if and only if M_i is a fully S_i-pure R_i-module for $i = 1, 2$.*

Proof. First assume that M is a fully S-pure R-module. It is enough to show that M_1 is a fully S_1-pure R_1-module. Let N_1 be a submodule of M_1 and I_1 be an ideal of R_1. Then $N_1 \times \{0\}$ is a submodule of M and $I_1 \times \{0\}$ is an ideal of R. As M is a fully S-pure R-module, there exists an $s = (s_1, s_2) \in S_1 \times S_2$ such that

$$(s_1, s_2)((N_1 \times \{0\}) \cap M(I_1 \times \{0\})) \subseteq (N_1 \times \{0\})(I_1 \times \{0\}).$$

By focusing on first coordinate, we have $s_1(N_1 \cap I_1M_1) \subseteq I_1N_1$. Hence, M_1 is a fully S_1-pure R_1-module. Now suppose that M_1 is a fully S_1-pure R_1-module and M_2 is a fully S_2-pure R_2-module. Let N be a submodule of M and I be an ideal of R. Then $N = N_1 \times N_2$ and $I = I_1 \times I_2$, where $N_1, \subseteq M_1, N_2 \subseteq M_2$, $I_1 \subseteq R_1$, and $I_2 \subseteq R_2$. As M_1 is a fully S_1-pure R_1-module, there exists an $s_1 \in S_1$ such that $s_1(N_1 \cap I_1M_1) \subseteq I_1N_1$. As M_2 is a fully S_2-pure R_2-module, there exists an element $s_2 \in S_2$ such that $s_2(N_2 \cap I_2M_2) \subseteq I_2N_2$. Set $s = (s_1, s_2) \in S$. Then we have

$$s(N \cap IM) = (s_1, s_2)((N_1 \times N_2) \cap (I_1 \times I_2)(M_1 \times M_2)) =$$
$$s_1(N_1 \cap I_1M_1) \times s_2(N_2 \cap I_2M_2) \subseteq I_1N_1 \times I_2N_2 =$$
$$(I_1 \times I_2)(N_1 \times N_2) = IN.$$

□

In the following theorem, we characterize the fully pure R-modules. This theorem is analogous to that for multiplication modules considered in [4, Theorem 1].

Theorem 2.11. *Let M be an R-module. Then the following statements are equivalent:*

(a) *M is a fully pure R-module;*
(b) *M is a fully $(R - \mathfrak{p})$-pure R-module for each prime ideal $\mathfrak{p}$ of R;*
(c) *M is a fully $(R - \mathfrak{m})$-pure R-module for each maximal ideal $\mathfrak{m}$ of R;*
(d) *M is a fully $(R - \mathfrak{m})$-pure R-module for each maximal ideal $\mathfrak{m}$ of R with $M_{\mathfrak{m}} \neq 0_{\mathfrak{m}}$.*

Proof. $(a) \Rightarrow (b)$ Let M be a fully pure R-module and $\mathfrak{p}$ be a prime ideal of R. Clearly, $R-\mathfrak{p}$ is multiplicatively closed set of R and hence M is a fully $(R-\mathfrak{p})$-pure R-module by Proposition 2.4.

$(b) \Rightarrow (c)$ The result follows from the fact that every maximal ideal is a prime ideal and the part (b).

$(c) \Rightarrow (d)$ This is clear.

$(d) \Rightarrow (a)$ Let N be a submodule of M, I an ideal of R, and $\mathfrak{m}$ be a maximal ideal of R with $M_{\mathfrak{m}} \neq 0_{\mathfrak{m}}$. As M is a fully $(R-\mathfrak{m})$-pure module, there exists an $s \notin \mathfrak{m}$ such that $s(N \cap IM) \subseteq IN$. It follows that

$$(N \cap IM)_{\mathfrak{m}} = (s(N \cap IM))_{\mathfrak{m}} \subseteq (IN)_{\mathfrak{m}}.$$

Now we have $(N \cap IM)_{\mathfrak{m}} \subseteq (IN)_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m}$ of R. This implies that $N \cap IM \subseteq IN$, as needed. □

References

1. H. Ahmed and H. Sana *Modules satisfying the S-Noetherian property and $S-ACCR$*, Comm. Algebra, **44** (5) (2016), 1941-1951.
2. M.M. Ali and D.J. Smith, *Pure submodules of multiplication modules*, *Beiträge Algebra Geom.* **45** (1) (2004) 61–74.
3. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Springer-Verlag, New York-Heidelberg-Berlin, 1974.
4. D.D. Anderson, T. Arabaci, Ü. Tekir and S. Koç, *On S-multiplication modules*, Comm. Algebra, **48** (8) (2020), 1-10.
5. D. D. Anderson and T. Dumitrescu, *S-Noetherian rings*, Comm. Algebra, **30** (9) 2002, 4407-4416.
6. H. Ansari-Toroghy and F. Farshadifar, *Fully idempotent and coidempotent modules*, Bull. Iranian Math. Soc. **38** (4) (2012), 987-1005.
7. A. Barnard, *Multiplication modules*, J. Algebra, **71** (1981), 174–178.
8. Z. Bilgin, M. L. Reyes and Ü. Tekir, *On right S-Noetherian rings and S-Noetherian modules* Comm. Algebra, **46** (2) (2018), 863-869.
9. P. M. Cohn, *On the free product of associative rings*, *Math. Z.* **71** (1959) 380–398.
10. F. Farshadifar, *S-second submodules of a module*, Algebra and Discrete Mathematics, to appear.
11. R. Gilmer, *Multiplicative Ideal Theory*, Queen's Papers in Pure and Applied Mathematics, No. 90. Kingston, Canada: Queen's University, 1992.
12. T. Y. Lam, *Lectures on Modules and Rings.* Springer 1999.
13. O. A. Naji, *On Classical $S-2$-absorbing Submodules*, Eurasian Bulletin of Mathematics, **3**(1) (2020), 12-19.
14. P. Ribenboim, *Algebraic Numbers.* Wiley 1972.
15. E.S. Sevim, T. Arabaci, $\ddot{U}$. Tekir and S. Koc, *On S-prime submodules*, Turkish Journal of Mathematics, **43** (2) (2019), 1036-1046.
16. G. Ulucak, $\ddot{U}$. Tekir and S. Koc, *On $S-2$-absorbing submodules and vn-regular modules*, An. St. Univ. Ovidius Constanta, to appear.
17. E. Yildiz, Ü. Tekir and S. Koç, *On S-comultiplication modules*, https://arxiv.org/abs/2004.01474.

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