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Stability for coupled systems on networks with Caputo-Hadamard fractional derivative

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Abstract. This paper discusses stability and uniform asymptotic stability of the trivial solution of the following coupled systems of fractional differential equations on networks

$$\begin{cases} {}^{cH}D^{\alpha}x_i = f_i(t, x_i) + \sum\limits_{j=1}^{n} g_{ij}(t, x_i, x_j), \quad t > t_0, \\ x_i(t_0) = x_{i0}, \end{cases}$$

where ${}^{cH}D^{\alpha}$ denotes the Caputo-Hadamard fractional derivative of order α, $1 < \alpha \leq 2$, $i = 1, 2, \ldots, n$, and $f_i : \mathbb{R}_+ \times \mathbb{R}^{m_i} \to \mathbb{R}^{m_i}$, $g_{ij} : \mathbb{R}_+ \times \mathbb{R}^{m_i} \times \mathbb{R}^{m_j} \to \mathbb{R}^{m_i}$ are given functions. Based on graph theory and the classical Lyapunov technique, we prove stability and uniform asymptotic stability under suitable sufficient conditions. We also provide an example to illustrate the obtained results.

Keywords: Fractional differential equation, Caputo-Hadamard, Coupled systems on networks, Lyapunov function.
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1 Introduction

In the past few decades, coupled systems of fractional differential equations on networks (CSFDENs) have been investigated extensively due to their wide applications in different fields such as engineering, physics, epidemiology, signal and image processing, artificial intelligence, pattern classification, etc [22, 27, 33, 34]. A network can be described as a directed graph consisting of vertices and directed arcs connecting them. At each vertex, the local dynamics are given by a system of differential equations called vertex system. The directed arcs indicate interactions between vertex systems.

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In 2010, Li et al. [22] introduced a new method based on graph theory and Lyapunov technique to study the stability and synchronization of neural networks. Since then, this technique has attracted considerable interest [11, 25].

On on other hand, thanks to their ability to model complex phenomena, fractional differential equations (FDEs) have been widely used in engineering, physics, chemistry, biology, and other fields [4–8, 15–20, 24, 28, 31]. FDEs involve fractional derivatives which generalize differentiation to any noninteger order. Many types of fractional derivatives have been proposed in the literature. In this work, we consider the Hadamard fractional derivative [9], modified by Jarad et al. [14] to get physically interpretable initial conditions, similar to the ones of the Caputo setting. For some recent work on the Hadamard fractional derivative and integral, see [1–3, 13, 29, 30].

Suo et al. [27] studied the stability of the following system:

$$\begin{cases} x_i' = f_i(t,x_i) + \sum\limits_{j=1}^{n} g_{ij}(t,x_i,x_j), & t \neq t_k, \\ \Delta x_i = I_k(x_i), & t = t_k,\ k = 1,2,\ldots, \\ x_i(t_0^+) = x_{i0}, & \end{cases}$$

where $i = 1,2,\ldots,n$, $0 < t_1 < t_2 < \cdots < t_k < \cdots$, and $t_k \to \infty$ as $k \to \infty$; f_i is continuous on $(t_{k-1},t_k] \times \mathbb{R}^{mi}$, g_{ij} is continuous on $(t_{k-1},t_k] \times \mathbb{R}^{mi} \times \mathbb{R}^{mj}$, and $I_k \in C[\mathbb{R}^{mi},\mathbb{R}^{mi}]$.

Zhang et al. [32] studied the global stability of the following impulsive coupled system on a digraph G

$$\begin{cases} D^{\mu} x_p = -\omega_p x_p + \sum\limits_{q=1}^{n} a_{pq} f_q(x_q(t)) + \sum\limits_{q=1}^{n} a_{pq}(x_p(t) - x_q(t)), & t \geq 0, \quad t \neq t_k, \\ \Delta x_p(t_k) = I_k(x_p(t_k)), & \\ x(t_k^-) = x_{t_k}, & k = 1,2,\ldots, \end{cases}$$

where D^{μ} is the Caputo fractional derivative of order $0 < \mu < 1$, $p,q = 1,2,\ldots,n$, $f_q(x)$ is a function satisfying the Lipschitz condition.

Li et al. [21] considered the stability of the coupled systems fractional differential equations on networks

$$\begin{cases} {}^{c}D^{q} x_i = f_i(t,x_i) + \sum\limits_{j=1}^{n} g_{ij}(t,x_i,x_j), & t \geq t_0,\ i = 1,2,\ldots,n, \\ x_i(t_0) = x_{i0}, & \end{cases}$$

where ${}^{c}D^{q}$ is the Caputos fractional derivative of order q, $0 < q < 1$, $x_i \in \mathbb{R}^{mi}$ and $f_i : \mathbb{R}_+ \times \mathbb{R}^{mi} \to \mathbb{R}^{mi}$ and $g_{ij} : \mathbb{R}_+ \times \mathbb{R}^{mi} \times \mathbb{R}^{mj} \to \mathbb{R}^{mi}$.

In this paper, we investigate the stability for coupled systems on networks with Caputo-Hadamard fractional derivative, of the form

$$\begin{cases} {}^{cH}D^{\alpha} x_i = f_i(t,x_i) + \sum\limits_{j=1}^{n} g_{ij}(t,x_i,x_j), & t > t_0, \\ x_i(t_0) = x_{i0}, & \end{cases} \tag{1}$$

where ${}^{cH}D^{\alpha}$ is the Caputo Hadamard fractional derivative of order α, $1 < \alpha \leq 2$, $i = 1,2,\ldots,n$, $f_i : \mathbb{R}_+ \times \mathbb{R}^{m_i} \to \mathbb{R}^{m_i}$, $g_{ij} : \mathbb{R}_+ \times \mathbb{R}^{m_i} \times \mathbb{R}^{m_j} \to \mathbb{R}^{m_i}$. We assume that the functions f_i and g_{ij} satisfy the Lipschitz conditions.

The rest of this paper is organized as follows. In Section 2, we provide Definitions, Propositions, Lemmas and some Theorems. Our main results are presented in Section 3. An example is presented in Section 4 to illustrate the feasibility of the obtained results.

2 Preliminaries

Definition 1. *[22] A function f belongs to K if $f \in C[\mathbb{R}_+, \mathbb{R}_+]$, $f(0) = 0$ and f is strictly increasing.*

Definition 2. *[14, 26] The Hadamard integral of order $\alpha > 0$ of a function f is defined for $t > 0$ as*

$$I^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s)\frac{ds}{s},$$

$$I^{\alpha}_{a^{b-}} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \left(\log \frac{s}{t}\right)^{\alpha-1} f(s)\frac{ds}{s}.$$

Definition 3. *[14,26] The Hadamard derivative of fractional order α for a function $f : [1, \infty) \to \mathbb{R}$ is defined as*

$${}^{H}D^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{t}{dt}\right)^n \int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} f(s)\frac{ds}{s}, \quad n-1 < \alpha < n, \ \ n = [\alpha] + 1,$$

where $[\alpha]$ denotes the integer part of the real number α.

Definition 4. *[9, 12] Let $Re(\alpha) \geq 0$, $n = [Re(\alpha) + 1]$ and $f \in AC^n_\delta[a, b]$, $0 < a < b < \infty$, the Caputo type Hadamard derivative of fractional order α is defined as :*

$${}^{cH}D^{\alpha}_{a^+} f(x) = D^{\alpha}_{a^+}\left[f(x) - \sum_{k=0}^{n-1} \frac{\delta^k f(a)}{k!} \log(\frac{t}{a})^k\right](x),$$

$${}^{cH}D^{\alpha}_{b^-} f(x) = D^{\alpha}_{b^-}\left[f(x) - \sum_{k=0}^{n-1} \frac{(-1)^k\delta^k f(b)}{k!} \log(\frac{b}{t})^k\right](x).$$

Here $Re(\alpha) \geq 0$, $n = [Re(\alpha) + 1]$, $0 < a < b < \infty$ and

$$f \in AC^n_\delta[a, b] = \left\{f : [a, b] \to \mathbb{C} : \delta^{(n-1)} f(x) \in AC[a, b], \delta = x\frac{d}{dx}\right\}.$$

In particular, if $0 < Re(\alpha) < 1$, then

$${}^{cH}D^{\alpha}_{a^+} f(x) = {}^{cH}D^{\alpha}_{a^+}[f(x) - f(a)](x),$$

$${}^{cH}D^{\alpha}_{b^-} f(x) = {}^{cH}D^{\alpha}_{b^-}[f(x) - f(b)](x).$$

Theorem 1. *[9, 12] Let $Re(\alpha) \geq 0$, $n = [Re(\alpha) + 1]$ and $f \in AC^n_\delta[a, b]$, $0 < a < b < \infty$. Then ${}^{cH}D^{\alpha}_{a^+} f(x)$ and ${}^{cH}D^{\alpha}_{b^-} f(x)$ exist everywhere on $[a, b]$ and*

1. if $\alpha \notin \mathbb{N}_0$,

$$
{}^{cH}D^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \left(\log\frac{x}{t}\right)^{n-\alpha-1}\delta^n f(t)\frac{dt}{t} = I^{n-\alpha}_{a^+}\delta^n f(x),
$$
$$
{}^{cH}D^{\alpha}_{b^-}f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b \left(\log\frac{t}{x}\right)^{n-\alpha-1}\delta^n f(t)\frac{dt}{t} = (-1)^n I^{n-\alpha}_{a^+}\delta^n f(x),
$$

2. if $\alpha = n \in \mathbb{N}_0$,

$$
{}^{cH}D^{\alpha}_{a^+}f(x) = \delta^n f(x), \quad {}^{cH}D^{\alpha}_{b^-}f(x) = (-1)^n\delta^n f(x).
$$

In particular,

$$
{}^{cH}D^{0}_{a^+}f(x) = {}^{cH}D^{0}_{b^-}f(x) = f(x).
$$

Lemma 1. *[9, 12] Let* $Re(\alpha) \geq 0$, $n = [Re(\alpha)+1]$ *and* $f \in C[a,b]$.
If $Re(\alpha) \neq 0$ *or* $\alpha \in \mathbb{N}$, *then*

$$
{}^{cH}D^{\alpha}_{a^+}(I^{\alpha}_{a^+}f)(x) = f(x), \quad {}^{cH}D^{\alpha}_{b^-}(I^{\alpha}_{b^-}f)(x) = f(x).
$$

Lemma 2. *[9, 12] Let* $f \in AC^n_\delta[a,b]$ *or* $C^n_\delta[a,b]$ *and* $\alpha \in C$, *then*

$$
I^{\alpha}_{a^+}({}^{cH}D^{\alpha}_{a^+}f)(x) = f(x) - \sum_{k=0}^{n-1}\frac{\delta^k f(a)}{k!}\log(\frac{t}{a})^k,
$$
$$
I^{\alpha}_{b^-}({}^{cH}D^{\alpha}_{b^-}f)(x) = f(x) - \sum_{k=0}^{n-1}\frac{(-1)^k\delta^k f(b)}{k!}\log(\frac{b}{t})^k.
$$

In the following, we gather together some basic concepts and theorems on graph theory (see [22, 27]). Let $G = [V, E]$ be a non-empty directed graph, i.e., $V = 1, 2, \ldots, n$ is a set of vertices and E is a set of edges whose elements are arcs (i, j) leading from initial vertex i to the terminal vertex j. A subgraph H of G is said to be spanning if H contains all vertices of G. A digraph G is weighted if each arc (i, j) is assigned a positive weight a_{ij} where $a_{ij} > 0$ if and only if there exists an arc from vertex j to vertex i in G. A directed path is a subgraph $P = (I, X)$ of the form $I = i_1, i_2 \ldots, i_m$, $X = \{(i_k, i_{k+1}) : k = 1, 2, \ldots, m\}$ where the i_k are all distinct. If P is closed, namely $i_m = i_1$, we say that P is a directed cycle. A connected subgraph T is a tree if it has no cycles. A tree T is rooted at vertex i, called the root, if i is not a terminal vertex of any arcs, and each of the remaining vertices is a terminal vertex of exactly one arc. A graph G is said to be strongly connected if from one vertex to other vertex there is a directed path.

Given a weighted digraph G with n vertices, we define the weight matrix $A = (a_{ij})_{n\times n}$ whose elements a_{ij} are the weight of arc (j, i), and we write (G, A). We define the weight $W(G)$ of G as the product of the weights of all its arcs. A weighted digraph (G, A) is said to be balanced if $W(C) = W(-C)$ for each directed cycle C. Here $-C$ denotes the reverse of C and is constructed by reversing the direction of all arcs in C. If (G, A) is balanced, then $W(Q) = W(-Q)$. The

Laplacian matrix of (G, A) is defined as

$$L = \begin{bmatrix} \sum\limits_{k \neq 1} a_{1k} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \sum\limits_{k \neq 2} a_{2k} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & \sum\limits_{k \neq n} a_{nk} \end{bmatrix}$$

Proposition 1. *[22] Assume $n \geq 2$. Then*

$$c_i = \sum_{\mathcal{T} \in \mathbb{T}_i} \omega(\mathcal{T}) \quad i = 1, 2, \ldots, n, \tag{2}$$

where $\mathbb{T}_i$ is the set of all spanning trees $\mathcal{T}$ of (G, A) that are rooted at vertex i, and $\omega(\mathcal{T})$ is the weight of $\mathcal{T}$. In particular, if (G, A) is strongly connected, then $c_i > 0$ for $1 \leq i \leq n$.

Theorem 2. *[22] Assume $n \geq 2$. Let c_i as defined in Proposition 1. Then the following identity holds:*

$$\sum_{i,j=1}^{n} c_i a_{ij} F_{ij}(x_i, x_j) = \sum_{Q \in \mathbb{Q}} w(Q) \sum_{(s,r) \in E_{c_Q}} F_{rs}(x_r, x_s). \tag{3}$$

Here $F_{ij}(x_i, x_j)$, $1 \leq i, j \leq n$, are arbitrary functions, Q is the set of all spanning unicyclic graphs of (G, A), $w(Q)$ is the weight of Q, and C_Q denotes the directed cycle of Q.

3 Stability analysis for coupled systems of fractional differential equations on networks

Consider a network represented by digraph G with n vertices $(n \geq 2)$. Assume that the i-th vertex dynamic is described by a system of fractional differential equations as follows:

$$\begin{cases} {}^{cH}D^{\alpha} x_i = f_i(t, x_i), \quad t > t_0, \;\; i \in I, \\ x_i(t_0) = x_{i0}, \end{cases} \tag{4}$$

where α, $1 < \alpha \leq 2$, $\;x_i \in \mathbb{R}^{mi}$ and $f_i \;: \mathbb{R}_+ \times \mathbb{R}^{mi} \to \mathbb{R}^{mi}$. Let $g_i \;: \mathbb{R}_+ \times \mathbb{R}^{mi} \times \mathbb{R}^{mj} \to \mathbb{R}^{mi}$ represents the influence of the vertex j on vertex i and $g_{ij} = 0$, if there exists no arc from j to i in G. Then we obtain the following coupled system on graph G:

$$\begin{cases} {}^{cH}D^{\alpha} x_i = f_i(t, x_i) + \sum\limits_{j=1}^{n} g_{ij}(t, x_i, x_j), \quad t > t_0, \\ x_i(t_0) = x_{i0}, \end{cases} \tag{5}$$

$i = 1, 2, \ldots, n$. Here functions f_i and g_{ij} satisfy the global Lipschitz conditions so that initial-value problem (5) has a unique solution. Equation (5) has a trivial solution $(x_1, x_2, \ldots, x_n) = 0$ for $t \geq t_0$.

Let $V_i(t,u_i)$ be a Lyapunov function for each vertex system (4). We are particularly interested in constructing Lyapunov functions for coupled system (5) of form

$$V(t,x)=\sum_{i=1}^{n}c_iV_i(t,x_i).$$

The following result gives a general and systematic approach for such construction.

Theorem 3. *Suppose that the following assumptions are satisfied.*

1. *There exist functions $V_i(t,x_i), F_{ij}(x_i,x_j)$, and a matrix $A=(a_{ij})_{n\times n}$ in which $a_{ij}>0$ such that*
$$^{cH}D^{\alpha}V_i(t,x_i)\leq\sum_{i=1}^{n}a_{ij}F_{ij}(x_i,x_j)\ \ t>t_0,\ x_i\in\mathbb{R}^{mi},\ 1<i<n. \tag{6}$$
2. *Along each directed cycle C of the weighted digraph (G,A)*
$$\sum_{(s,r)\in E(c)}a_{ij}F_{rs}(x_r,x_s)\leq 0\ ;\ t\geq t_0\ ,x_r\in\mathbb{R}^{mr},x_s\in\mathbb{R}^{ms}. \tag{7}$$
3. *Constants c_i are given in (2).*

Then function $V(t,x)=\sum\limits_{i=1}^{n}c_iV_i(t,x_i)$ is a Lyapunov function for (5) and $V(t,x)$ satisfies $^{cH}D^{\alpha}V(t,x)\leq 0$ for $t\geq t_0$ and $x\in\mathbb{R}^m$.

Proof. For $V(t,x)=\sum\limits_{i=1}^{n}c_iV_i(t,x_i)$, we have

$$\begin{aligned}^{cH}D^{\alpha}V(t,x)&={}^{cH}D^{\alpha}\sum_{i=1}^{n}c_i.V_i(t,x_i),\\&\leq\sum_{i=1}^{n}c_i^{cH}D^{\alpha}V_i(t,x_i).\end{aligned}$$

According to condition (1), we have

$$^{cH}D^{\alpha}V(t,x)\leq\sum_{i=1}^{n}\sum_{j=1}^{n}c_ia_{ij}F_{ij}(t,x_i,x_j).$$

Applying Theorem 2

$$^{cH}D^{\alpha}V(t,x)=\sum_{Q\in\mathbb{Q}}w(Q)\sum_{(j,i)\in E_{c_Q}}F_{ij}(t,x_i,x_j).$$

According to condition (2) and $w(Q)>0$, we have

$$^{cH}D^{\alpha}V(t,x)\leq 0.$$

The proof is therefore complete. □

Note that if (G, A) is balanced, then

$$\sum_{i,j=1}^{n} c_i a_{ij} F_{ij}(x_i, x_j) \leq \frac{1}{2} \sum_{Q \in \Omega} w(Q) \sum_{(s,r) \in E_{c_Q}} [F_{rs}(x_r, x_s) + F_{sr}(x_s, x_r)] .$$

Proposition 2. *Suppose that (G, A) is balanced. Then the conclusion of Theorem 3 holds if condition (2) is replaced by the following one:*

$$[F_{rs}(x_r, x_s) + F_{sr}(x_s, x_r)] \leq 0 \quad t \geq t_0 \, , x_r \in \mathbb{R}^{mr}, x_s \in \mathbb{R}^{ms}. \tag{8}$$

Theorem 4. *Assume the following conditions are satisfied.*

- *There exist functions $V_i \in C^1[\mathbb{R}^+ \times D_i, \mathbb{R}^+]$, $F_{ij}(t, x_i, x_j)$ a matrix $A = (a_{ij})_{n \times n}$ in which $a_{ij} \leq 0$ and $b_i > 0$ such that for $i = 1, 2, \ldots, n$*

$$^{cH}D^{\alpha} V_i(t, x_i) \leq -b_i V_i(t, x_i) + \sum_{j=1}^{n} a_{ij} F_{ij}(x_i, x_j) \ \ t > t_0. \tag{9}$$

- *Either (2) holds, or if (G, A) is balanced and (8) holds.*
- *There exists a $\delta_0^{(i)} > 0$ and a function $d_i \in K$ such that*

$$V_i(t, x_i) \leq d_i(\| x_i \|), \quad \textit{provided} \quad \| x_i \| < \delta_0^{(i)}. \tag{10}$$

- *Constants c_i are given in (2).*

Then, the function $V(t, x) = \sum_{i=1}^{n} c_i V_i(t, x_i)$ is a Lyapunov function for (5) and the trivial solution of (5) is uniformly asymptotically stable.

Proof. For $V(t, x) = \sum_{i=1}^{n} c_i V_i(t, x_i)$, according to condition (2) and (9), we have

$$\begin{aligned}
^{cH}D^{\alpha} V(t, x) &= ^{cH} D^{\alpha} \sum_{i=1}^{n} c_i V_i(t, x_i), \\
&\leq \sum_{i=1}^{n} c_i^{cH} D^{\alpha} V_i(t, x_i), \\
&\leq \sum_{i=1}^{n} c_i \left[-b_i V_i(t, x_i) + \sum_{j=1}^{n} a_{ij} F_{ij}(x_i, x_j) \right], \\
&\leq - \sum_{i=1}^{n} c_i b_i V_i(t, x_i), \\
&\leq -b V(t, x),
\end{aligned}$$

where $b = \min\{b_1, b_2, \ldots, b_n\}$. So that the trivial solution is asymptotically stable. On the other hand, as $d_i \in K$, we get (10) independent of t. Therefore the number δ can be chosen independent of t_0. Define

$$\begin{aligned}\delta_0 &= \min\{\delta_0^{(1)}, \delta_0^{(2)}, \ldots, \delta_0^{(n)}\},\\ b(\| x \|) &= n \min\{c_1 b_1(\| x_1 \|), c_2 b_2(\| x_2 \|), \ldots, c_n b_n(\| x_n \|)\},\end{aligned}$$

and

$$d(\| x \|) = n \min\{c_1 d_1(\| x_1 \|), c_2 d_2(\| x_2 \|), \ldots, c_n d_n(\| x_n \|)\}.$$

For every $\varepsilon > 0$, there exists $0 < \delta(\varepsilon) < \delta_0$ such that $d(\| x \|) < b(\varepsilon)$ provided that $\| x \| < \delta$. If $\| x \| < \delta$, then according to (10), we have

$$V(t, x) = \sum_{i=1}^{n} c_i V_i(t, x_i) \leq \sum_{i=1}^{n} c_i d_i(\| x_i \|) \leq \sum_{i=1}^{n} \frac{1}{n} d(\| x_0 \|) \leq b(\varepsilon).$$

Since $V_i(t, x_i)$ is a positive definite function, we deduce that there exists $b_i(.) \in K$ such that

$$V_i(t, x_i) \geq b_i(\| x_i \|).$$

Then

$$V(t, x) = \sum_{i=1}^{n} c_i V_i(t, x_i) \geq \sum_{i=1}^{n} c_i b_i(\| x_i \|) \geq \sum_{i=1}^{n} \frac{1}{n} b(\| x \|) = b(\| x \|).$$

So, we have

$$b(\| x \|) \leq V(t, x) \leq b(\varepsilon).$$

Then $\| x \| \leq \varepsilon$. This implies that the trivial solution of (5) is uniformly stable. We conclude that the trivial solution of (5) is uniformly asymptotically stable. □

4 Example

We consider the following coupled system of fractional differential equation on digraph G:

$$\begin{cases} {}^{cH}D^{\alpha} x_i = -\omega_i x_i + f_i(x_i) + \sum\limits_{j=1}^{n} \beta_{ij}(x_i - |x_j|), \\ x_i(t_0) = x_{i0}, \end{cases} \tag{11}$$

$i, j = 1, \ldots, n$, $0 < \alpha < 1$, $\omega_i > 0$, where x_i is n-dimensional column vectors, f_i is continuous and there exists a Lipschitz constant $L_i > 0$ such that $|f_i(x_i) - f_i(y_i) \leq L_i |x_i - y_i|$ for all $x_i \neq y_i$ In addition, $f_i(0) = 0$, $\beta_{ij} \leq 0$, $\beta_{ij} = -\beta_{ji}$ and $\beta_{ij} \neq 0$ if $i \neq j$.

Suppose that the following conditions hold:

1. (G, A) is strongly connected and balanced.

2. $\gamma_i = \omega_i - L_i > 0$, $i = 1, 2, \ldots, n$.

Then the trivial solution of system (11) is uniformly asymptotically stable.

Proof. Let us consider $V_i(t, x_i(t)) = |x_i(t)|$, then we get

$$\mu|x_i(t)| \geq |x_i(t)| \text{ for all } \quad \mu < 1,$$

and, there exists $d_i \in K$ such that

$$V_i(t, x_i(t)) \leq d_i(|x_i(t)|) = \mu|x_i(t)|, \ \mu \geq 1.$$

Therefore (10) holds.

If $x_i(t) = 0$, then ${}^{cH}D^{\alpha}|x_i| = 0$. If $x_i(t) > 0$, then

$$\begin{aligned}
{}^{cH}D^{\alpha}|x_i(t)| =& \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \left(s\frac{d}{ds}\right)^n x(s)\frac{ds}{s}, \\
=& {}^{cH}D^{\alpha}x_i(t).
\end{aligned}$$

If $x_i(t) < 0$, then

$$\begin{aligned}
{}^{cH}D^{\alpha}|x_i(t)| =& -\frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \left(s\frac{d}{ds}\right)^n x(s)\frac{ds}{s}, \\
=& -{}^{cH}D^{\alpha}x_i(t).
\end{aligned}$$

Therefore ${}^{cH}D^{\alpha}|x_i(t)| = sgn(x_i(t)){}^{cH}D^{\alpha}x_i(t)$. According to (11), we have

$$\begin{aligned}
{}^{cH}D^{\alpha}|x_i(t)| =& sgn(x_i(t)){}^{cH}D^{\alpha}x_i(t), \\
=& sgn(x_i(t)) \left(-\omega_i x_i + f_i(x_i) + \sum_{j=1}^{n} \beta_{ij}(x_i - |x_j|)\right), \\
=& -\omega_i|x_i| + f_i(|x_i|) + \sum_{j=1}^{n} \beta_{ij}(|x_i| - |x_j|), \\
\leq& -\omega_i|x_i| + L_i|x_i| + \sum_{j=1}^{n} \beta_{ij}(|x_i| - |x_j|), \\
\leq& \left(-\omega_i + L_i\right)|x_i| + \sum_{j=1}^{n} \beta_{ij}(|x_i| - |x_j|), \\
\leq& -\gamma_i V_i(t, x_i) + \sum_{j=1}^{n} a_{ij} F_{ij}(x_i, x_j),
\end{aligned}$$

where $F_{ij}(x_i, x_j) = sgn(\beta_{ij})(|x_i| - |x_j|)$.

It is easy to show that

$$\begin{aligned}
F_{ij}(x_i, x_j) =& sgn(\beta_{ij})(|x_i| - |x_j|), \\
=& -sgn(\beta_{ji})(|x_j| - |x_i|), \\
=& -F_{ji}(x_j, x_i).
\end{aligned}$$

Thus along each directed cycle C of the weighted digraph (G, A)

$$\sum_{(i,j)\in E(c_Q)} [F_{ij}(x_i, x_j) + F_{ji}(x_j, x_i)] = 0.$$

According to Theorem 4, we can conclude that (11) is uniformly asymptotically stable. □

5 Conclusion

In this work, we applied results from graph theory and the Lyapunov method to study stability and uniform asymptotic stability of the trivial solution. We adapted a systematic approach that allowed us to construct global Lyapunov functions for large-scale coupled systems from building blocks of individual vertex systems. The approach is successfully applied to coupled systems on networks with Caputo-Hadamard fractional derivative

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