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CHARACTERIZATION OF SOME SPECIAL RINGS VIA LINKAGE

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Abstract. Providing a description of linked ideals in a commutative Noetherian ring in terms of some associated prime ideals, we make a characterization of Cohen-Macaulay, Gorenstein and regular local rings in terms of their linked ideals. More precisely, it is shown that the local ring $(R, \mathfrak{m})$ is Cohen-Macaulay if and only if any linked ideal is unmixed. Also, $(R, \mathfrak{m})$ is Gorenstein if and only if any unmixed ideal $\mathfrak{a}$ is linked by every maximal regular sequence in $\mathfrak{a}$.

We also compute the annihilator of top local cohomology modules in some special cases.

1. introduction

Throughout the paper, R denotes a non-trivial commutative Noetherian ring, $\mathfrak{a}$ and $\mathfrak{b}$ are non-zero proper ideals of R and M will denote a finitely generated R-module.

The theory of linkage is an important topic in commutative algebra and algebraic geometry. It refers to Halphen (1870) and M. Noether [11](1882) who worked to classify space curves. In 1974 the significant

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work of Peskine and Szpiro [12] stated this theory in the modern algebraic language; $\mathfrak{a}$ and $\mathfrak{b}$ are said to be linked if there exists a regular sequence $\underline{\mathfrak{x}}$ in their intersection such that $\mathfrak{a} = (\underline{\mathfrak{x}}) :_R \mathfrak{b}$ and $\mathfrak{b} = (\underline{\mathfrak{x}}) :_R \mathfrak{a}$.

In a recent paper [7], inspired by the works in the ideal case, the authors present the concept of the linkage of ideals over a module. Let $I \subseteq \mathfrak{a} \cap \mathfrak{b}$ be an ideal of R which is generated by an M-regular sequence and $\mathfrak{a}M \neq M \neq \mathfrak{b}M$. Then, $\mathfrak{a}$ and $\mathfrak{b}$ are said to be linked by I over M if $\mathfrak{b}M = IM :_M \mathfrak{a}$ and $\mathfrak{a}M = IM :_M \mathfrak{b}$. This is a generalization of the classical concept of linkage when $M = R$.

In this paper, first we consider the above generalization and define the set $S_{(I;M)}$, which contains the set of linked radical ideals of R over M by I. In Section 2, we study some basic properties of this set and, using them, we characterize the linked radical ideals. Indeed, it is shown that a radical ideal $\mathfrak{a}$ is linked if and only if $\mathfrak{a} = \bigcap_{\mathfrak{p} \in \Lambda} \mathfrak{p}$ for some R-regular sequence $\mathfrak{x}$ and some $\Lambda \subseteq \operatorname{Ass} \frac{R}{(\mathfrak{x})}$ (Corollary 2.8).

Computing the annihilator of local cohomology modules is one of the main problems in this theory and attracts lots of interest, see for example [1], [13] and [14]. As an application of the above result, we study the annihilator of local cohomology modules. More precisely, we show that the radical of the annihilator of top local cohomology module is a linked ideal (see Proposition 2.14 and Example 3.3).

In Section 3, using the results provided in Section 2, we characterize Cohen-Macaulay, Gorenstein and regular local rings in terms of their linked ideals. In particular, it is shown that the finitely generated module M over a local ring is Cohen-Macaulay if and only if $\frac{M}{\mathfrak{a}M}$ is an unmixed module for all ideals $\mathfrak{a}$ which are linked over M (see Theorem 3.5). Also, the local ring R is Gorenstein if and only if any unmixed ideal $\mathfrak{a}$ is linked by every maximal R-regular sequence in $\mathfrak{a}$. A characterization of Gorenstein local rings in term of the "horizontally linked modules" is provided, too (Theorem 3.7).

2. LINKED IDEALS OVER A MODULE

In this section, first, we study some basic properties of "linked ideals over a module" and provide some characterization of them. Then, using this characterization, we make a description of linked ideals in R.

Definition 2.1. *Assume that $\mathfrak{a}M \neq M \neq \mathfrak{b}M$ and let $I \subseteq \mathfrak{a} \cap \mathfrak{b}$ be an ideal generated by an M-regular sequence. Then we say that the ideals $\mathfrak{a}$ and $\mathfrak{b}$ are linked by I over M, denoted $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$, if $\mathfrak{b}M = IM :_M \mathfrak{a}$ and $\mathfrak{a}M = IM :_M \mathfrak{b}$. The ideals $\mathfrak{a}$ and $\mathfrak{b}$ are said to be geometrically linked by I over M if $\mathfrak{a}M \cap \mathfrak{b}M = IM$. Also, we say that the ideal $\mathfrak{a}$ is linked over M if there exist ideals $\mathfrak{b}$ and I of R such that $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$. If $M = R$, for abbreviation, $\mathfrak{a}$ is said to be linked.*

Remark 2.2.
1. *Note that in the case where $M = R$, the above concept is the classical concept of linkage of ideals in [12]. In general case, linkedness of two ideals over M does not imply linkedness of them over R and vice versa (see M. Jahangiri and Kh. Sayyari, Linkage of ideals over a module, [7]).*
2. *One can see that by the above definition, I is actually generated by a maximal M-regular sequence in $\mathfrak{a} \cap \mathfrak{b}$.*

Lemma 2.3. *Let I be a proper ideal of R such that $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$. Then, $\frac{M}{\mathfrak{a}M}$ can be embedded in finite copies of $\frac{M}{IM}$.*

Proof. It follows from [7, 2.8]. □

In the following, we construct the set of ideals $S_{(I;M)}$ which plays a fundamental role in the paper. This set, as we show, contains all linked radical ideals by I over M.

Convention 2.4. *Assume that I is an ideal of R which is generated by an M-regular sequence. Set*

$$S_{(I;M)} := \{\mathfrak{a} \lhd R | I \subsetneqq \mathfrak{a}, \mathfrak{a} = IM :_R (IM :_M \mathfrak{a})\}.$$

Note that $S_{(I;R)}$ actually contains all linked ideals by I.

Some basic properties of the set $S_{(I;M)}$ are presented in the following Lemma.

Lemma 2.5. *Let I be an ideal of R which is generated by an M-regular sequence. Then*

(i) $\mathrm{Ass}\ \frac{R}{\mathfrak{a}} \subseteq \mathrm{Ass}\ \frac{M}{IM}$, *for all* $\mathfrak{a} \in S_{(I;M)}$.

(ii) $\mathrm{Ass}\ \frac{M}{IM} - \{I\} = \mathrm{Spec}\ S_{(I;M)}$, *the set of prime ideals of* $S_{(I;M)}$ *(note that I need not to be a prime ideal).*

(iii) $S_{(I;M)}$ *is closed under finite intersection. More precisely,* $\mathfrak{a}_1 \cap \mathfrak{a}_2 \in S_{(I;M)}$, *for all* $\mathfrak{a}_1, \mathfrak{a}_2 \in S_{(I;M)}$ *with* $\mathfrak{a}_1 \cap \mathfrak{a}_2 \neq I$.

(iv) $S_{(I;M)}$ *is closed under radical. In particular, if* $\mathfrak{a} \in S_{(I;M)}$ *then* $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p} \in \Lambda} \mathfrak{p} \in S_{(I;M)}$ *for some* $\Lambda \subseteq \mathrm{Ass}\ \frac{M}{IM} - \{I\}$.

(v) $\sqrt{\mathfrak{a} + \mathrm{Ann}\ M} \in S_{(I;M)}$, *for all ideals $\mathfrak{a}$ of R which are linked by I over M.*

Proof. Note that it is enough to consider the case where $I = 0$.

(i) Let $\mathfrak{a} \in S_{(0;M)}$, $N := 0 :_M \mathfrak{a}$ and assume that $N = \Sigma_{i=1}^t R\alpha_i$ for some $\alpha_1, ..., \alpha_t \in N$. Then, by the assumption, $\mathfrak{a} = 0 :_R N$ and using the natural monomorphism $\frac{R}{0:_R N} \to \oplus_{i=1}^t R\alpha_i$, we get

$$\mathrm{Ass}\ \frac{R}{\mathfrak{a}} \subseteq \bigcup_{i=1}^t \mathrm{Ass}\ R\alpha_i \subseteq \mathrm{Ass}\ M.$$

(ii) Let $\alpha \in M$ such that $\mathfrak{p} = 0 :_R \alpha \in \mathrm{Ass}\ M - \{0\}$. Hence,

$$\mathfrak{p} = 0 :_R \alpha = 0 :_R (0 :_M \mathfrak{p}) \in S_{(0;M)}.$$

The converse follows from (i).

(iii) It follows from the fact that

$$\mathfrak{a}_1 \cap \mathfrak{a}_2 \subseteq 0 :_R (0 :_M (\mathfrak{a}_1 \cap \mathfrak{a}_2)) \subseteq (0 :_R (0 :_M \mathfrak{a}_1)) \cap (0 :_R (0 :_M \mathfrak{a}_2)).$$

(iv) First note that $\sqrt{\mathfrak{a}}$ is a non-zero ideal. Hence $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p} \in \mathrm{Min}\ \mathfrak{a}} \mathfrak{p} \in S_{(0;M)}$, by (iii) and (ii).

(v) First, note that $\sqrt{\mathfrak{a} + \mathrm{Ann}\ M} = \bigcap_{\mathfrak{p} \in \mathrm{Ass}\ \frac{M}{\mathfrak{a}M}} \mathfrak{p}$. Therefore, by the assumption and 2.3, $\sqrt{\mathfrak{a} + \mathrm{Ann}\ M} = \bigcap_{\mathfrak{p} \in \Lambda} \mathfrak{p}$ for some $\Lambda \subseteq \mathrm{Ass}\ M$ and the result follows from (i), (ii) and the fact that $I \subsetneq \mathfrak{a}$.

□

In the following proposition we study when the set $S_{(I;M)}$ is empty.

Proposition 2.6. *Let I be a proper ideal of R which is generated by an M-regular sequence. Then, the following statements hold.*

(i) *If $S_{(I;M)} = \emptyset$ then I is prime.*
(ii) *If I is prime and M is flat or $\frac{M}{IM}$ is torsion-free then $S_{(I;M)} = \emptyset$.*
(iii) *If $S_{(I;M)} \neq \emptyset$ then* Max $S_{(I;M)} =$ Max Ass $\frac{M}{IM}$.

Proof. (i) Note that Ass $\frac{M}{IM} \neq \emptyset$ and so, by the assumption and 2.5(ii), $I \in$ Ass $\frac{M}{IM}$.

(ii) If M is flat then Ass $\frac{M}{IM} = \{I\}$, by [10, 23.2].

Also, if $\frac{M}{IM}$ is torsion-free then, as $\frac{M}{IM}$ has finite rank, it embeds in a finite copies of $\frac{R}{I}$. Hence Ass $\frac{M}{IM} \subseteq$ Ass $\frac{R}{I} = \{I\}$.

Now, assume that there exists $\mathfrak{a} \in S_{(I;M)}$. Then, by 2.5,

$$\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p}\in\Lambda} \mathfrak{p} \in S_{(I;M)}$$

for a subset Λ of Ass $\frac{M}{IM} - \{I\} = \emptyset$, and this is a contradiction.

(iii) Let $\mathfrak{a} \in$ Max $S_{(I;M)}$. Then, by 2.5(iv), $\mathfrak{a} = \sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p}\in\Lambda} \mathfrak{p} \in S_{(I;M)}$ for some $\Lambda \subseteq$ Ass $\frac{M}{IM} - \{I\}$. Therefore, by 2.5(ii), $\mathfrak{a} \in$ Max Ass $\frac{M}{IM}$.

Now, let $\mathfrak{p} \in$ Max Ass $\frac{M}{IM}$. Then, $\mathfrak{p} \neq I$, otherwise, as

$$\text{Max } S_{(I;M)} \subseteq \text{Max Ass } \frac{M}{IM} \subseteq V(I),$$

$\mathfrak{p} = I \subseteq \mathfrak{q}$ for all $\mathfrak{q} \in$ Max $S_{(I;M)}$. So, $I = \mathfrak{q} \in S_{(I;M)}$, which is a contradiction.

Therefore, $\mathfrak{p} \in$ Ass $\frac{M}{IM} - \{I\}$ and, by 2.5(ii), $\mathfrak{p} \in$ Max $S_{(I;M)}$. □

As a corollary of the above proposition, we have a criterion for the existence of an ideal of R linked by a fixed R-regular sequence I.

Corollary 2.7. *Let I be a proper non-prime ideal of R which is generated by a R-regular sequence. Then,*

$$\mathrm{Max}\,\{\mathfrak{a} \lhd R | \mathfrak{a} \textit{ is linked by } I\} = \mathrm{Max}\ \mathrm{Ass}\ \frac{R}{I}.$$

We can also characterize linked ideals, as follows.

Corollary 2.8. *The following statements hold.*

(i) *A radical ideal $\mathfrak{a}$ is linked if and only if $\mathfrak{a} = \bigcap_{\mathfrak{p}\in\Lambda} \mathfrak{p}$ for some R-regular sequence $\mathfrak{x}$ and some $\Lambda \subseteq \mathrm{Ass}\ \frac{R}{(\mathfrak{x})}$.*

(ii) *Non-zero elements of $\mathrm{Max}\,(R)$ and $\mathrm{Min}\ R$ are linked ideals.*

(iii) *If R isn't reduced then the nilradical ideal $\sqrt{0}$ is a linked ideal.*

Proof. (i) Assume that $\mathfrak{a}$ is a linked ideal by I. Hence, by [9, Proposition 5. p594], $\mathrm{Ass}\ \frac{R}{\mathfrak{a}} \subseteq \mathrm{Ass}\ \frac{R}{I}$ and $\mathfrak{a} = \bigcap_{\mathfrak{p}\in\Lambda} \mathfrak{p}$ for some $\Lambda \subseteq \mathrm{Ass}\ \frac{R}{I}$.

Now, let $\underline{\mathfrak{x}} = x_1, ..., x_n$ be a regular sequence and $\mathfrak{a} = \bigcap_{\mathfrak{p}\in\Lambda} \mathfrak{p}$ for some $\Lambda \subseteq \mathrm{Ass}\ \frac{R}{(\mathfrak{x})}$. Then, in view of [10, 6.3 and Exersice 6.7] and considering the regular sequence $x_1, ..., x_{n-1}, x_n^2$, we may assume that $(\mathfrak{x}) \notin \mathrm{Ass}\ \frac{R}{(\mathfrak{x})}$ and that $\mathfrak{a} \supsetneqq (\mathfrak{x})$. Now, the result follows from 2.5(ii) and (iii).

(ii) and (iii) follow from (i).

□

One may ask whether linking over a module implies linking over the ring and vice versa. In the following corollary, we consider a case where linking over the canonical module implies linking over R. For some other cases, we refer the reader to "M. Jahangiri and Kh. Sayyari, Linkage of ideals over a module, [7], Sec.4".

Corollary 2.9. *Let $(R, \mathfrak{m})$ be an unmixed complete local ring with the canonical module w_R and $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R such that $\mathfrak{a} \sim_{(0;w_R)} \mathfrak{b}$. Then $\sqrt{\mathfrak{a}}$ is a linked ideal over R.*

Proof. By local duality theorem [2, 11.2.6] and [2, 11.2.7(iii)],

$$w_R \cong \mathrm{Hom}\ _R(H_{\mathfrak{m}}^{\dim R}(R), E(\frac{R}{\mathfrak{m}})).$$

Therefore, in view of [2, 7.3.2 and 10.2.20],

$$\mathrm{Ass}\ w_R = \mathrm{Att}\ H_{\mathfrak{m}}^{\dim R}(R) = \mathrm{Assh}\ R. \qquad (1)$$

Also, using [6, 2.2(e)], $\mathrm{Ann}\, w_R = Ker(R \to \mathrm{Hom}_R(w_R, w_R)) = 0$. Hence, by 2.5, $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p}\in\Lambda} \mathfrak{p}$ for some $\Lambda \subsetneq \mathrm{Ass}\, w_R$. Now, the result follows from (1) and 2.8(i).

□

Definition 2.10. Let $(-)^* := \mathrm{Hom}_R(-, R)$ and consider an exact sequence $F_2 \xrightarrow{f} F_1 \xrightarrow{g} M \to 0$, where F_1 and F_2 are free R-modules. Setting $\mathrm{Tr}\, M := \mathrm{Coker}\, f^*$ and $\lambda M := \Omega\, \mathrm{Tr}\, M$, where "$\Omega$" is the first syzygy module, we get the exact sequences

$$0 \to M^* \xrightarrow{g^*} (F_1)^* \to \lambda M \to 0$$

and

$$0 \to M^* \xrightarrow{g^*} (F_1)^* \xrightarrow{f^*} (F_2)^* \to \mathrm{Tr}\, M \to 0.$$

Now, following [9], a finitely generated R-module M is said to be horizontally linked if $\lambda(\lambda M) \cong M$.

As another example of linked ideals we have the following corollary. Also, in [4], the authors study the relation between linkdness of M and that of $\mathrm{Ann}\, M$ as an ideal. In the following, we consider this problem, too.

Corollary 2.11. *Let M be a horizontally linked R-module such that $\mathrm{Ann}\, M \neq 0$ or R is not reduced. Then $\sqrt{\mathrm{Ann}\, M}$ is a linked ideal.*

Proof. As M is a syzygy, $\mathrm{Ass}\, M \subseteq \mathrm{Ass}\, R$ and $\sqrt{\mathrm{Ann}\, M} = \cap_{\mathfrak{p}\in\Lambda}\mathfrak{p}$ for some $\Lambda \subseteq \mathrm{Ass}\, R$. Also, using the assumption, $0 \subsetneqq \sqrt{0} \subseteq \sqrt{\mathrm{Ann}\, M}$ and the result follows from 2.8. □

In the next two items we describe the ideals that are linked by a radical ideal and show that they are, actually, geometrically linked.

Theorem 2.12. *Let I be an ideal of R which is generated by an R-regular sequence and $\mathrm{Ass}\, \frac{R}{I} = \mathrm{Min}\, \mathrm{Ass}\, \frac{R}{I}$. Let $I = \cap_{i=1}^{n}\mathfrak{q}_i$ be a minimal primary decomposition of I. Then*

(i) *$\cap_{i\in\Lambda}\mathfrak{q}_i$ and $\cap_{i\in\{1,\dots,n\}-\Lambda}\mathfrak{q}_i$ are linked by I, where $\Lambda \subset \{1, ..., n\}$.*
(ii) *If I is radical then all ideals which are linked by I are radical.*

Proof. (i) Let $\mathfrak{q}_i$ be $\mathfrak{p}_i$-primary, for all $i = 1, ..., n$ and let $r \in \{1, ..., n\}$. We have

$$\begin{aligned} I : (\cap_{i=1}^{r} \mathfrak{q}_i) &= \cap_{j=1}^{n} (\mathfrak{q}_j : (\cap_{i=1}^{r} \mathfrak{q}_i)) \\ &= \cap_{j=r+1}^{n} (\mathfrak{q}_j : (\cap_{i=1}^{r} \mathfrak{q}_i)) = \cap_{j=r+1}^{n} \mathfrak{q}_j \end{aligned}$$

The last equality follows from the fact that $\cap_{i=1}^{r} \mathfrak{q}_i \nsubseteq \mathfrak{p}_j$ for all $j > r$.

(ii) Assume that $\mathfrak{a}$ is linked by I. If $\mathfrak{a} = \cap_{j=1}^{r} Q_j$ is a minimal primary decomposition of $\mathfrak{a}$ then, by [9, Proposition 5. p594], for all $j = 1, ..., r$, Q_j is $\mathfrak{p}_j$-primary for some $\mathfrak{p}_j \in \operatorname{Ass} \frac{R}{I}$. For all $j = 1, ..., r$, we have

$$I \subseteq Q_j \cap \cap_{\mathfrak{p} \in \operatorname{Ass} \frac{R}{I} - \{\mathfrak{p}_j\}} \mathfrak{p} \subseteq \cap_{\mathfrak{p} \in \operatorname{Ass} \frac{R}{I}} \mathfrak{p} = I.$$

Therefore, $Q_j \cap \cap_{\mathfrak{p} \in \operatorname{Ass} \frac{R}{I} - \{\mathfrak{p}_j\}} \mathfrak{p}$ is another minimal decomposition of I. Via of $\operatorname{Ass} \frac{R}{I} = \operatorname{Min} \operatorname{Ass} \frac{R}{I}$ and second uniqueness theorem, $Q_j = \mathfrak{p}_j$ for all $j = 1, ..., r$. Therefore, $\mathfrak{a}$ is radical. □

Corollary 2.13. *Let I be a radical ideal of R which is generated by an R-regular sequence. Then the ideal $\mathfrak{a}$ is linked by I if and only if $\mathfrak{a} = \cap_{\mathfrak{p} \in \Lambda} \mathfrak{p}$ for some $\Lambda \subset \operatorname{Ass} \frac{R}{I}$. In this case, $\mathfrak{a}$ and $\mathfrak{b} := \cap_{\mathfrak{p} \in \operatorname{Ass} \frac{R}{I} - \Lambda} \mathfrak{p}$ are geometrically linked.*

In the theory of local cohomology modules, computing the annihilator of these modules attracts lots of interest, see for example [1], [13] and [14].

The following proposition consider a case where the annihilator of some local cohomology modules are linked. For another case see example 3.3.

Proposition 2.14. *Let $(R, \mathfrak{m})$ be a complete local ring of dimension $d > 0$ and $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R such that $\mathfrak{a} \sim_{(0;R)} \mathfrak{b}$ and $\operatorname{cd}(\mathfrak{a}, R) = d$. Then, the following statements hold.*

(i) $\sqrt{0 : H_{\mathfrak{a}}^{d}(R)}$ *is a linked ideal.*

(ii) *If R is unmixed and $\mathfrak{a}+\mathfrak{b}$ is $\mathfrak{m}$-primary then* $0 : H^d_{\mathfrak{a}}(R) = \mathfrak{b}$ *and* $0 : H^d_{\mathfrak{b}}(R) = \mathfrak{a}$.

Proof. (i) First we claim that $0 : H^d_{\mathfrak{a}}(R) \neq 0$. Suppose the contrary. Then, in view of [8, 2.4], Ass R = Assh R and $\sqrt{\mathfrak{a}+\mathfrak{p}} = \mathfrak{m}$ for all $\mathfrak{p} \in$ Assh R. On the other hand, by [9, Proposition 5. p594], there are some $\mathfrak{p} \in$ Ass R such that $\mathfrak{p} \supseteq \mathfrak{a}$. This implies that $\mathfrak{p} = \mathfrak{m}$ which is a contradiction.

Now, let $0 = \cap^n_{i=1}\mathfrak{q}_i$ be a minimal primary decomposition of 0 such that $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary, for all $i = 1, ..., n$. Then, by [8, 2.4],

$$0 : H^d_{\mathfrak{a}}(R) = \cap^r_{j=1}\mathfrak{q}_{i_j}, \tag{2}$$

for some $\{i_1, ..., i_r\} \subset \{1, ..., n\}$. Therefore, by 2.8, $\sqrt{0 : H^d_{\mathfrak{a}}(R)}$ is a linked ideal.

(ii) Let R be unmixed. Hence, by theorem 2.12 and (2), $0 : H^d_{\mathfrak{a}}(R)$ is a linked ideal. Also, by [2, 8.2.6] and the fact that $d > 0$, Att $H^d_{\mathfrak{a}}(R) \subseteq$ Ass $R - V(\mathfrak{a})$. Let $\mathfrak{p} \in$ Ass $R - V(\mathfrak{a})$. Then, $\mathfrak{p} \supseteq \mathfrak{b}$ and by the assumption $\sqrt{\mathfrak{a}+\mathfrak{p}} = \mathfrak{m}$. This implies that $\mathfrak{p} \in$ Att $H^d_{\mathfrak{a}}(R)$. Therefore, Att $H^d_{\mathfrak{a}}(R)$ = Ass $R - V(\mathfrak{a})$ and

$$0 : H^d_{\mathfrak{a}}(R) = \bigcap^{n}_{i=1,\mathfrak{p}_i\not\supseteq\mathfrak{a}} \mathfrak{q}_i. \tag{3}$$

Similarly, Att $H^d_{\mathfrak{b}}(R)$ = Ass $R - V(\mathfrak{b})$. We claim that $H^d_{\mathfrak{b}}(R) \neq 0$. Suppose the contrary, i.e. Ass $R = V(\mathfrak{b})$. So, by [9, Proposition 5. p594], there are some $\mathfrak{p} \in$ Ass R such that $\mathfrak{p} \supseteq \mathfrak{a}+\mathfrak{b}$. It follows from the assumption that $\mathfrak{p} = \mathfrak{m}$ which is a contradiction. Then,

$$0 : H^d_{\mathfrak{b}}(R) = \bigcap^{n}_{i=1,\mathfrak{p}_i\not\supseteq\mathfrak{b}} \mathfrak{q}_i. \tag{4}$$

On the other hand, let $\mathfrak{a} = \cap^k_{i=1}Q_i$ and $\mathfrak{b} = \cap^l_{j=1}Q'_j$ be the minimal primary decompositions of $\mathfrak{a}$ and $\mathfrak{b}$. Then, by the fact that Ass $R \cap V(\mathfrak{a}+\mathfrak{b}) = \emptyset$, $\mathfrak{a}$ and $\mathfrak{b}$ are geometrically linked and so $\mathfrak{a} \cap \mathfrak{b} = 0$. Hence, $0 = \cap^r_{i=1}Q_i \bigcap \cap^l_{j=1}Q'_j$ is another minimal primary decompositions of 0 and using the second uniqueness theorem, without lose of generality, one may assume that $\mathfrak{a} = \cap_{i=1}^{r}\mathfrak{q}_i$ and $\mathfrak{b} = \cap_{i=r+1}^{n}\mathfrak{q}_i$. Now, let $\mathfrak{p}_i \not\supseteq \mathfrak{a}$, for some $i = 1, ..., n$. Then, $\mathfrak{q}_i \not\supseteq \mathfrak{a}$ and so $i > r$ and $\mathfrak{q}_i \supseteq \mathfrak{b}$. Also, if $\mathfrak{q}_i \supseteq \mathfrak{b}$, for some $i = 1, ..., n$, then $\mathfrak{q}_i \not\supseteq \mathfrak{a}$, else $\mathfrak{p}_i \supseteq \mathfrak{a}$ and $\mathfrak{p}_i \in \text{Ass } R \cap V(\mathfrak{a}+\mathfrak{b}) = \emptyset$. Hence

$$\{\mathfrak{q}_i | \mathfrak{q}_i \supseteq \mathfrak{b}\} = \{\mathfrak{q}_i | \mathfrak{q}_i \not\supseteq \mathfrak{a}\}.$$

This implies that $\mathfrak{b} = \cap_{i=1,\mathfrak{p}_i \not\supseteq \mathfrak{a}}^{n}\mathfrak{q}_i$ and $\mathfrak{a} = \cap_{i=1,\mathfrak{p}_i \not\supseteq \mathfrak{b}}^{n}\mathfrak{q}_i$. Now, the result follows from (3) and (4).

□

3. CHARACTERIZATION OF SOME SPECIAL RINGS IN TERMS OF LINKAGE

In this section, we characterize Cohen-Macaulay, Gorenstein and regular local rings in terms of the linked ideals.

Proposition 3.1. *Let R be a Cohen-Macaulay ring. Then*

(i) $\mathfrak{p}$ *is a linked ideal, for all* $\mathfrak{p} \in \text{Spec } R - \{0\}$.

(ii) $\mathfrak{p}_1 \cap \mathfrak{p}_2$ *is a linked ideal, for all* $\mathfrak{p}_1, \mathfrak{p}_2 \in \text{Spec } R$ *with* $\mathfrak{p}_1 \cap \mathfrak{p}_2 \neq 0$ *and* $\text{ht } \mathfrak{p}_1 = \text{ht } \mathfrak{p}_2$.

Proof. (i) Let $\mathfrak{p} \in \text{Spec } R - \{0\}$ and $t := \text{ht } \mathfrak{p}$. Then there exists an R-regular sequence $x_1, ..., x_t$ in $\mathfrak{p}$ and $\mathfrak{q} \in \text{Ass }_R(\frac{R}{(x_1,...,x_t)})$ such that $\mathfrak{p} \subseteq \mathfrak{q}$. Via the assumption, $\mathfrak{p} = \mathfrak{q}$ and, by 2.8, $\mathfrak{p}$ is a linked ideal.

(ii) Let $\mathfrak{p}_1, \mathfrak{p}_2 \in \text{Spec } R$ such that $\mathfrak{p}_1 \cap \mathfrak{p}_2 \neq 0$ and $t := \text{ht } \mathfrak{p}_1 = \text{ht } \mathfrak{p}_2$. Then there exists a R-regular sequence $x_1, ..., x_t \in \mathfrak{p}_1 \cap \mathfrak{p}_2$. By the proof of (i), $\mathfrak{p}_1, \mathfrak{p}_2 \in \text{Ass }_R(\frac{R}{(x_1,...,x_t)})$ and the assertion follows, again, from 2.8.

□

As a corollary of the above proposition and 2.8(i), one can characterize the linked radical ideals in a Cohen-Macaulay ring. Recall that M is said to be relative Cohen-Macaulay with respect to $\mathfrak{a}$ if $H_{\mathfrak{a}}^{i}(M) = 0$ for all $i \neq \text{grade }_M\mathfrak{a}$.

Corollary 3.2. *Let R be a Cohen-Macaulay ring. Then*

(i) *A radical ideal $\mathfrak{a}$ is a linked ideal in R if and only if $\mathfrak{a}$ is unmixed.*

(ii) *If R is relative Cohen-Macaulay with respect to the radical ideal $\mathfrak{a}$ then $\mathfrak{a}$ is linked.*

In theorem 3.5, we will show that, in a certain case, part (i) of the above corollary characterize Cohen-Macaulay rings.

Example 3.3. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay complete local ring and M be a finitely generated R-module. Then, by 3.1(i), every non-zero prime ideal of* Supp M *is a linked ideal. Also, by* [2, 7.2.11(ii) and 7.3.2]*,*

$$\sqrt{\operatorname{Ann} H_{\mathfrak{m}}^{\dim M}(M)} = \bigcap_{\mathfrak{p} \in \operatorname{Att} H_{\mathfrak{m}}^{\dim M}(M)} \mathfrak{p} = \bigcap_{\mathfrak{p} \in \operatorname{Assh} M} \mathfrak{p}$$

is an unmixed ideal. Therefore, if $\operatorname{Ann} H_{\mathfrak{m}}^{\dim M}(M) \neq 0$ *then, by 3.2(i),* $\sqrt{\operatorname{Ann} H_{\mathfrak{m}}^{\dim M}(M)}$ *is a linked ideal.*

In spite of the proposition 3.1, there are non-Cohen-Macaulay rings for which every prime ideal is linked.

Example 3.4. *Let R be a one dimensional ring with* $\operatorname{depth} R = 0$. *Then* $\operatorname{Spec} R = \operatorname{Max} R \cup \operatorname{Min} R$ *and, by 2.8(ii) and (iii), every prime ideal of R is linked.*

In the rest of this section, we classify regular, Gorenstein and Cohen-Macaulay rings in terms of their linked ideals.

Theorem 3.5. *Let $(R, \mathfrak{m})$ be a local ring and* $\operatorname{Ass} M = \operatorname{Min} \operatorname{Ass} M$. *Then the following statements are equivalent.*

(i) *M is Cohen-Macaulay.*

(ii) *$\frac{M}{\mathfrak{a}M}$ is an unmixed module for all ideals $\mathfrak{a}$ which are linked over M.*

Proof. "*(i)* $\Rightarrow$ *(ii)*" Let $\mathfrak{a}$ be an ideal which is linked by the ideal I generating by an M-regular sequence over M and let $\mathfrak{p} \in \operatorname{Ass}_R \frac{M}{\mathfrak{a}M}$. Then, by 2.3, $\mathfrak{p} \in \operatorname{Ass}_R \frac{M}{IM}$. Via Cohen-Macaulayness of $\frac{M}{IM}$, $\dim \frac{R}{\mathfrak{p}} =$

$\dim \frac{M}{IM}$. On the other hand $\dim \frac{M}{IM} \geq \dim \frac{M}{\mathfrak{a}M}$. Putting together both of the estimates, the desired equality is shown to be true.

"$(ii) \Rightarrow (i)$" In the case where $\mathfrak{m} \in \operatorname{Ass} M$, clearly, M is Cohen-Macaulay. So, assume that depth $M > 0$. Let $t \in \mathbb{N}$, $x_1, ..., x_t$ be an M-regular sequence and $\mathfrak{p} \in \operatorname{Ass}_R(\frac{M}{(x_1,...,x_t)M})$. As $(x_1, ..., x_t) \sim_{((x_1^2,...,x_t);M)} (x_1, ..., x_t)$, by the assumption, $\frac{M}{(x_1,...,x_t)M}$ is unmixed. Therefore,

$$\dim M - t \geq \dim M - \operatorname{ht}_M \mathfrak{p} \geq \dim \frac{R}{\mathfrak{p}} = \dim \frac{M}{(x_1, ..., x_t)M} = \dim M - t.$$

This implies that $\operatorname{ht}_M \mathfrak{p} = t$ and, hence, M is Cohen-Macaulay.

$\square$

Corollary 3.6. *Let* $(R, \mathfrak{m})$ *be a local ring and* $\operatorname{Ass} R = \operatorname{Min} R$. *Then the following statements are equivalent.*

(i) *R is Cohen-Macaulay.*
(ii) *For any ideal I which is generated by an R-regular sequence, the horizontally linked $\frac{R}{I}$-modules are unmixed.*
(iii) *Any linked ideal is unmixed.*

In "M.T. Dibaei and Y. Khalatpour, Characterizations of generically Gorenstein and Gorenstein local rings, *arxiv: 1708.07948*" a characterization of Gorenstein local rings is presented in terms of the "generically linked" ideals, provided R is a Cohen-Macaulay ring.

In the following, we have a general characterization without the assumption that R is Cohen-Macaulay.

Theorem 3.7. *Let* $(R, \mathfrak{m})$ *be a local ring. Then the following are equivalent.*

(i) *R is Gorenstein.*
(ii) *For any ideal I which is generated by an R-regular sequence, the unmixed stable $\frac{R}{I}$-module M with* $\operatorname{grade} \operatorname{Ann} M = \operatorname{grade} I$ *is horizontally linked as $\frac{R}{I}$-module.*
(iii) *Any unmixed ideal $\mathfrak{a}$ is linked by every maximal R-regular sequence $(\mathfrak{x})$ with $(\mathfrak{x}) \subset \mathfrak{a}$.*

Proof. "$(i) \Rightarrow (ii)$" It follows from [9, Corollary 9. p.601].

"$(ii) \Rightarrow (iii)$" It is clear.

"$(iii) \Rightarrow (i)$" We proceed by induction on $d := \dim R$. Let $d = 0$. Then every non-zero ideal $\mathfrak{a}$ of R is unmixed of grade zero and, by the assumption, $0 :_R (0 :_R \mathfrak{a}) = \mathfrak{a}$. Therefore, in view of [3, 3.2.15], R is Gorenstein.

Now assume that $d > 0$ and the assertion has been proved for all local ring of dimension $< d$. We claim that depth $R > 0$. Assume to the contrary that $\mathfrak{m} \in \operatorname{Ass} R$. Then, by the assumption, $\mathfrak{m}^j$ is linked by the zero ideal and $0 :_R (0 :_R \mathfrak{m}^j) = \mathfrak{m}^j$ for all $j \in \mathbb{N}$. On the other hand, there is $i \in \mathbb{N}$ such that $0 :_R \mathfrak{m}^i = 0 :_R \mathfrak{m}^{i+1}$. This implies that $\mathfrak{m}^i = 0$ and $d = 0$, which is a contradiction.

Now, let $x \in \mathfrak{m} - Z(R)$, $\overline{\mathfrak{a}}$ be an unmixed ideal of grade l and $\overline{y_1}, ..., \overline{y_l}$ be an arbitrary maximal $\overline{R}$-regular sequence in $\overline{\mathfrak{a}}$ such that $\overline{\mathfrak{a}} \neq (\overline{y_1}, ..., \overline{y_l})$, where $- : R \to \frac{R}{Rx}$ is the natural homomorphism. Then, by [10, Exersice 6.7], $\mathfrak{a}$ is an unmixed ideal of grade $l+1$, and, by the assumption,

$$(x, y_1, ..., y_l) :_R ((x, y_1, ..., y_l) :_R \mathfrak{a}) = \mathfrak{a}.$$

In other words,

$$(\overline{y_1}, ..., \overline{y_l}) :_{\overline{R}} ((\overline{y_1}, ..., \overline{y_l}) :_{\overline{R}} \overline{\mathfrak{a}}) = \overline{\mathfrak{a}}.$$

This means that $\overline{\mathfrak{a}}$ is a linked ideal by $(\overline{y_1}, ..., \overline{y_l})$. Now, using the inductive hypothesis, $\overline{R}$, and so R, is Gorenstein.

□

As another consequence of 2.5, one can also characterize the regular local rings.

Proposition 3.8. *A local ring $(R, \mathfrak{m})$ is regular if and only if there exists a maximal R-regular sequence $x_1, ..., x_t$ such that $\mathfrak{m}$ is not linked by $(x_1, ..., x_t)$.*

Proof. Let R be a regular local ring and set $t := \dim R$. Then, there exists a R-regular sequence $x_1, ..., x_t$ such that $\mathfrak{m} = (x_1, ..., x_t)$. Therefore, $\mathfrak{m}$ is not linked by $(x_1, ..., x_t)$.

Now, assume that there exists a maximal R-regular sequence $x_1, ..., x_t$ such that $\mathfrak{m}$ is not linked by $(x_1, ..., x_t)$. As $\mathfrak{m} \in \operatorname{Ass} \frac{R}{(x_1,...,x_t)}$, by 2.5 (i), $\mathfrak{m} = (x_1, ..., x_t)$. Therefore, R is a regular ring.

□

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