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Mathematical models for the variable weights version of the inverse minimax circle location problem

Mehraneh Gholami, Jafar Fathali*
Faculty of Mathematical Sciences, Shahrood University of Technology, University Blvd., Shahrood, Iran

Email(s): mehraneh.gholami@gmail.com, fathali@shahroodut.ac.ir

Abstract. This paper deals with the case of variable weights of the inverse model of the minimax circle location problem. The goal of the classic minimax circle location problem is finding a circle in the plane such that the maximum weighted distance from a given set of existing points to the circumference of the circle is minimized. In the corresponding inverse model, a circle is given and we should modify the weights of existing points with minimum cost, such that the given circle becomes optimal. The radius of the given circle can be fixed or variable. In this paper, both of these cases are investigated and mathematical models are presented for solving them.

Keywords: Minimax circle location, inverse facility location, variable weights.
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1 Introduction

In the most single facility location problems a set of points, represent the location of clients, are given and the goal is finding the location of a facility such that the cost of servicing clients by the facility is minimized. There are different criteria and objective functions in facility location problems. Among them, the minisum and minimax single facility location problems are two basic models in location theory. The goal of minisum problem is minimizing the sum of weighted distances between clients and the facility, whereas in the minimax problem the goal is minimizing the maximum weighted distances between clients and the facility. In the circle location models, the facility is a circle. The radius of circle can be fixed or variable.

There are many researches devoted on the circle location problem. Among them, Brimberg et al. [3] showed that in the minisum circle location problem with variable radius, there exists an optimal circle passing through two of the existing facilities. The discrete case of minisum circle location problem is investigated by Labb et al. [10]. Gholami and Fathali [9] studied the

*Corresponding author.
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minisum circle location problem with positive and negative weights of the existing points and developed a meta-heuristic algorithm to solve this problem.

The minimax case of the circle location problems with fixed and variable radius have been considered by Brimberg et al. [2]. Drezner et al. [7] applied the minimax circle location model to the out-of-roundness problem and suggested an approximate approach to solve the problem.

In some real applications of location models, the facilities may already exist and the aim is modifying the parameters of the problem with minimum cost, such that the given locations are optimal. These kind of location models are called inverse location problems.
Among many researches in inverse location models, Cai et al. [6] showed that the inverse minimax location problem is NP-hard. The inverse minisum single facility location problem has been considered by Burkard et al. [5] and [4]. Burkard et al. [4] solved the Euclidean case of this problem with variable vertex weights in $O(nlogn)$ time. Baroughi-Bonab et al. [1] investigated the inverse minisum single facility location problem with variable coordinates. They showed the problem with rectilinear and Chebyshev norms are NP-hard. Nazari et al. [11] studied the inverse backup of minisum facility location problem with variable coordinates on the plane. Recently, Fathali [8] proposed a row generation method for solving the general case of continuous inverse location problem.

The classic circle location problems and their inverse models have been applied to make many decisions in the real world such as locating circular facilities, e.g., a circular irrigation pipe, circular conveyor belts, or ring roads and out-of-roundness problem (see e.g. Drezner et al. [7]).

In this paper, we consider the inverse minimax circle location problem with variable weights. Let a circle with radius r_0 is given. Two cases are considered: 1- The fixed radius problem. In this case, we want to modify the weights of existing points with minimum cost such that the given circle becomes the optimal with comparing to all other circles with radius r_0. 2- The variable radius problem. This problem asks to modify the weights of existing points with minimum cost such that the given circle becomes optimal with compare to all other circles with positive radius. Using some properties on the classical minimax circle location problems we presented mathematical models for solving the inverse models.

In what follows, the inverse minimax circle location problem with fixed radius is investigated in Section 2. In Section 3, a mathematical model for the variable radius case of inverse model is presented. Section 4 contains the summary and the conclusion on this paper.

2 The inverse fixed radius minimax circle location

In this section we consider the fixed radius case of the inverse circle location problem.

First consider the traditional Fixed radius Minimax Circle Location Problem (FMCLP). In the FMCLP, a fix radius r_0 and n points $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ are given in the plane. Each point represents the location of a client and for $i = 1, \ldots, n$, the point p_i has the weight w_i. The problem is finding a circle $C = C(\mathbf{x}, r_0)$ with center $\mathbf{x}$ and radius r_0 such that the maximum weighted distance between the circumference of the circle C and the existing points is minimized, i.e.

$$\min_{\mathbf{x}} g(\mathbf{x}) = \max_{j=1,\ldots,n} \{w_i|d(\mathbf{x}, \mathbf{p}_j) - r_0|\}, \tag{1}$$

where $d(\mathbf{x}, \mathbf{p}_j)$ represents the distance between point $\mathbf{p}_j$ and the center of circle C.

The proof of the following theorem can be found in [2].

Theorem 1. *Consider any pair* $(\mathbf{p}_i, \mathbf{p}_j)$ *with* $d_{ij} = d(\mathbf{p}_i, \mathbf{p}_j) > 2r_0$, *and let* $\mathbf{x}_{ij}$ *be the weighted midpoint on segment* $[\mathbf{p}_i, \mathbf{p}_j]$ *such that*

$$w_i(d(\mathbf{x}_{ij}, \mathbf{p}_i) - r_0) = w_j(d(\mathbf{x}_{ij}, \mathbf{p}_j) - r_0). \tag{2}$$

Let the circle C^* *with center* $\mathbf{p}^*$ *be the optimal solution of FMCLP and*

$$g_{ij} = w_i(d(\mathbf{x}_{ij}, \mathbf{p}_i) - r_0), \;\; for\ i, j = 1, \ldots, n,$$

$$g_L = \begin{cases} g_{rs} = \max\{g_{ij};\ \forall i, j \ such\ that\ d_{ij} > 2r_0\}, & If\ there\ is\ (i,j) such\ that\ d_{ij} > 2r_0, \\ 0, & Otherwise, \end{cases}$$

and if $g_L \neq 0$ *let* $\mathbf{x}_m = \mathbf{x}_{rs}$.

1. *If* $g(\mathbf{x}_m) = g_L$, *then* $\mathbf{p}^* = \mathbf{x}_m$ *and* $g^* = g(\mathbf{x}_m)$.
2. *If* $g(\mathbf{x}_m) > g_L$, *then at least three extreme points* $\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k$, *exist such that* $w_i d_i(C^*) = w_j d_j(C^*) = w_k d_k(C^*)$, *where* $d_j(C) = |d(\mathbf{x}, \mathbf{p}_j) - r_0|$, *for* $j = 1, \ldots, n$.

Unfortunately the sub-problem (2) for an arbitrary triple $(\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k)$ does not appear to have a close-form solution. Therefore Brimberg et al. [2] proposed an algorithm for solving FMCLP, by using Theorem 1 and the following mathematical programming instead of sub-problem (2).

$$\begin{aligned} &\min\ K \\ &\quad K \geq w_i^2(r_0 - d(\mathbf{x}, \mathbf{p}_i))^2 \quad \forall\ i \in S, \end{aligned} \tag{3}$$

where S is the associated set of extreme points.

Now consider the inverse case. Let the circle $C^* = C(\mathbf{p}^*, r_0)$ be given and we want to modify the weight of existing points w_j, for $j = 1, \ldots, n$, to $w_j^* = w_j + w_j^+ - w_j^-$ with minimum cost, such that the circle C^* be the optimal with comparing to any other circle with radius r_0. $w_j^+ \geq 0$ and $w_j^- \geq 0$ are the values of augmenting and reduction of the weight of point $\mathbf{p}_j$ and bounded from above by u_j^+ and u_j^-, respectively. Suppose that c_j^+ and c_j^- denote the cost of augmenting and reduction of per unit of w_j, respectively. Then the inverse fixed radius minimax circle location problem with variable weights (IFMCLPW) can be modeled as follow.

$$\min \sum_{i=1}^{n} (c_i^+ w_i^+ + c_i^- w_i^-) \tag{4}$$

$$\max_i \{w_i^* |d(\mathbf{p}^*, \mathbf{p}_i) - r_0|\} \leq \max_i \{w_i^* |d(\mathbf{x}, \mathbf{p}_i) - r_0|\} \qquad \forall \mathbf{x} \in \mathbb{R}^2 \tag{5}$$

$$w_i^* = w_i + w_i^+ - w_i^- \qquad i = 1, \ldots, n, \tag{6}$$

$$0 \leq w_i^+ \leq u_i^+ \qquad i = 1, \ldots, n, \tag{7}$$

$$0 \leq w_i^- \leq u_i^- \qquad i = 1, \ldots, n. \tag{8}$$

This is a nonlinear model with infinity constraints. Therefor, we are going to find a simpler model and its optimal solution by using Theorem 1. According to the Theorem 1 the center of optimal circle is either one of the weighted midpoint on segment $[\mathbf{p}_i, \mathbf{p}_j]$ i.e. $\mathbf{x}_{ij}$, or a point $\mathbf{p}^*$ with at least three extreme points such that the value of the objective function for $\mathbf{p}^*$ is better than any other point with three extreme points.

First consider the case that $\mathbf{p}^*$ is in the weighted midpoints of segments $[\mathbf{p}_i, \mathbf{p}_j]$. In this case, we form the set

$$L = \{(i,j)|\ i < j,\ d_{ij} > 2r_0\},$$

and find $\mathbf{x}_{ij}$ for all $(i,j) \in L$. Then $\mathbf{p}^*$ can be equal to one of the $\mathbf{x}_{ij}$'s. Let

$$y_{ij} = \begin{cases} 1, & \text{If } \mathbf{x}_{ij} \text{ is the optimal solution,} \\ 0, & \text{Otherwise.} \end{cases}$$

Therefore, the inverse problem is converted to the following model:

$$(P_0) \quad \min \sum_{i=1}^{n} (c_i^+ w_i^+ + c_i^- w_i^-) \tag{9}$$
$$g_{ij} = w_i^*(d(\mathbf{x}_{ij}, \mathbf{p}_i) - r_0), \qquad \forall\ (i,j) \in L, \tag{10}$$
$$g_{ij} = w_i^*(d(\mathbf{x}_{ij}, \mathbf{p}_j) - r_0), \qquad \forall\ (i,j) \in L, \tag{11}$$
$$\max_{(i,j)\in L} \{w_i^*(d(\mathbf{x}_{ij}, \mathbf{p}_i) - r_0)\} = \sum_{(i,j)\in L} y_{ij} g_{ij}, \tag{12}$$
$$\max_{i=1,\cdots,n} \{w_i^* |d(\mathbf{p}^*, \mathbf{p}_i) - r_0|\} = \sum_{(i,j)\in L} y_{ij} g_{ij}, \tag{13}$$
$$\mathbf{p}^* = \sum_{(i,j)\in L} y_{ij} \mathbf{x}_{ij}, \tag{14}$$
$$\sum_{(i,j)\in L} y_{ij} = 1, \tag{15}$$
$$y_{ij} \in \{0,1\}, \qquad (i,j) \in L, \tag{16}$$
$$w_i^* = w_i + w_i^+ - w_i^-, \quad i = 1, \ldots, n, \tag{17}$$
$$0 \le w_i^+ \le u_i^+, \qquad i = 1, \ldots, n, \tag{18}$$
$$0 \le w_i^- \le u_i^-, \qquad i = 1, \ldots, n. \tag{19}$$

In the second case, we consider all triples $(\mathbf{p}_j, \mathbf{p}_k, \mathbf{p}_l)$ and find the candidate point $\mathbf{x}_{jkl}$ of each one by using the problem 3. Note that $\mathbf{p}^*$ can only be equal to one of candidate points, so we define the following binary variable

$$y_{jkl} = \begin{cases} 0, \text{ if } \mathbf{x}_{jkl} \text{ is the optimal solution,} \\ 1, \text{ Otherwise.} \end{cases}$$

Therefore, in this case the inverse problem can be modeled as follows:

$$(P_1)\ \ \min \sum_{i=1}^{n} (c_i^+ w_i^+ + c_i^- w_i^-) \tag{20}$$

$$\left.\begin{array}{l} \min\ K \\ \quad K \geq {w_j^*}^2 (r_0 - d(\mathbf{x}_{jkl}, \mathbf{p}_j))^2 \\ \quad K \geq {w_k^*}^2 (r_0 - d(\mathbf{x}_{jkl}, \mathbf{p}_k))^2 \\ \quad K \geq {w_l^*}^2 (r_0 - d(\mathbf{x}_{jkl}, \mathbf{p}_l))^2 \end{array}\right\}, \quad \forall\ j < k < l, \tag{21}$$

$$\left| \max_{i=1,\cdots,n} \{w_i^* |d(\mathbf{p}^*, \mathbf{p}_i) - r_0|\} - w_j^* |d(\mathbf{x}_{jkl}, \mathbf{p}_j) - r_0| \right| \leq M y_{jkl}, \quad \forall\ j < k < l, \tag{22}$$

$$\max_{i=1,\cdots,n} \{w_i^* |d(\mathbf{p}^*, \mathbf{p}_i) - r_0|\} \leq \max_{i=1,\cdots,n} \{w_i^* |d(\mathbf{x}_{jkl}, \mathbf{p}_i) - r_0|\}, \quad \forall\ j < k < l, \tag{23}$$

$$\mathbf{p}^* = \sum_{j=1}^{n-2} \sum_{k=j+1}^{n-1} \sum_{l=k+1}^{n} (1 - y_{jkl}) \mathbf{x}_{jkl}, \tag{24}$$

$$\sum_{j=1}^{n-2} \sum_{k=j+1}^{n-1} \sum_{l=k+1}^{n} (1 - y_{jkl}) = 1, \tag{25}$$

$$y_{jkl} \in \{0, 1\}, \qquad \forall\ j < k < l, \tag{26}$$

$$w_i^* = w_i + w_i^+ - w_i^-, \qquad i = 1, \ldots, n, \tag{27}$$

$$0 \leq w_i^+ \leq u_i^+, \qquad i = 1, \ldots, n, \tag{28}$$

$$0 \leq w_i^- \leq u_i^-, \qquad i = 1, \ldots, n. \tag{29}$$

Finally the solution of inverse problem can be obtained by finding the solutions of the problems (P_0) and (P_1) with the least objective value.

3 The inverse variable radius minimax circle location

In this section the variable radius case of the inverse circle location problem is investigated.

Consider the same notations as Section 2. The Variable radius Minimax Circle Location Problem (VMCLP) asks to find the location of a circle $C = C(x, r)$ such that the maximum weighted distances between the circumference of C and the existing points is minimized, i.e.

$$\min_{\mathbf{x}, r} g(\mathbf{x}, r) = \max_{j=1}^{n} \{w_j |d(\mathbf{x}, \mathbf{p}_j) - r|\}, \tag{30}$$

The Theorem 1 has been extended to the variable radius case Brimberg et al. [2] as follows.

Theorem 2. *Let $C^* = (\mathbf{x}^*, r^*)$ be an Optimal solution of VMCLP, and S be the associated set of extreme points with maximum weighted distance to C^*, that is $w_i d_i(C^*) = w_j d_j(C^*) = w_k d_k(C^*) = w_h d_h(C^*)$. Then, $|S| \geq 4$, with at least two extreme points located on each side of C^*.*

In this case, Brimberg et al. [2] used the Theorem 2 and presented an algorithm for solving VMCLP, by considering the following sub-problem:

$$\begin{aligned} &\min K \\ &\quad K \geq w_i^2(r - d(\mathbf{x}, \mathbf{p}_i))^2, \quad \forall\, i \in S. \end{aligned} \tag{31}$$

In the inverse problem, a circle $C^* = C(\mathbf{p}^*, r^*)$ be given and we want to modify the weight of existing points w_j, for $j = 1, \ldots, n$, to

$$w_j^* = w_j + w_j^+ - w_j^-,$$

with minimum cost, such that the circle C^* be the optimal with comparing to any other circle with positive radius. The other notations are the same as Section 2.

The inverse variable radius minimax circle location problem (IVMCLP) can be modeled as follows.

$$\min \sum_{i=1}^{n} (c_i^+ w_i^+ + c_i^- w_i^-) \tag{32}$$

$$\max_i \{w_i^* |d(\mathbf{p}^*, \mathbf{p}_i) - r^*|\} \leq \max_i \{w_i^* |d(\mathbf{x}, \mathbf{p}_i) - r|\}, \qquad \forall \mathbf{x} \in \mathbb{R}^2,\ r \in \mathbb{R}^+ \tag{33}$$

$$w_i^* = w_i + q_i^+ - q_i^-, \qquad i = 1, \ldots, n, \tag{34}$$

$$0 \leq w_i^+ \leq u_i^+, \qquad i = 1, \ldots, n, \tag{35}$$

$$0 \leq w_i^- \leq u_i^-, \qquad i = 1, \ldots, n. \tag{36}$$

To find a simpler model, note that according to Theorem 2 the center of optimal circle is a point with at least four extreme points such that the value of the objective function for C^* is better than any other circle whose center has four extreme points.

Therefore, we should check all quadruples $(\mathbf{p}_j, \mathbf{p}_k, \mathbf{p}_l, \mathbf{p}_h)$ where $j < k, l < h,\ j \neq l,\ k \neq h$ and $l, h \in J_+,\ \ j, k \in J_-$, where

$$J_- = \{i | d(\mathbf{x}, \mathbf{p}_i) < r\},$$

$$J_+ = \{i | d(\mathbf{x}, \mathbf{p}_i) > r\}.$$

Then the candidate circle associated of each extreme point $C_{jklh} = (\mathbf{x}_{jklh}, r_{jklh})$ is obtained by solving the mathematical programming problem (31).

Not that, C^* can be only equal to one of these cadidate circles so we define the following binary varable,

$$y_{jklh} = \begin{cases} 0, & \text{If } C_{jklh} \text{ is an optimal circle,} \\ 1, & \text{Otherwise.} \end{cases}$$

Therefore the inverse problem can be modeled as follows

$$\min \sum_{i=1}^{n} (c_i^+ w_i^+ + c_i^- w_i^-) \tag{37}$$

$$\left.\begin{array}{l} \min\ K \\ \quad K \geq {w_j^*}^2 (r_{jklh} - d(\mathbf{x}_{jklh}, \mathbf{p}_j))^2 \\ \quad K \geq {w_k^*}^2 (r_{jklh} - d(\mathbf{x}_{jklh}, \mathbf{p}_k))^2 \\ \quad K \geq {w_l^*}^2 (r_{jklh} - d(\mathbf{x}_{jklh}, \mathbf{p}_l))^2 \\ \quad K \geq {w_h^*}^2 (r_{jklh} - d(\mathbf{x}_{jklh}, \mathbf{p}_h))^2 \end{array}\right\}, \qquad \forall\ j < k,\ l < h,\ j \neq l,\ k \neq h, \tag{38}$$

$$\left| \max_{i=1,\cdots,n} \{w_i^* |d(\mathbf{p}^*, \mathbf{p}_i) - r^*|\} - w_j^* |d(\mathbf{x}_{jklh}, \mathbf{p}_j) - r_{jklh}| \right| \leq M y_{jklh}, \quad \forall\ j < k,\ l < h,\ j \neq l,\ k \neq h, \tag{39}$$

$$\max_{i=1,\cdots,n} \{w_i^* |d(\mathbf{p}^*, \mathbf{p}_i) - r_{jklh}|\} \leq \max_{i=1,\cdots,n} \{w_i^* |d(\mathbf{x}_{jklh}, \mathbf{p}_i) - r_{jklh}|\}, \quad \forall\ j < k,\ l < h,\ j \neq l,\ k \neq h, \tag{40}$$

$$\mathbf{p}^* = \sum_{j=1}^{n-2} \sum_{k=j+1}^{n-1} \sum_{l=k+1}^{n} (1 - y_{jklh}) \mathbf{x}_{jklh}, \tag{41}$$

$$r^* = \sum_{j=1,\ l=1}^{n-1} \sum_{\substack{k=j+1,\ h=l+1 \\ j\neq l,\ k\neq h}}^{n} (1 - y_{jklh}) r_{jklh}, \tag{42}$$

$$\sum_{j=1}^{n-2} \sum_{k=j+1}^{n-1} \sum_{l=k+1}^{n} (1 - y_{jklh}) = 1, \tag{43}$$

$$y_{jkl} \in \{0, 1\}, \qquad \forall\ j < k < l, \tag{44}$$

$$w_i^* = w_i + w_i^+ - w_i^-, \qquad i = 1, \ldots, n, \tag{45}$$

$$0 \leq w_i^+ \leq u_i^+, \qquad i = 1, \ldots, n, \tag{46}$$

$$0 \leq w_i^- \leq u_i^-, \qquad i = 1, \ldots, n. \tag{47}$$

4 Summary and conclusion

In this paper, we presented the mathematical models for the fixed and variable radius inverse minimax circle location problems with variable weights of existing points. As the future works, the other kind of inverse circle location problems such as inverse circle location problem with Hamming norm and inverse circle location on networks with variable edge lengths can be considered.

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