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Caputo-Hadamard fractional differential equation with impulsive boundary conditions

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Abstract. This manuscript is concerned about the study of the existence and uniqueness of solutions for fractional differential equation involving Caputo Hadamard fractional operator of order $1 < \vartheta \leq 2$ with impulsive boundary conditions. The existence results are established firstly through the Banach Contraction Principle and then using Schauder's fixed point theorem. We present some examples to demonstrate the application of our main results.

Keywords: Boundary value problem, impulses, Caputo-Hadamard fractional derivative, fixed point theorem.

AMS Subject Classification 2010: 26A33, 34B15.

1 Introduction

The theory of fractional differential equations is an advanced and more generalized version of differential equation theory. Boundary value problems of fractional order have numerous applications in applied physics, biological, engineering, and chemical background. For the fundamental study of fractional systems, one can go through the books [6,10,13,19,21] and references therein. There are an immense number of papers related to differential equations of arbitrary order with initial and boundary conditions being published but still far less work has been done to develop the existence of solutions for fractional order differential equations with boundary conditions. The differential equations of fractional order involving Riemann-Liouville and Caputo fractional derivatives have recently been studied by many authors [2–4,14,18,22]. On the other hand, the literature on fractional differential equations involving the Hadamard derivative has been comparatively less explored. Formally Riemann-Liouville fractional derivative is a fractional power $(\frac{d}{dx})^{\alpha}$ of the differentiation $\frac{d}{dx}$ and is invariant with respect to translation on the whole axis [20].

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Later, Hadamard [11] suggested a fractional power of the form $(x\frac{d}{dx})^{\alpha}$. This construction is well suited to the case of the half-axis and is invariant relative to dilation [20]. Furthermore, Jarad et al. [12] modified the Hadamard fractional derivative into a more suitable one having physically interpretable initial conditions similar to the ones in the Caputo setting. For more theories and properties of Hadamard integral and derivative, we refer the reader to [8, 16, 17, 25, 27] and the references cited therein.

The study of the impulsive boundary value problems has been evolved over the last few decades. Also, it has been very helpful in the development of the various applied mathematical models of real-world processes occurring in engineering and applied sciences. An Abundance of literature is available on impulsive boundary value problems, for instance, see the monographs [5,7,9,15,24,26] and references therein. Impulsive differential equations with boundary conditions are used to understand the processes in which abrupt changes and discontinuous jumps occur.

In 2013, Tian and Bai [23] studied a sufficient condition for the existence of solutions to the impulsive boundary value problem involving the Caputo fractional derivative. They established the existence and uniqueness results using Krasnoselskii fixed point theorem and Banach fixed point theorem. Recently, W. Yukunthorn et. al. [28] considered impulsive multi-order boundary value problem involving Caputo-Hadamard fractional differential operator. The necessary conditions for the existence of solutions are established by Rothe Fixed Point theorem, Banach fixed point theorem and degree theory. In these papers, existence results of various types of dynamical systems with fractional-order have been established. However, to the study of the existence of solutions of dynamical systems, fixed-point technique has been effectively utilized. Motivated by this fact, we use Schauder's fixed point theorem to study the existence of solutions of Caputo-Hadamard fractional differential equation with impulsive boundary condition of the following form

$$\begin{cases} {}^{CH}D^{\vartheta}_{\zeta_k}\mathfrak{u}(\zeta) = \mathfrak{f}(\zeta, \mathfrak{u}(\zeta)), \quad \zeta,\, \zeta_k \in [1, e],\ \zeta \neq \zeta_k, \\ \triangle \mathfrak{u}(\zeta_k) = \mathfrak{A}_k(\mathfrak{u}(\zeta_k)),\ k = 1, 2, \cdots, m, \\ \triangle \delta \mathfrak{u}(\zeta_k) = \overline{\mathfrak{A}_k}(\mathfrak{u}(\zeta_k)), \\ \mathfrak{u}(1) = \mathfrak{h}(\mathfrak{u}),\ \ \mathfrak{u}(e) = \mathfrak{g}(\mathfrak{u}), \end{cases} \tag{1}$$

where ${}^{CH}D^{\vartheta}_{\zeta_k}$ is the Caputo-Hadamard fractional derivative of order $1 < \vartheta \leq 2$. Let $\mathfrak{f} : [1, e]\times\mathbb{R} \to \mathbb{R}$, $\mathfrak{A}_k, \overline{\mathfrak{A}_k} : PC(J, \mathbb{R}) \to \mathbb{R}$ be continuous functions and $\triangle \mathfrak{u}(\zeta_k) = \mathfrak{u}(\zeta_k^+) - \mathfrak{u}(\zeta_k^-)$, $\triangle\delta\mathfrak{u}(\zeta_k) = \delta\mathfrak{u}(\zeta_k^+) - \delta\mathfrak{u}(\zeta_k^-)$ and $\delta = \zeta\frac{d}{d\zeta}$. Also, $\mathfrak{u}(\zeta_k^+)$ and $\mathfrak{u}(\zeta_k^-)$ are the right limit and left limit of $\mathfrak{u}(\zeta)$ at $\zeta = \zeta_k$ respectively and $\mathfrak{g}, \mathfrak{h} : PC(J, \mathbb{R}) \to \mathbb{R}$ are any fixed continuous functionals, where $PC(J, \mathbb{R})$ is the space of piece-wise continuous functions.

This manuscript is organized as follows. In Section 2, we present a few useful definitions and lemmas. In Section 3, we present two main results based on Banach Contraction Principle and Schauder's fixed point theorem. In Section 4, we present some examples to demonstrate the application of our main results.

2 Some definitions and lemmas

This section is devoted to some notations, basic definitions and some results which are useful for validating the main results.

Let us denote $\mathfrak{J}=[1,e]$, $1=t_0<t_1<t_2<\cdots<t_{m+1}=e$, $\mathfrak{J}_0=[1,t_1],\mathfrak{J}_1=(t_1,t_2],\ldots,\mathfrak{J}_m=(t_m,e]$, and the Banach space

$$\mathbb{PC}(J,\mathbb{R})=\{\mathfrak{u}:\mathfrak{J}\to\mathbb{R};\ \mathfrak{u}(\cdot)\in C((t_k,t_{k+1}],\mathbb{R}),k=0,1,\ldots,m$$

$$\text{and } \mathfrak{u}(t_k^+),\mathfrak{u}(t_k^-) \text{ exist with } \ \mathfrak{u}(t_k^-)=\mathfrak{u}(t_k),k=1,2,\ldots m\},$$

with the norm $\|\mathfrak{u}\|_{PC}:=\sup\{|\mathfrak{u}(t)|:t\in\mathfrak{J}\}$.

Definition 1. *[11, 12] The Hadamard derivative of fractional order q for a function $g:[a,b]\to\mathbb{R},\ 0<a<b$ is defined as*

$${}^{H}D_a^q g(t)=\frac{1}{\Gamma(n-q)}\left(t\frac{d}{dt}\right)^n\int_a^t\left(\log\frac{t}{s}\right)^{n-q-1}\frac{g(s)}{s}ds \quad n-1<q\le n\ ,\ n=[q]+1,$$

where $[q]$ denotes the integer part of the real number q.

Definition 2. *[11, 12] The Hadamard fractional integral of order q for a function g is defined as*

$${}^{H}I_a^q g(t)=\frac{1}{\Gamma(q)}\int_a^t\left(\log\frac{t}{s}\right)^{q-1}\frac{g(s)}{s}ds.$$

Definition 3. *[11, 12] For an n-times differentiable function $g:[a,b]\to\mathbb{R},\quad 0<a<b$, the Caputo type Hadamard derivative of fractional order α is defined as*

$${}^{CH}D_a^\alpha g(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t\left(\log\frac{t}{s}\right)^{n-\alpha-1}\delta^n g(s)\frac{ds}{s},\qquad n-1<\alpha<n,n=[\alpha]+1,$$

where $\delta=t\frac{d}{dt}$ $t\in[1,e]$ and $[\alpha]$ denotes the integer part of the real number α and $\log(\cdot)=\log_e(\cdot)$.

Lemma 1. *[1] **Schauder Fixed Point Theorem:** Let C be a closed, convex subset of a normed linear space E. Then every compact and continuous map $F:C\to C$ has at least one fixed point.*

Lemma 2. *[12] Let $y\in AC_\delta^n[1,e]$ or $C_\delta^n[1,e]$ and $\alpha\in\mathbb{C}$. Then*

$${}^{H}I_a^q({}^{CH}D_a^\alpha y(x))=y(x)-\sum_{k=0}^{n-1}c_k\left(\log\frac{x}{a}\right)^k.$$

Lemma 3. *Let $1<\vartheta\le 2$ and $a\in C[J,\mathbb{R}]$. Then the nonlinear system*

$$\begin{cases}{}^{CH}D_{t_k}^{\vartheta}\mathfrak{u}(\zeta)=a(\zeta), \quad \zeta,\zeta_k\in[1,e],\zeta\neq\zeta_k,\\ \triangle\mathfrak{u}(\zeta_k)=\mathfrak{A}_k(\mathfrak{u}(\zeta_k)), \quad k=1,2,\ldots,m,\\ \triangle\delta\mathfrak{u}(\zeta_k)=\overline{\mathfrak{A}_k}(\mathfrak{u}(\zeta_k)),\\ \mathfrak{u}(1)=\mathfrak{h}(\mathfrak{u}),\ \mathfrak{u}(e)=\mathfrak{g}(\mathfrak{u})\end{cases}\tag{2}$$

is equivalent to the following integral equation

$$\mathfrak{u}(\zeta)=\begin{cases} c\log\zeta+\mathfrak{h}(\mathfrak{u})+{}^{H}I_{1}^{\vartheta}a(\zeta); \quad \zeta\in J_0,\\ c\log\zeta+\mathfrak{h}(\mathfrak{u})+{}^{H}I_{\zeta_k}^{\vartheta}a(\zeta)+\sum_{j=1}^{k}{}^{H}I_{\zeta_{j-1}}^{\vartheta}a(\zeta_j)\\ +\sum_{j=1}^{k}\mathfrak{A}_j(\mathfrak{u}(\zeta_j))+\sum_{j=1}^{k}\left(\log\frac{\zeta}{\zeta_j}\right){}^{H}I_{\zeta_{j-1}}^{\vartheta-1}a(\zeta_j)\\ +\sum_{j=1}^{k}\left(\log\frac{\zeta}{\zeta_j}\right)\overline{\mathfrak{A}_j}(\mathfrak{u}(\zeta_j)); \quad \zeta\in J_k \quad k=1,2,\dots,m. \end{cases} \tag{3}$$

where

$$\begin{aligned} c=\mathfrak{g}(\mathfrak{u})-\mathfrak{h}(\mathfrak{u})-\sum_{j=1}^{m+1}{}^{H}I_{\zeta_{j-1}}^{\vartheta}a(\zeta_j)-\sum_{j=1}^{m}\mathfrak{A}_j(\mathfrak{u}(\zeta_j))\\ -\sum_{j=1}^{m}\left(\log\frac{e}{\zeta_j}\right){}^{H}I_{\zeta_{j-1}}^{\vartheta-1}a(\zeta_j)-\sum_{j=1}^{m}\left(\log\frac{e}{\zeta_j}\right)\overline{\mathfrak{A}_j}(\mathfrak{u}(\zeta_j)). \end{aligned} \tag{4}$$

Proof. By Lemma 2, for $\zeta\in J_0=[1=\zeta_0,\zeta_1]$, we have

$$\mathfrak{u}(\zeta)={}^{H}I_{1}^{\vartheta}a(\zeta)+c_0+c_1\log\zeta=\frac{1}{\Gamma(\vartheta)}\int_1^{\zeta}\left(\log\frac{\zeta}{s}\right)^{\vartheta-1}a(s)\frac{ds}{s}+c_0+c_1\log\zeta,$$

and

$$\delta\mathfrak{u}(\zeta)=\zeta\frac{d}{d\zeta}[\mathfrak{u}(\zeta)]=\zeta\left[\frac{1}{\Gamma(\vartheta-1)}\int_1^{\zeta}\left(\log\frac{\zeta}{s}\right)^{\vartheta-2}a(s)\frac{ds}{s}\frac{1}{\zeta}+c_1\frac{1}{\zeta}\right]={}^{H}I_{1}^{\vartheta-1}a(\zeta)+c_1.$$

As $\mathfrak{u}(1)=\mathfrak{h}(\mathfrak{u})$ implies $c_0=\mathfrak{h}(\mathfrak{u})$, it follows that

$$\mathfrak{u}(\zeta)={}^{H}I_{1}^{\vartheta}a(\zeta)+\mathfrak{h}(\mathfrak{u})+c_1\log\zeta=c\log\zeta+\mathfrak{h}(\mathfrak{u})+{}^{H}I_{1}^{\vartheta}a(\zeta),$$

where $c=c_1$.

Now for $\zeta\in J_1=(\zeta_1,\zeta_2]$, we have

$$\begin{aligned} {}^{H}I_{\zeta_1}^{\vartheta}({}^{CH}D_{\zeta_1}^{\vartheta})\mathfrak{u}(\zeta)&=\mathfrak{u}(\zeta)-d_0-d_1\left(\log\frac{\zeta}{\zeta_1}\right),\\ \mathfrak{u}(\zeta)&={}^{H}I_{\zeta_1}^{\vartheta}a(\zeta)+d_0+d_1\left(\log\frac{\zeta}{\zeta_1}\right), \end{aligned}$$

and

$$\delta\mathfrak{u}(\zeta)=\zeta\frac{d}{d\zeta}(\mathfrak{u}(\zeta))=\zeta\left[\frac{1}{\zeta}\frac{1}{\Gamma(\vartheta-1)}\int_{\zeta_1}^{\zeta}\left(\log\frac{\zeta}{s}\right)^{\vartheta-2}a(s)\frac{ds}{s}+\frac{d_1}{\zeta}\right]={}^{H}I_{\zeta_1}^{\vartheta-1}a(\zeta)+d_1.$$

Also, $\mathfrak{u}(\zeta_1^+)-\mathfrak{u}(\zeta_1^-)=\triangle\mathfrak{u}(\zeta_1)=\mathfrak{A}_1(\mathfrak{u}(\zeta_1))$ implies

$$\mathfrak{A}_1(\mathfrak{u}(\zeta_1))=d_0-{}^{H}I_{1}^{\vartheta}a(\zeta_1)-\mathfrak{h}(\mathfrak{u})-c\log\zeta_1,$$

$$d_0 = {}^H I_1^{\vartheta} a(\zeta_1) + \mathfrak{h}(\mathfrak{u}) + c \log \zeta_1 + \mathfrak{A}_1(\mathfrak{u}(\zeta_1)),$$

and $\delta\mathfrak{u}(\zeta_1^+) - \delta\mathfrak{u}(\zeta_1^-) = \triangle\delta\mathfrak{u}(\zeta_1) = \overline{\mathfrak{A}_1}(\mathfrak{u}(\zeta_1))$ implies

$$\begin{aligned}\overline{\mathfrak{A}_1}(\mathfrak{u}(\zeta_1)) &= d_1 - {}^H I_1^{\vartheta-1} a(\zeta_1) - c,\\ d_1 &= \overline{\mathfrak{A}_1}(\mathfrak{u}(\zeta_1)) + {}^H I_1^{\vartheta-1} a(\zeta_1) + c.\end{aligned}$$

Applying the above arguments, we get

$$\begin{aligned}\mathfrak{u}(\zeta) = {}&{}^H I_{\zeta_1}^{\vartheta} a(\zeta) + {}^H I_{\zeta_1}^{\vartheta} a(\zeta_1) + \left(\log \frac{\zeta}{\zeta_1}\right) {}^H I_1^{\vartheta-1} a(\zeta_1) + \left(\log \frac{\zeta}{\zeta_1}\right) \overline{\mathfrak{A}_1}(\mathfrak{u}(\zeta_1)) + \mathfrak{h}(\mathfrak{u})\\ &+ \mathfrak{A}_1(\mathfrak{u}(\zeta_1)) + c \log \zeta.\end{aligned}$$

Continuing in the same manner, we obtain for $\zeta \in J_m = (\zeta_m, e]$,

$$\begin{aligned}\mathfrak{u}(\zeta) = {}& c \log \zeta + \mathfrak{h}(\mathfrak{u}) + \sum_{j=1}^{m} \mathfrak{A}_j(\mathfrak{u}(\zeta_j)) + \sum_{j=1}^{m} \left(\log \frac{\zeta}{\zeta_j}\right) \overline{\mathfrak{A}_j}(\mathfrak{u}(\zeta_j)) + {}^H I_{\zeta_m}^{\vartheta} a(\zeta)\\ &+ \sum_{j=1}^{m} {}^H I_{\zeta_{j-1}}^{\vartheta} a(\zeta_j) + \sum_{j=1}^{m} \left(\log \frac{\zeta}{\zeta_j}\right) {}^H I_{\zeta_{j-1}}^{\vartheta-1} a(\zeta_j),\end{aligned}$$

and using $\mathfrak{u}(e) = \mathfrak{g}(\mathfrak{u})$, we have

$$\begin{aligned}\mathfrak{g}(\mathfrak{u}) = {}& c + \mathfrak{h}(\mathfrak{u}) + \sum_{j=1}^{m} \mathfrak{A}_j(\mathfrak{u}(\zeta_j)) + \sum_{j=1}^{m} \left(\log \frac{e}{\zeta_j}\right) \overline{\mathfrak{A}_j}(\mathfrak{u}(\zeta_j))\\ &+ {}^H I_{\zeta_m}^{\vartheta} a(e) + \sum_{j=1}^{m} {}^H I_{\zeta_{j-1}}^{\vartheta} a(\zeta_j) + \sum_{j=1}^{m} \left(\log \frac{e}{\zeta_j}\right) {}^H I_{\zeta_{j-1}}^{\vartheta-1} a(\zeta_j).\end{aligned}$$

This implies that

$$\begin{aligned}c = {}& \mathfrak{g}(\mathfrak{u}) - \mathfrak{h}(\mathfrak{u}) - \sum_{j=1}^{m} \mathfrak{A}_j(\mathfrak{u}(\zeta_j)) - \sum_{j=1}^{m} \left(\log \frac{e}{\zeta_j}\right) \overline{\mathfrak{A}_j}(\mathfrak{u}(\zeta_j))\\ &- \sum_{j=1}^{m+1} {}^H I_{\zeta_{j-1}}^{\vartheta} a(\zeta_j) - \sum_{j=1}^{m} \left(\log \frac{e}{\zeta_j}\right) {}^H I_{\zeta_{j-1}}^{\vartheta-1} a(\zeta_j),\end{aligned}$$

which completes the proof. □

3 Main Results

Now we construct an operator $\mathfrak{T} : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ by

$$\mathfrak{T}(\mathfrak{u}(\zeta)) = a \log \zeta + (1 - \log \zeta)\mathfrak{h}(\mathfrak{u}) + (\log \zeta)\mathfrak{g}(\mathfrak{u}) + \sum_{0<\zeta_k<\zeta} \mathfrak{A}_k(\mathfrak{u}(\zeta_k)$$

$$
\begin{aligned}
&+\sum_{0<\zeta_k<\zeta}\left(\log\frac{\zeta}{\zeta_k}\right)\overline{\mathfrak{A}_k}(\mathfrak{u}(\zeta_k))+\frac{1}{\Gamma(\vartheta)}\int_{\zeta_k}^{\zeta}\left(\log\frac{\zeta}{s}\right)^{\vartheta-1}\mathfrak{f}(s,\mathfrak{u}(s))\frac{ds}{s}\\
&+\sum_{0<\zeta_k<\zeta}\frac{1}{\Gamma(\vartheta)}\int_{\zeta_{k-1}}^{\zeta_k}\left(\log\frac{\zeta}{s}\right)^{\vartheta-1}\mathfrak{f}(s,\mathfrak{u}(s))\frac{ds}{s}\\
&+\sum_{0<\zeta_k<\zeta}\left(\log\frac{\zeta}{\zeta_k}\right)\frac{1}{\Gamma(\vartheta-1)}\int_{\zeta_{k-1}}^{\zeta_k}\left(\log\frac{\zeta}{s}\right)^{\vartheta-2}\mathfrak{f}(s,\mathfrak{u}(s))\frac{ds}{s}, \qquad (5)
\end{aligned}
$$

where

$$
\begin{aligned}
a=&-\sum_{k=1}^{m}\mathfrak{A}_k(\mathfrak{u}(\zeta_k))-\sum_{k=1}^{m}\left(\log\frac{e}{\zeta_k}\right)\frac{1}{\Gamma(\vartheta-1)}\int_{\zeta_{k-1}}^{\zeta_k}\left(\log\frac{\zeta_k}{s}\right)^{\vartheta-2}\mathfrak{f}(s,\mathfrak{u}(s))\frac{ds}{s}\\
&-\sum_{k=1}^{m+1}\frac{1}{\Gamma(\vartheta)}\int_{\zeta_{k-1}}^{\zeta_k}\left(\log\frac{\zeta_k}{s}\right)^{\vartheta-1}\mathfrak{f}(s,\mathfrak{u}(s))\frac{ds}{s}-\sum_{k=1}^{m}\left(\log\frac{e}{\zeta_k}\right)\overline{\mathfrak{A}_k}(\mathfrak{u}(\zeta_k)). \qquad (6)
\end{aligned}
$$

Here, we take some suitable assumptions for establishing our main results.

(H$_1$) $\mathfrak{f}: J\times\mathbb{R}\to\mathbb{R}$ is continuous and there exists constant $L_1>0$ such that

$$|\mathfrak{f}(\zeta,z_1)-\mathfrak{f}(\zeta,z_2)|\le L_1|z_1-z_2|,$$

for each $\zeta\in J$ and all $z_1,z_2\in\mathbb{R}$.

(H$_2$) There exist constants $L_2,L_3>0$ such that

$$|\mathfrak{A}_k(\mathfrak{u})-\mathfrak{A}_k(v)|\le L_2|\mathfrak{u}-v|,\quad |\overline{\mathfrak{A}_k}(\mathfrak{u})-\overline{\mathfrak{A}_k}(v)|\le L_3|\mathfrak{u}-v|,$$

for $\mathfrak{u},v\in\mathbb{R},\ k=1,2,\cdots,m$.

(H$_3$) There exist constants $L_4,L_5>0$ such that

$$|\mathfrak{g}(\mathfrak{u})-\mathfrak{g}(v)|\le L_4\|\mathfrak{u}-v\|_{PC},\quad |\mathfrak{h}(\mathfrak{u})-\mathfrak{h}(v)|\le L_5\|\mathfrak{u}-v\|_{PC}.$$

We firstly establish the existence and uniqueness result using Banach Contraction Principle.

Theorem 1. *Assume that (H$_1$),(H$_2$) and (H$_3$) hold along with*

$$\left(L_1\frac{4m+2}{\Gamma(\vartheta)}+2m(L_2+L_3)+L_4+L_5\right)<1 \qquad (7)$$

Then the problem given by (1) has a unique solution on [1,e].

Proof. Let $\sup_{\zeta\in J}|\mathfrak{f}(\zeta,0)|=M_1$, $\max_k|\mathfrak{A}_k(0)|=M_2$, $\max_k|\overline{\mathfrak{A}_k}(0)|=M_3$, $|\mathfrak{g}(0)|=M_4$, $|\mathfrak{h}(0)|=M_5$ and

$$U_0=\left\{\mathfrak{u}(\zeta)\in PC(J,\mathbb{R});\|\mathfrak{u}\|_{PC}\le R_0\right\},$$

where

$$R_0 = \frac{(4m+2)M_1 + 2m\Gamma(\vartheta)(M_2+M_3) + \Gamma(\vartheta)(M_4+M_5)}{\Gamma(\vartheta) - (4m+2)L_1 - 2m\Gamma(\vartheta)(L_2+L_3) - \Gamma(\vartheta)(L_4+L_5)}.$$

Firstly we prove that $\mathfrak{T}$ maps U_0 into itself

$$\begin{aligned}
|a| \leq & \frac{1}{\Gamma(\vartheta)} \sum_{k=1}^{m+1} \int_{\zeta_{k-1}}^{\zeta_k} \left(\log \frac{\zeta_k}{s} \right)^{\vartheta-1} |\mathfrak{f}(s, \mathfrak{u}(s)) - \mathfrak{f}(s,0)| \frac{ds}{s} \\
& + \frac{1}{\Gamma(\vartheta)} \sum_{k=1}^{m+1} \int_{\zeta_{k-1}}^{\zeta_k} \left(\log \frac{\zeta_k}{s} \right)^{\vartheta-1} |\mathfrak{f}(s,0)| \frac{ds}{s} \\
& + \sum_{k=1}^{m} \left(\log \frac{e}{\zeta_k} \right) \frac{1}{\Gamma(\vartheta-1)} \int_{\zeta_{k-1}}^{\zeta_k} \left(\log \frac{\zeta_k}{s} \right)^{\vartheta-2} |\mathfrak{f}(s, \mathfrak{u}(s)) - \mathfrak{f}(s,0)| \frac{ds}{s} \\
& + \sum_{k=1}^{m} \left(\log \frac{e}{\zeta_k} \right) \frac{1}{\Gamma(\vartheta-1)} \int_{\zeta_{k-1}}^{\zeta_k} \left(\log \frac{\zeta_k}{s} \right)^{\vartheta-2} |\mathfrak{f}(s,0)| \frac{ds}{s} \\
& + \sum_{k=1}^{m} |\mathfrak{A}_k(\mathfrak{u}(\zeta_k)) - \mathfrak{A}_k(\mathfrak{u}(0))| + \sum_{k=1}^{m} |\mathfrak{A}_k(\mathfrak{u}(0))| \\
& + \sum_{k=1}^{m} \left(\log \frac{e}{\zeta_k} \right) |\overline{\mathfrak{A}_k}(\mathfrak{u}(\zeta_k)) - \overline{\mathfrak{A}_k}(\mathfrak{u}(0))| + \sum_{k=1}^{m} \left(\log \frac{e}{\zeta_k} \right) |\overline{\mathfrak{A}_k}(\mathfrak{u}(0))| \\
\leq & \sum_{k=1}^{m+1} \frac{L_1\|\mathfrak{u}\|_{PC} + M_1}{\Gamma(\vartheta+1)} + \sum_{k=1}^{m} \frac{L_1\|\mathfrak{u}\|_{PC} + M_1}{\Gamma(\vartheta)} \\
& + m(L_2\|\mathfrak{u}\|_{PC} + M_2) + m(L_3\|\mathfrak{u}\|_{PC} + M_3) \\
\leq & \frac{2m+1}{\Gamma(\vartheta)} (L_1\|\mathfrak{u}\|_{PC} + M_1) + m(L_2\|\mathfrak{u}\|_{PC} + M_2) + m(L_3\|\mathfrak{u}\|_{PC} + M_3).
\end{aligned}$$

Again

$$\begin{aligned}
|\mathfrak{T}(\mathfrak{u}(\zeta))| \leq & |a| + |\mathfrak{h}(\mathfrak{u})| + |\mathfrak{g}(\mathfrak{u})| + m(L_2\|\mathfrak{u}\|_{PC} + M_2) + m(L_3\|\mathfrak{u}\|_{PC} + M_3) \\
& + \frac{L_1\|\mathfrak{u}\|_{PC} + M_1}{\Gamma(\vartheta+1)} + \sum_{k=1}^{m} \frac{L_1\|\mathfrak{u}\|_{PC} + M_1}{\Gamma(\vartheta+1)} + \sum_{k=1}^{m} \frac{L_1\|\mathfrak{u}\|_{PC} + M_1}{\Gamma(\vartheta)} \\
\leq & \frac{2m+1}{\Gamma(\vartheta)} (L_1\|\mathfrak{u}\|_{PC} + M_1) + 2m(L_2\|\mathfrak{u}\|_{PC} + M_2) + 2m(L_3\|\mathfrak{u}\|_{PC} + M_3) \\
& + (L_4\|\mathfrak{u}\|_{PC} + M_4) + (L_5\|\mathfrak{u}\|_{PC} + M_5) + \frac{2m+1}{\Gamma(\vartheta)} (L_1\|\mathfrak{u}\|_{PC} + M_1) \\
\leq & \frac{4m+2}{\Gamma(\vartheta)} (L_1\|\mathfrak{u}\|_{PC} + M_1) + 2m(L_2\|\mathfrak{u}\|_{PC} + M_2) + 2m(L_3\|\mathfrak{u}\|_{PC} + M_3) \\
& + (L_4\|\mathfrak{u}\|_{PC} + M_4) + (L_5\|\mathfrak{u}\|_{PC} + M_5) \\
\leq & R_0.
\end{aligned}$$

This implies that $\mathfrak{T}$ maps U_0 into itself.

Next we shall prove that the map $\mathfrak{T}$ is a contraction. Let $\mathfrak{u}, v \in PC(J,\mathbb{R})$. Then for any $\zeta \in J_k, k = 1, 2, \dots, m$, we have

$$\begin{aligned}|\mathfrak{T}(\mathfrak{u}(\zeta)) - \mathfrak{T}(v(\zeta))| \leq & \left(\frac{2m+1}{\Gamma(\vartheta+1)} L_1 + \frac{2m+1}{\Gamma(\vartheta)} L_1 + 2mL_2 + 2mL_3 + L_4 + L_5 \right) \|\mathfrak{u} - v\|_{PC} \\ \leq & \left(\frac{4m+2}{\Gamma(\vartheta)} L_1 + 2mL_2 + 2mL_3 + L_4 + L_5 \right) \|\mathfrak{u} - v\|_{PC}.\end{aligned}$$

It follows that the map $\mathfrak{T}$ is a contraction map. Hence, from Banach fixed point theorem, the operator $\mathfrak{T}$ has a unique fixed point. That is, the problem (1) has a unique solution on [1,e]. □

The next result is established using Schauder fixed point theorem.

Theorem 2. *Assume that $\mathfrak{f}$, $\mathfrak{A}_k$, $\overline{\mathfrak{A}_k}$, $\mathfrak{g}$, $\mathfrak{h}$ are continuous functions and there exists an non-negative function $\mathfrak{a}(\cdot) \in L(J,\mathbb{R})$ and constants $\mathfrak{z}_1 \geq 0$ and $\mathfrak{z}_i > 0$ for $i = 2, 3, 4, 5$ s.t., $|\mathfrak{f}(\zeta, \mathfrak{u})| \leq \mathfrak{a}(\zeta) + \mathfrak{z}_1 |\mathfrak{u}|^\sigma$, $|\mathfrak{A}_k(\mathfrak{u})| \leq \mathfrak{z}_2 |\mathfrak{u}|^\mu$, $|\overline{\mathfrak{A}_k}(\mathfrak{u})| \leq \mathfrak{z}_3 |\mathfrak{u}|^\nu$, $|\mathfrak{g}(\mathfrak{u})| \leq \mathfrak{z}_4 |\mathfrak{u}|^\theta$, $|\mathfrak{h}(\mathfrak{u})| \leq \mathfrak{z}_5 |\mathfrak{u}|^\gamma, k = 1, 2, \cdots, m$, for any $\mathfrak{u} \in \mathbb{R}$ and for some $0 < \sigma, \mu, \nu, \theta, \gamma \leq 1$. If*

$$\frac{2(2m+1)\mathfrak{z}_1}{\Gamma(\vartheta)} + 2m\mathfrak{z}_2 + 2m\mathfrak{z}_3 + \mathfrak{z}_4 + \mathfrak{z}_5 < 1,$$

then the impulsive boundary value problem (1) has at least one solution on $[1, e]$.

Proof. The proof of this result is divided into three steps.
Step I. Operator $\mathfrak{T}$ is continuous.
As $\mathfrak{f}, g, \mathfrak{h}, \mathfrak{A}_k, \overline{\mathfrak{A}_k}$; $k = 1, 2, \dots, m$, are continuous functions, we conclude that the operator $\mathfrak{T}$ is continuous.
Step II. $\mathfrak{T}$ maps bounded sets into uniformly bounded sets in $PC(J,\mathbb{R})$.
Let

$$U = \{\mathfrak{u}(\zeta) \in PC(J,\mathbb{R}) : \|\mathfrak{u}\|_{PC} \leq R\},$$

where

$$R \geq \max \left\{ 1, \frac{\frac{4}{\Gamma(\vartheta-1)} \sum_{k=1}^{m+1} \int_{\zeta_{k-1}}^{\zeta_k} \left(\log \frac{\zeta_k}{s} \right)^{\vartheta-2} \mathfrak{a}(s) \frac{ds}{s}}{1 - \left(\frac{4m+2}{\Gamma(\vartheta)} \mathfrak{z}_1 + 2m\mathfrak{z}_2 + 2m\mathfrak{z}_3 + \mathfrak{z}_4 + \mathfrak{z}_5 \right)} \right\}.$$

For any arbitrary $\mathfrak{u} \in U$, we have

$$\begin{aligned}|a| \leq & \sum_{k=1}^{m} |\mathfrak{A}_k(\mathfrak{u}(\zeta_k))| + \sum_{k=1}^{m} \left(\log \frac{e}{\zeta_k} \right) \frac{1}{\Gamma(\vartheta-1)} \int_{\zeta_{k-1}}^{\zeta_k} \left(\log \frac{\zeta_k}{s} \right)^{\vartheta-2} |\mathfrak{f}(s, \mathfrak{u}(s))| \frac{ds}{s} \\ & \sum_{k=1}^{m+1} \frac{1}{\Gamma(\vartheta)} \int_{\zeta_{k-1}}^{\zeta_k} \left(\log \frac{\zeta_k}{s} \right)^{\vartheta-1} |\mathfrak{f}(s, \mathfrak{u}(s))| \frac{ds}{s} + \sum_{k=1}^{m} \left(\log \frac{e}{\zeta_k} \right) |\overline{\mathfrak{A}_k}(\mathfrak{u}(t_k))| \\ \leq & m\mathfrak{z}_2 R^\mu + m\mathfrak{z}_3 R^\nu\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\vartheta)}\sum_{k=1}^{m+1}\int_{\zeta_{k-1}}^{\zeta_k}\left(\log\frac{\zeta_k}{s}\right)^{\vartheta-1}\mathfrak{a}(s)\frac{ds}{s}+\frac{\mathfrak{z}_1R^\sigma}{\Gamma(\vartheta)}\sum_{k=1}^{m+1}\int_{\zeta_{k-1}}^{\zeta_k}\left(\log\frac{\zeta_k}{s}\right)^{\vartheta-1}\frac{ds}{s}\\
&+\frac{1}{\Gamma(\vartheta-1)}\sum_{k=1}^{m}\left(\log\frac{e}{\zeta_k}\right)\int_{\zeta_{k-1}}^{\zeta_k}\left(\log\frac{\zeta_k}{s}\right)^{\vartheta-2}\mathfrak{a}(s)\frac{ds}{s}\\
&+\frac{\mathfrak{z}_1R^\sigma}{\Gamma(\vartheta-1)}\sum_{k=1}^{m}\left(\log\frac{e}{\zeta_k}\right)\int_{\zeta_{k-1}}^{\zeta_k}\left(\log\frac{\zeta_k}{s}\right)^{\vartheta-2}\frac{ds}{s}\\
\leq&\frac{1}{\Gamma(\vartheta)}\sum_{k=1}^{m+1}\int_{\zeta_{k-1}}^{\zeta_k}\left(\log\frac{\zeta_k}{s}\right)^{\vartheta-1}\mathfrak{a}(s)\frac{ds}{s}\\
&+\frac{1}{\Gamma(\vartheta-1)}\sum_{k=1}^{m}\left(\log\frac{e}{\zeta_k}\right)\int_{\zeta_{k-1}}^{\zeta_k}\left(\log\frac{\zeta_k}{s}\right)^{\vartheta-2}\mathfrak{a}(s)\frac{ds}{s}+\frac{(m+1)\mathfrak{z}_1R^\sigma}{\Gamma(\vartheta+1)}\\
&+\frac{m\mathfrak{z}_1R^\sigma}{\Gamma(\vartheta)}+m\mathfrak{z}_2R^\mu+m\mathfrak{z}_3R^\nu\\
\leq&\frac{1}{\Gamma(\vartheta-1)}\sum_{k=1}^{m+1}\int_{\zeta_{k-1}}^{\zeta_k}\left(\left(\log\frac{\zeta_k}{s}\right)^{\vartheta-2}+\left(\log\frac{\zeta_k}{s}\right)^{\vartheta-2}\right)\mathfrak{a}(s)\frac{ds}{s}\\
&+\frac{(2m+1)\mathfrak{z}_1R^\sigma}{\Gamma(\vartheta)}+m\mathfrak{z}_2R^\mu+m\mathfrak{z}_3R^\nu\\
=&\frac{2}{\Gamma(\vartheta-1)}\sum_{k=1}^{m+1}\int_{\zeta_{k-1}}^{\zeta_k}\left(\log\frac{\zeta_k}{s}\right)^{\vartheta-2}\mathfrak{a}(s)\frac{ds}{s}+\frac{(2m+1)\mathfrak{z}_1R^\sigma}{\Gamma(\vartheta)}+m\mathfrak{z}_2R^\mu+m\mathfrak{z}_3R^\nu
\end{aligned}
$$

and

$$
\begin{aligned}
|\mathfrak{T}(\mathfrak{u}(\zeta))|\leq{}&|a|+|\mathfrak{g}(\mathfrak{u})|+|\mathfrak{h}(\mathfrak{u})|+\sum_{k=1}^{m}|\mathfrak{A}_k(\mathfrak{u}(\zeta_k)|+\sum_{k=1}^{m}\left(\log\frac{\zeta}{\zeta_k}\right)|\overline{\mathfrak{A}_k}(\mathfrak{u}(\zeta_k))|\\
&+\frac{1}{\Gamma(\vartheta)}\int_{\zeta_k}^{\zeta}\left(\log\frac{\zeta}{s}\right)^{\vartheta-1}\mathfrak{a}(s)\frac{ds}{s}+\frac{\mathfrak{z}_1R^\sigma}{\Gamma(\vartheta)}\int_{\zeta_k}^{\zeta}\left(\log\frac{\zeta}{s}\right)^{\vartheta-1}\frac{ds}{s}\\
&+\frac{1}{\Gamma(\vartheta)}\sum_{k=1}^{m}\int_{\zeta_{k-1}}^{\zeta_k}\left(\log\frac{\zeta}{s}\right)^{\vartheta-1}\mathfrak{a}(s)\frac{ds}{s}+\frac{\mathfrak{z}_1R^\sigma}{\Gamma(\vartheta)}\sum_{k=1}^{m}\int_{\zeta_{k-1}}^{\zeta_k}\left(\log\frac{\zeta}{s}\right)^{\vartheta-1}\frac{ds}{s}\\
&+\frac{1}{\Gamma(\vartheta-1)}\sum_{k=1}^{m}\left(\log\frac{\zeta}{\zeta_k}\right)\int_{\zeta_{k-1}}^{\zeta_k}\left(\log\frac{\zeta}{s}\right)^{\vartheta-2}\mathfrak{a}(s)\frac{ds}{s}\\
&+\frac{\mathfrak{z}_1R^\sigma}{\Gamma(\vartheta-1)}\sum_{k=1}^{m}\left(\log\frac{\zeta}{\zeta_k}\right)\int_{\zeta_{k-1}}^{\zeta_k}\left(\log\frac{\zeta}{s}\right)^{\vartheta-2}\frac{ds}{s}\\
\leq{}&|a|+m\mathfrak{z}_4R^\theta+m\mathfrak{z}_5R^\gamma+m\mathfrak{z}_2R^\mu+m\mathfrak{z}_3R^\nu+\frac{(m+1)\mathfrak{z}_1R^\sigma}{\Gamma(\vartheta+1)}+\frac{(m)\mathfrak{z}_1R^\sigma}{\Gamma(\vartheta+1)}\\
&+\frac{1}{\Gamma(\vartheta)}\sum_{k=1}^{m+1}\int_{\zeta_{k-1}}^{\zeta_k}\left(\log\frac{\zeta}{s}\right)^{\vartheta-1}\mathfrak{a}(s)\frac{ds}{s}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\vartheta-1)}\sum_{k=1}^{m}\left(\log\frac{\zeta}{\zeta_k}\right)\int_{\zeta_{k-1}}^{\zeta_k}\left(\log\frac{\zeta}{s}\right)^{\vartheta-2}\mathfrak{a}(s)\frac{ds}{s}\\
\leq&\frac{2}{\Gamma(\vartheta-1)}\sum_{k=1}^{m+1}\int_{\zeta_{k-1}}^{\zeta_k}\left(\log\frac{\zeta_k}{s}\right)^{\vartheta-2}\mathfrak{a}(s)\frac{ds}{s}+\frac{(2m+1)\mathfrak{z}_1R^{\sigma}}{\Gamma(\vartheta+1)}+m\mathfrak{z}_2R^{\mu}\\
&+m\mathfrak{z}_3R^{\nu}+m\mathfrak{z}_4R^{\theta}+m\mathfrak{z}_5R^{\gamma}+m\mathfrak{z}_2R^{\mu}+m\mathfrak{z}_3R^{\nu}+\frac{(m+1)\mathfrak{z}_1R^{\sigma}}{\Gamma(\vartheta+1)}\\
&+\frac{(m)\mathfrak{z}_1R^{\sigma}}{\Gamma(\vartheta+1)}+\frac{2}{\Gamma(\vartheta-1)}\sum_{k=1}^{m+1}\int_{\zeta_{k-1}}^{\zeta_k}\left(\log\frac{\zeta_k}{s}\right)^{\vartheta-2}\mathfrak{a}(s)\frac{ds}{s}\\
\leq&\frac{4}{\Gamma(\vartheta-1)}\sum_{k=1}^{m+1}\int_{\zeta_{k-1}}^{\zeta_k}\left(\log\frac{\zeta_k}{s}\right)^{\vartheta-2}\mathfrak{a}(s)\frac{ds}{s}+\frac{2(2m+1)\mathfrak{z}_1R}{\Gamma(\vartheta+1)}\\
&+2m\mathfrak{z}_2R+2m\mathfrak{z}_3R+m\mathfrak{z}_4R+m\mathfrak{z}_5R\\
\leq&R
\end{aligned}
$$

So $\mathfrak{T}$ maps U into itself.

Step III. $\mathfrak{T}$ maps bounded sets into equicontinuous sets of $PC(J,\mathbb{R})$.

Let $U\subset PC(J,\mathbb{R})$ be any arbitrary bounded set as assumed in STEP II. We fix, $N_{\mathfrak{f}}=\max\{|\mathfrak{f}(\zeta,\mathfrak{u}):\zeta\in J\ \&\ u\in U\}|+1,\ \mathfrak{u}\in U$ and let $\zeta,\tau\in[1,e]$ with $\zeta<\tau$. So we obtain

$$
\begin{aligned}
|a|\leq&m\mathfrak{z}_2R^{\mu}+m\mathfrak{z}_3R^{\nu}+\frac{N_{\mathfrak{f}}}{\Gamma(\vartheta)}\sum_{k=1}^{m}\int_{\zeta_{k-1}}^{\zeta_k}\left(\log\frac{\zeta_k}{s}\right)^{\vartheta-1}\frac{ds}{s}\\
&+\frac{N_{\mathfrak{f}}}{\Gamma(\vartheta-1)}\sum_{k=1}^{m}\left(\log\frac{e}{\zeta_k}\right)\int_{\zeta_{k-1}}^{\zeta_k}\left(\log\frac{\zeta_k}{s}\right)^{\vartheta-2}|\mathfrak{f}(s,\mathfrak{u}(s))|\frac{ds}{s}\\
\leq&\frac{N_{\mathfrak{f}}(2m+1)}{\Gamma(\vartheta)}+m\mathfrak{z}_2R^{\mu}+m\mathfrak{z}_3R^{\nu},
\end{aligned}
$$

and

$$
\begin{aligned}
|\mathfrak{T}(\mathfrak{u}(\tau))-\mathfrak{T}(\mathfrak{u}(\zeta))|\leq&|a|(\log\tau-\log\zeta)+\mathfrak{z}_4R^{\theta}(\log\tau-\log\zeta)+\mathfrak{z}_5R^{\theta}(\log\tau-\log\zeta)\\
&+\frac{N_{\mathfrak{f}}}{\Gamma(\vartheta)}\bigg|\int_{\zeta_k}^{\tau}\left(\log\frac{\tau}{s}\right)^{\vartheta-1}\frac{ds}{s}-\int_{\zeta_k}^{\zeta}\left(\log\frac{\zeta}{s}\right)^{\vartheta-1}\frac{ds}{s}\bigg|\\
&+\frac{|\log\tau-\log\zeta|N_{\mathfrak{f}}}{\Gamma(\vartheta-1)}\sum_{k=1}^{m}\int_{\zeta_{k-1}}^{\zeta_k}\left(\log\frac{\zeta_k}{s}\right)^{\vartheta-2}\frac{ds}{s}+(\log\tau-\log t)m\mathfrak{z}_3R^{\nu}\\
\leq&(|a|+\mathfrak{z}_4R^{\theta}+\mathfrak{z}_5R^{\theta}+m\mathfrak{z}_3R^{\nu})(\log\tau-\log\zeta)\\
&+\frac{N_{\mathfrak{f}}}{\Gamma(\vartheta)}\bigg\|\int_{\zeta_k}^{\tau}\left(\log\tau-\log s\right)^{\vartheta-1}\frac{ds}{s}-\int_{\zeta_k}^{\zeta}\left(\log\zeta-\log s\right)^{\vartheta-1}\frac{ds}{s}\bigg\|_{PC}\\
&+\frac{(\log\tau-\log\zeta)N_{\mathfrak{f}}}{\Gamma(\vartheta-1)}\sum_{k=1}^{m}\int_{\zeta_{k-1}}^{\zeta_k}\left(\log\frac{\zeta_k}{s}\right)^{\vartheta-2}\frac{ds}{s}\\
\leq&(|a|+\mathfrak{z}_4R^{\theta}+\mathfrak{z}_5R^{\theta}+m\mathfrak{z}_3R^{\nu})(\log\tau-\log\zeta)
\end{aligned}
$$

$$+\frac{(\log\tau-\log\zeta)N_{\mathfrak{f}}}{\Gamma(\vartheta)}\sum_{k=1}^{m}(\log\zeta_k-\log\zeta_{k-1})^{\vartheta-1}$$
$$+\frac{N_{\mathfrak{f}}}{\Gamma(\vartheta+1)}[(\log\tau-\log\zeta_k)^{\vartheta}-(\log\zeta-\log\zeta_k)^{\vartheta}],$$

$\to 0$ as $\tau\to\zeta$. Therefore, for any arbitrary $U\subset PC(J,\mathbb{R})$, $\mathfrak{T}$ maps U into equicontinuous set of $PC(J,\mathbb{R})$. Combining Step I, II, III along with the Arzela-Ascoli theorem, it can be found that $\mathfrak{T}$ is a completely continuous operator. Therefore from the fixed point theorem given by Schauder, operator $\mathfrak{T}$ has at least one fixed point which is solution of the problem (1) in $[1,e]$. □

In the next section we present some numerical results.

4 Numerical examples

Example 1. Consider the following non-linear boundary value problem

$$\begin{cases} {}^{CH}D^{\frac{3}{2}}_{\zeta_k}\mathfrak{u}(\zeta)=\frac{\sin(\mathfrak{u}(\zeta))}{24+\zeta^2}+\frac{1}{1+\zeta^2}, \quad \zeta\in[1,e], \quad \zeta\neq\frac{13}{7},\\ \triangle\mathfrak{u}(\frac{13}{7})=\frac{1}{10}\mathfrak{u}(\frac{13}{7}),\\ \triangle\delta\mathfrak{u}(\frac{13}{7})=\frac{1}{30}\mathfrak{u}(\frac{13}{7}),\\ \mathfrak{u}(1)=\mathfrak{h}(\mathfrak{u});\mathfrak{u}(e)=\mathfrak{g}(\mathfrak{u}),\end{cases} \tag{8}$$

where

$$\mathfrak{h}(\mathfrak{u})=\sum_{i=1}^{n}\alpha_i\mathfrak{u}(\xi_i), \quad \mathfrak{g}(\mathfrak{u})=\sum_{i=1}^{n}\beta_i\mathfrak{u}(\eta_i), \quad \xi_i,\eta_i\neq\frac{13}{7}\in(1,e),$$

and

$$\sum_{i=1}^{n}\alpha_i<\frac{1}{10}, \quad \sum_{i=1}^{n}\beta_i<\frac{1}{10}.$$

Here

$$\mathfrak{f}(\zeta,\mathfrak{u})=\frac{\sin(\mathfrak{u}(\zeta))}{24+\zeta^2}+\frac{1}{1+\zeta^2}.$$

We can easily see that,

$$|\mathfrak{f}(\zeta,\mathfrak{u})-\mathfrak{f}(\zeta,v)|\leq\frac{1}{24}|\mathfrak{u}-v|.$$

So $L_1=\frac{1}{24}$. Similarly, $L_2=\frac{1}{10}$, $L_3=\frac{1}{30}$, $L_4=\frac{1}{10}$, $L_5=\frac{1}{10}$. Here $\vartheta=\frac{3}{2},\zeta_1=\frac{13}{7},m=1$ and therefore

$$\left(L_1\frac{4m+2}{\Gamma(\vartheta)}+2m(L_2+L_3)+L_4+L_5\right)=0.749<1.$$

Thus all the postulates of Theorem 1 are satisfied. Therefore, the boundary value problem (8) has a unique solution on $[1,e]$.

Example 2. Consider the following boundary value problem

$$\begin{cases} {}^{CH}D^{\frac{3}{2}}_{\zeta_k}\mathfrak{u}(\zeta) = \frac{2+(\mathfrak{u}(\zeta))}{108e^{\zeta+3}}, \quad \zeta \in [1,e], \quad \zeta \neq \frac{15}{7}, \\ \triangle\mathfrak{u}(\frac{15}{7}) = \frac{\mathfrak{u}(\frac{15}{7})}{13+\mathfrak{u}(\frac{15}{7})}, \\ \triangle\delta\mathfrak{u}(\frac{15}{7}) = \frac{\mathfrak{u}(\frac{15}{7})}{25+\mathfrak{u}(\frac{15}{7})}, \\ \mathfrak{u}(1) = \mathfrak{h}(\mathfrak{u}); \mathfrak{u}(e) = \mathfrak{g}(\mathfrak{u}), \end{cases} \tag{9}$$

where

$$\mathfrak{h}(\mathfrak{u}) = \min_j \frac{(\mathfrak{u}(\vartheta_j))^{\frac{1}{4}}}{15+\mathfrak{u}(\vartheta_j)}, \quad \mathfrak{g}(\mathfrak{u}) = \max_j \frac{(\mathfrak{u}(\beta_j))^{\frac{1}{3}}}{15+\mathfrak{u}(\beta_j)},$$

with $j = 1,2,\ldots,10$ ϑ_j, $\beta_j \neq \frac{15}{7} \in (1,e)$. Here $\mathfrak{z}_1 = \frac{1}{108e^4}$, $\mathfrak{z}_2 = \frac{1}{13}$, $\mathfrak{z}_3 = \frac{1}{25}$, $\mathfrak{z}_4 = \frac{1}{15}$, $\mathfrak{z}_5 = \frac{1}{15}$.

Clearly

$$\frac{2(2m+1)\mathfrak{z}_1}{\Gamma(\vartheta)} + 2m\mathfrak{z}_2 + 2m\mathfrak{z}_3 + \mathfrak{z}_4 + \mathfrak{z}_5 = 0.368 < 1.$$

Therefore, by Theorem 2 the nonlinear system (9) has at least one solution on $[1,e]$.

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