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A GENERALIZATION OF GRADED PRIME SUBMODULES OVER NON-COMMUTATIVE GRADED RINGS

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Abstract. Let G be a group with identity e. Let R be an associative G-graded ring and M be a G-graded R-module. In this article, we introduce the concept of graded 2-absorbing submodules as a generalization of graded prime submodules over non-commutative graded rings. Moreover, we get some properties of such graded submodules.

1. Introduction

In this article, all rings are assumed to be associative rings with identity, and all modules are unitary left R-modules. Let G be a group with identity e and R be a ring. Then R is said to be a G-graded if $R = \bigoplus_{g\in G} R_g$ such that $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$, where R_g is an additive subgroup of R for all $g \in G$. The elements of R_g are homogeneous of degree g. Consider $supp(R, G) = \{g \in G \mid R_g \neq 0\}$. For simplicity, we will denote the graded ring (R, G) by R. If $r \in R$, then r can be written as $\sum_{g\in G} r_g$, where r_g is the component r in R_g. Moreover, R_e is a subring of R and if R contains a unitary 1, then $1 \in R_e$. Furthermore, $h(R) = \bigcup_{g\in G} R_g$.
Let I be a left ideal of a graded ring R. Then I is said to be a graded left ideal of R, if $I = \bigoplus_{g\in G}(I \cap R_g)$, i. e., for $x \in I$, $x = \sum_{g\in G} x_g$,

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where $x_g \in I$ for all $g \in G$. The following example from [1] shows that a left ideal of a graded ring need not to be graded.

Example 1.1. Consider $R = M_2(K)$ (the ring of all 2×2 matrices with entries from a field K) and $G = \mathbb{Z}_4$ (the group of integers modulo 4). Then R is G-graded by

$$R_0 = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}, R_2 = \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix} \text{ and } R_1 = R_3 = \{0\}$$

Consider the left ideal $I = \left\langle \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\rangle$ of R. Note that, $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in I$ such that $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. If I is a graded left ideal of R, then $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in I$ which is a contradiction. So I is not a graded left ideal of R.

Assume that M is a left R-module. Then M is said to be G-graded if $M = \bigoplus_{g \in G} M_g$ with $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$, where M_g is an additive subgroup of M for all $g \in G$. The elements of M_g are called homogeneous of degree g. Also, we consider $supp(M, G) = \{g \in G \mid M_g \neq 0\}$. It is clear that M_g is an R_e-submodule of M for all $g \in G$. Moreover $h(M) = \bigcup_{g \in G} M_g$. Let N be an R-submodule of a graded R-module M. Then N is said to be a graded R-submodule if $N = \bigoplus_{g \in G}(N \cap M_g)$, i. e., for $m \in N$, $m = \sum_{g \in G} m_g$, where $m_g \in N$ for all $g \in G$. Moreover, M/N becomes a graded R-module with $(M/N)_g = (M_g + N)/N$ for $g \in G$. The following example shows that an R-submodule of a graded R-module need not be graded (see [1]).

Example 1.2. Consider $R = M = M_2(K)$ and $G = \mathbb{Z}_4$. Then R and M are G-graded as in Example 1.1 and similarly $N = \left\langle \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\rangle$ is an R-submodule of M which is not graded.

Graded prime ideals play an important role in the theory of graded rings. A proper graded ideal I of a non-commutative graded ring R is said to be graded prime if whenever J and K are graded ideals of R such that $JK \subseteq I$, then either $J \subseteq I$ or $K \subseteq I$ (see [1]). If R is commutative, this definition is equivalent to: a proper graded ideal I of a graded ring R is said to be graded prime if whenever $r_g s_h \in I$ for some $r_g, s_h \in h(R)$, then $r_g \in I$ or $s_h \in I$. Graded prime ideals over commutative graded rings have been studied in [13].

Graded prime submodules over non-commutative graded rings have been introduced and studied by R. Abu-Dawwas and et al. in [1]. A proper graded submodule N of a graded R-module M is said to be graded prime, if $IK \subseteq N$ for some graded ideal I of R and graded submodule K of M, then $K \subseteq N$ or $I \subseteq (N : M)$. It is easy to show that this definition is equivalent to: a proper graded submodule N of a graded R-module M is graded prime, if $r_g R m_h \subseteq N$ where $r_g \in h(R)$ and $m_h \in h(M)$, then $m_h \in N$ or $r_g \in (N : M)$. If N is a graded prime submodule of M, then $(N : M) = P$ is a graded prime ideal of R. In this case, we say that N is a P-graded prime submodule of M. A graded R-module M is called graded prime, if the zero graded submodule is graded prime in M. If R is commutative, this definition is equivalent to: a proper graded submodule N of a graded R-module M is said to be graded prime, if $r_g m_h \in N$ where $r_g \in h(R)$ and $m_h \in h(M)$, then $m_h \in N$ or $r_g \in (N : M)$. For more information about graded prime submodules over commutative graded rings see [4, 5, 6, 7, 8].
The notion of 2-absorbing ideals over commutative rings which is a generalization of prime ideals has been introduced and investigated by A. Badawi in [3]. A proper ideal I of a commutative ring R is said to be 2-absorbing, if $abc \in I$, where $a, b, c \in R$, then $ab \in I$ or $ac \in I$ or $bc \in I$. Also, the concept of 2-absorbing submodules have been studied in [12, 15]. A proper submodule N of an R-module M is called 2-absorbing, if whenever $a, b \in R$ and $m \in M$ with $abm \in N$, then $am \in N$ or $bm \in N$ or $abM \subseteq N$. After that in [2, 14], the authors extened the notion of 2-absorbing ideals to graded rings. A proper graded ideal I of a graded ring R is said to be graded 2-absorbing, if $abc \in I$, where $a, b, c \in h(R)$, then $ab \in I$ or $ac \in I$ or $bc \in I$. The concept of 2-absorbing submodules over non-commutative rings has been studied in [9]. In this paper, we introduce and study the notions of graded 2-absorbing and graded strongly 2-absorbing submodules over non-commutative graded rings. Several results concerning such graded submodules are given.

2. Graded 2-absorbing submodules

In this section, we introduce the concepts of graded 2-absorbing and graded strongly 2-absorbing submodules of graded modules over an associative graded ring.

The following Lemma is known, but we write it here for the sake of references.

Lemma 2.1. *Let M be a graded module over a graded ring R. Then the following hold:*

(i) *If N is a graded submodule of M, I a graded ideal of R, $r \in h(R)$ and $x \in h(M)$, then Rx, IN and rN are graded submodules of M.*
(ii) *If N and K are graded submodules of M, then $N + K$ and $N \bigcap K$ are also graded submodules of M and $(N : M)$ is a graded ideal of R.*
(iii) *Let $\{N_\lambda\}$ be a collection of graded submodules of M. Then $\sum_\lambda N_\lambda$ and $\bigcap_\lambda N_\lambda$ are graded submodues of M.*

Definition 2.2. Let R be a non-commutative graded ring and N be a proper graded submodule of a graded R-module M. Then N is a graded 2-absorbing submodule of M if $a_g R b_h R m_k \subseteq N$ implies $a_g b_h M \subseteq N$ or $a_g m_k \in N$ or $b_h m_k \in N$ for all $a_g, b_h \in h(R)$ and $m_k \in h(M)$.

Definition 2.3. Let R be a non-commutative graded ring and N be a proper graded submodule of a graded R-module M. Then N is a graded strongly 2-absorbing submodule of M if whenever $a_g, b_h \in h(R)$ and $m_k \in h(M)$ with $a_g b_h m_k \in N$, then $a_g b_h M \subseteq N$ or $a_g m_k \in N$ or $b_h m_k \in N$.

Example 2.4. Consider $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in \mathbb{Z}_2 \right\}$, $M = R$ and $G = \mathbb{Z}_4$ (the group of integers modulo 4). Then R is G-graded by $R_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in \mathbb{Z}_2 \right\}$, $R_2 = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in \mathbb{Z}_2 \right\}$ and $R_1 = R_3 = \{0\}$. Clearly, M is G-graded by $M_g = R_g$ for all $g \in G$. It is clear that $\{0\}$ is a graded strongly 2-absorbing submodule of M, and so it is a graded 2-absorbing submodule of M.

Lemma 2.5. *Let $M = \bigoplus_{g \in G} M_g$ be a graded R-module and $N = \bigoplus_{g \in G} N_g$ be a proper graded submodule of M. If N be a graded 2-absorbing submodule of M, then N_g is a g-2-absorbing R_e-submodule of M_g for all $g \in G$.*

Proof. Let $a, b \in R_e$ and $m \in M_g$ with $aR_e bR_e m \subseteq N_g$. Since N is a graded 2-absorbing submodule of M and $N_g = N \cap M_g \subset N$, we get either $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$. If $ab \in (N :_R M)$, then $ab \in (N_g :_{R_e} M_g)$ as $(N : M) \subset ((N \cap M_g) :_{R_e} M_g) = (N_g :_{R_e} M_g)$. Suppose that $am \in N$. Since $am \in M_g$ and $am \in N$, we have $am \in N \cap M_g = N_g$. If $bm \in N$, then similarly we conclude $bm \in N_g$. Therefore N_g is a g-2-absorbing R_e-submodule of M_g. $\square$

Proposition 2.6. *Let R be a graded ring and N be a graded prime submodule of a graded R-module M. If $a_g R b_h R m_k \subseteq N$ and $a_g m_k \notin N$, then $b_h M \subseteq N$ for all $a_g, b_h \in h(R)$ and $m_k \in h(M)$.*

Proof. Let $a_g, b_h \in h(R)$ and $m_k \in h(M)$. Assume that $a_g R b_h R m_k \subseteq N$ and $a_g m_k \notin N$. First, we show that $b_h R m_k \subseteq N$. Let $r = \sum_{g' \in G} r_{g'}$ be any element of the graded ring R. Then $a_g R(b_h r_{g'} m_k) \subseteq a_g R b_h R m_k \subseteq N$. Since N is a graded prime submodule, $a_g M \subseteq N$ or $b_h r_{g'} m_k \in N$. Then $b_h r_{g'} m_k \in N$ because $a_g m_k \notin N$. Hence $b_h r m_k = \sum_{g' \in G} b_h r_{g'} m_k \in N$, that is $b_h R m_k \subseteq N$. Since N is a graded prime submodule and $a_g m_k \notin N$, it follows that $m_k \notin N$ so that $b_h M \subseteq N$, as needed. □

Definition 2.7. Let R be a non-commutative graded ring and M be a graded R-module. A proper graded submodule N of M is called graded completely prime, if $a_g m_h \in N$ where $a_g \in h(R)$ and $m_h \in h(M)$ implies $m_h \in N$ or $a_g M \subseteq N$.

A graded R-module M is graded completely prime if the zero graded submodule of M is a graded completely prime submodule of M.

Proposition 2.8. *Let N be a proper graded submodule of a graded R-module M. Then the following hold:*

(i) *If N is a graded prime submodule of a graded R-module M, then N is a graded 2-absorbing submodule of M.*
(ii) *If N is a graded completely prime submodule of a graded R-module M, then N is a graded strongly 2-absorbing submodule of M.*

Proof. (i) Assume that N is a graded prime submodule of M. Let $a_g R b_h R m_k \subseteq N$ and $a_g m_k \notin N$ where $a_g, b_h \in h(R)$ and $m_k \in h(M)$. Thus $b_h M \subseteq N$ by Proposition 2.6. Then $b_h m_k \in N$ and $a_g b_h M \subseteq a_g M \subseteq N$. Hence N is a graded 2-absorbing submodule of M.
(ii) Assume that N is a graded completely prime submodule of M and let $a_g b_h m_k \in N$ but $a_g m_k \notin N$ for some $a_g, b_h \in h(R)$ and $m_k \in h(M)$. Hence $a_g M \subseteq N$. Therefore, $a_g b_h M \subseteq a_g M \subseteq N$, and so N is a graded strongly 2-absorbing submodule of M. □

Definition 2.9. Let M be a graded left R-module. M is called a semi-commutative graded module, if whenever $a_g m_h = 0$ for $a_g \in h(R)$ and $m_h \in h(M)$, we have $a_g R m_h = 0$. A graded submodule N of a graded R-module M is a semi-commutative graded submodule, if whenever $a_g m_h \in N$ for $a_g \in h(R)$ and $m_h \in h(M)$, we have $a_g R m_h \subseteq N$.

Proposition 2.10. *Let M be a graded left R-module. If N is a graded 2-absorbing submodule which is also a semi-commutative graded submodule, then N is a graded strongly 2-absorbing submodule.*

Proof. Let $a_g, b_h \in h(R)$ and $m_k \in h(M)$ be such that $a_g b_h m_k \in N$. Since N is a semi-commutative graded submodule, we have $a_g R b_h R m_k \subseteq$

N. Since N is graded 2-absorbing, so $a_g m_k \in N$ or $b_h m_k \in N$ or $a_g b_h M \subseteq N$. Hence N is a graded strongly 2-absorbing submodule of M. □

Lemma 2.11. *Let N be a proper graded submodule of a graded R-module M. Let $g \in G$. If N_g is a g-2-absorbing R_e-submodule of M_g, then $(N_g :_{R_e} M_g)$ is a 2-absorbing ideal of R_e.*

Proof. Let $a, b, c \in R_e$ with $aR_e bR_e c \subseteq (N_g :_{R_e} M_g)$ and suppose that $ac \notin (N_g :_{R_e} M_g)$ and $bc \notin (N_g :_{R_e} M_g)$. We show that $ab \in (N_g :_{R_e} M_g)$. Since $ac, bc \notin (N_g :_{R_e} M_g)$, there exist $m_g, m'_g \in M_g$ such that $acm_g \notin N$ and $bcm'_g \notin N_g$. Now $aR_e bR_e c(m_g + m'_g) \subseteq N_g$. So $ab \in (N_g :_{R_e} M_g)$ or $ac(m_g+m'_g) \in N_g$ or $bc(m_g+m'_g) \in N_g$. If $ac(m_g+m'_g) \in N_g$, then $acm'_g \notin N_g$ since $acm_g \notin N_g$. Similarly, $bcm'_g \notin N_g$. Since $aR_e bR_e cm'_g \subseteq N_g$ and $bcm'_g \notin N_g$ and $acm'_g \notin N_g$ we have $ab \in (N_g :_{R_e} M_g)$. Hence $(N_g :_{R_e} M_g)$ is a 2-absorbing ideal of R_e. □

Proposition 2.12. *The intersection of each pair of graded prime (graded completely prime) submodules of a graded R-module M is a graded 2-absorbing (graded strongly 2-absorbing) submodule of M.*

Proof. Let N and K be two graded prime (graded completely prime) submodules of M. If $N = K$, then $N \cap K$ is a graded prime (graded completely prime) submodule of M, so that $N \cap K$ is a graded 2-absorbing (graded strongly 2-absorbing) submodule of M. Assume that N and K are distinct. Since N and K are proper submodules of M, then $N \cap K$ is a proper submodule of M. Now, let $a_g, b_{g'} \in h(R)$ and $m_h \in h(M)$ be such that $a_g R b_{g'} R m_h \subseteq N \cap K$ ($a_g b_{g'} m_h \in N \cap K$) but $a_g m_h \notin N \cap K$ and $a_g b_{g'} M \nsubseteq N \cap K$. Then we can conclude that

(a) $a_g m_h \notin N$ or $a_g m_h \notin K$, and
(b) $a_g b_{g'} M \nsubseteq N$ or $a_g b_{g'} M \nsubseteq K$.

These two conditions give 4 cases:

(1) $a_g m_h \notin N$ and $a_g b_{g'} M \nsubseteq N$;
(2) $a_g m_h \notin N$ and $a_g b_{g'} M \nsubseteq K$;
(3) $a_g m_h \notin K$ and $a_g b_{g'} M \nsubseteq N$;
(4) $a_g m_h \notin K$ and $a_g b_{g'} M \nsubseteq K$;

We first consider Case (1). Since $a_g R b_{g'} R m_h \subseteq N \cap K \subseteq N$ and $a_g m_h \notin N$, it follows from Proposition 2.6 that $b'_g M \subseteq N$. This is a contradiction because $a_g b'_g M \nsubseteq N$. Hence Case (1) does not occur. Similarly, Case (4) is not possible. Next, Case (2) is considered. Again, we obtain that $b'_g M \subseteq N$ and then $b_{g'} m_h \in N$. Let $r = \sum_{g'' \in G} r_{g''} \in R$. Since $a_g R b_{g'} R m_h \subseteq N \cap K \subseteq K$, it follows that $a_g R(b_{g'} r_{g''} m_h) \subseteq a_g R(b_{g'} R m_h) \subseteq K$. Hence $a_g M \subseteq K$ or $b_{g'} r_{g''} m_h \in K$ because K is a graded prime submodule of M. If $a_gM \subseteq K$, then $a_gb'_gM \subseteq a_gM \subseteq K$ contradicts $a_gb'_gM \nsubseteq K$. Then $b_{g'}r_{g''}m_h \in K$. That is $b_{g'}Rm_h \subseteq K$. Since K is a graded prime submodule, $b_{g'}M \subseteq K$ or $m_h \in K$. If $b_{g'}M \subseteq K$, then $a_gb'_gM \subseteq K$ leading to the same contradiction. Therefore, $m_h \in K$ and then $b_{g'}m_h \in K$. Hence $b_{g'}m_h \in N \cap K$. The proof of Case (3) is similar to that of Case (2). Now, let N and K are graded completely prime submodules of M and $a_gb_{g'}m_h \in N \cap K \subseteq N$. We consider Case (1). Since $a_gb_{g'}m_h \in N$ and N is a graded completely prime, we have $a_gM \subseteq N$ or $b_{g'}m_h \in N$. If $a_gM \subseteq N$, then $a_gb'_gM \subseteq a_gM \subseteq N$ which is not possible. So suppose that $b_{g'}m_h \in N$. Therefore $b'_gM \subseteq N$ or $m_h \in N$. This is not possible and hence Case (1) does not occur. Similarly, Case (4) is not possible. Next, Case (2) is considered. We have $a_gb_{g'}m_h \in N \cap K \subseteq K$ and since K is a graded completely prime submodule of M, it follows that $a_gM \subseteq K$ or $b_{g'}m_h \in K$. If $a_gM \subseteq K$, then $a_gb'_gM \subseteq a_gM \subseteq K$ which contradicts $a_gb'_gM \nsubseteq K$, thus $b_{g'}m_h \in K$. From $a_gb_{g'}m_h \in N \cap K \subseteq N$ we have $a_gM \subseteq N$ or $b_{g'}m_h \in N$. Since $b_{g'}m_h \notin N$, $a_gM \subseteq N$ is not possible. Hence $b_{g'}m_h \in N \cap K$. The proof of Case (3) is similar to that of Case (2). □

Proposition 2.13. *Let N and K be two graded submodules of a graded R-module M and $N \subseteq K$. If N is a graded 2-absorbing (graded strongly 2-absorbing) submodule of M, then N is a graded 2-absorbing (graded strongly 2-absorbing) submodule of K.*

Proof. If $K = M$, then there is nothing to prove. Let $a_gRb_{g'}Rm_h \subseteq N$ ($a_gb_{g'}m_h \in N$) where $a_g, b_{g'} \in h(R)$ and $m_h \in h(K)$. Since N is a graded 2-absorbing (graded strongly 2-absorbing) submodule of M, so either $a_gm_h \in N$ or $b_{g'}m_h \in N$ or $a_gb_{g'} \in (N : M)$. Since $N \subseteq K$, implies $(N : M) \subseteq (N : K)$, then either $a_gm_h \in N$ or $b_{g'}m_h \in N$ or $a_gb_{g'} \in (N : K)$. Therefore N is a graded 2-absorbing (graded strongly 2-absorbing) submodule of K. □

Lemma 2.14. *Let N be a proper graded submodule of a graded R-module M. Let $g \in G$. N_g is a g-2-absorbing (g-strongly 2-absorbing) R_e-submodule of M_g if and only if $aR_ebK \subseteq N_g$ ($abK \subseteq N_g$) implies $ab \in (N_g :_{R_e} M_g)$ or $aK \subseteq N_g$ or $bK \subseteq N_g$ for each $a, b \in R_e$ and R_e-submodule K of M_g.*

Proof. Let N_g be a g-2-absorbing submodule of M_g. Suppose that $ab \notin (N_g :_{R_e} M_g)$ and $aK \nsubseteq N_g$ and $bK \nsubseteq N_g$ for some $a, b \in R_e$ and a submodule K of M_g. Then there exist $m_g, m'_g \in K$ such that $am_g \notin N_g$ and $bm'_g \notin N_g$. Since $aR_ebR_em_g \subseteq aR_ebK \subseteq N_g$ ($abm_g \in abK \subseteq N_g$), $ab \notin (N_g :_{R_e} M_g)$ and $am_g \notin N$ we get $bm_g \in N_g$. Also, since

$aR_ebR_em'_g \subseteq aR_ebK \subseteq N_g$ ($abm'_g \in abK \subseteq N_g$), $ab \notin (N_g :_{R_e} M_g)$ and $bm'_g \notin N_g$ we get $am'_g \in N_g$. Now, since $aR_ebR_e(m_g + m'_g) \subseteq aR_ebK \subseteq N_g$ ($ab(m_g + m'_g) \in abK \subseteq N_g$) and $ab \notin (N_g :_{R_e} M_g)$ we have $a(m_g + m'_g) \in N_g$ or $b(m_g + m'_g) \in N_g$. If $a(m_g + m'_g) \in N_g$, i. e. $(am_g + am'_g) \in N_g$, then since $am'_g \in N_g$ we get $am_g \in N_g$ which is a contradiction. If $b(m_g + m'_g) \in N_g$, i. e. $(bm_g + bm'_g) \in N_g$, then since $bm_g \in N_g$ we get $bm'_g \in N_g$ which is a contradiction. Thus $ab \in (N_g :_{R_e} M_g)$ or $aK \subseteq N_g$ or $bK \subseteq N_g$. The converse is clear. □

Proposition 2.15. *If N_g is a g-2-absorbing submodule of an R_e-module M_g, then $(N_g :_{R_e} R_em_g)$ is a 2-absorbing ideal of R_e for every $m_g \in M_g - N_g$.*

Proof. Let $a, b, c \in R_e$ and $m_g \in M_g - N_g$ be such that $aR_ebR_ec \subseteq (N_g :_{R_e} R_em_g)$. $aR_eb(R_ecR_e)m_g \subseteq N_g$. Since R_ecR_e is an ideal of R_e, we have $(R_ecR_e)m_g$ is an R_e-submodule of M_g. It follows from Lemma 2.14 that $a(R_ecR_e)m_g \subseteq N_g$ or $b(R_ecR_e)m_g \subseteq N_g$ or $abM_g \subseteq N_g$. Thus $ac \in (N_g : R_em_g)$ or $bc \in (N_g : R_em_g)$ or $abR_em_g \subseteq abM_g \subseteq N_g$ i. e. $ab \in (N_g : R_em_g)$, as needed. □

Theorem 2.16. *If N is a proper graded submodule of a graded R-module M. Let $g \in G$. If N_g is a g-2-absorbing (g-strongly 2-absorbing) R_e-submodule of M_g and I and J are ideals of R_e and K an R_e-submodule of M_g such that $IJK \subseteq N_g$, then $IK \subseteq N_g$ or $JK \subseteq N_g$ or $IJ \subseteq (N_g :_{R_e} M_g)$.*

Proof. Suppose $IJK \subseteq N_g$ and $IJ \nsubseteq (N_g : M_g)$. We show that $IK \subseteq N_g$ or $JK \subseteq N_g$. Suppose $IK \nsubseteq N_g$ and $JK \nsubseteq N_g$. There exist $a_1 \in I$ and $a_2 \in J$ such that $a_1K \nsubseteq N_g$ and $a_2K \nsubseteq N_g$. But $a_1R_ea_2K \subseteq IJK \subseteq N_g$ ($a_1a_2K \subseteq IJK \subseteq N_g$). Since N_g is a g-2-absorbing (g-strongly 2-absorbing) R_e-submodule of M_g it follows from Lemma 2.14 that $a_1a_2 \in (N_g : M_g)$. Since $IJ \nsubseteq (N_g : M_g)$, there exist $b_1 \in I$ and $b_2 \in J$ such that $b_1b_2M_g \nsubseteq N_g$. Now since N_g is a g-2-absorbing (g-strongly 2-absorbing) R_e-submodule of M_g and $b_1R_eb_2K \subseteq IJK \subseteq N_g$ ($b_1b_2K \subseteq IJK \subseteq N_g$) and also $b_1b_2M_g \nsubseteq N_g$ it follows from Lemma 2.14 that $b_1K \subseteq N_g$ or $b_2K \subseteq N_g$. We have the following cases:

Case (1): $b_1K \subseteq N_g$ and $b_2K \nsubseteq N_g$.

Since $a_1R_eb_2K \subseteq IJK \subseteq N_g$ ($a_1b_2K \subseteq IJK \subseteq N_g$) and $a_1K \nsubseteq N_g$ and $b_2K \nsubseteq N_g$ it follows from Lemma 2.14 that $a_1b_2 \in (N_g : M_g)$. Since $b_1K \subseteq N_g$ and $a_1K \nsubseteq N_g$, we conclude $(a_1 + b_1)K \nsubseteq N_g$. On the other hand, $(a_1 + b_1)R_eb_2K \subseteq N_g$ ($(a_1 + b_1)b_2K \subseteq N_g$) and neither $(a_1 + b_1)K \subseteq N_g$ nor $b_2K \subseteq N_g$, we get that $(a_1 + b_1)b_2 \in (N_g : M_g)$ by Lemma 2.14. But since $(a_1 + b_1)b_2 = (a_1b_2 + b_1b_2) \in (N_g : M_g)$ and $a_1b_2 \in (N_g : M_g)$, we get $b_1b_2 \in (N_g : M_g)$ which is a contradiction.

Case (2): $b_2K \subseteq N_g$ and $b_1K \nsubseteq N_g$.
By a similar argument to case (1), we get a contradiction.

Case (3): $b_1K \subseteq N_g$ and $b_2K \subseteq N_g$.
$b_2K \subseteq N_g$ and $a_2K \nsubseteq N_g$ gives $(a_2+b_2)K \nsubseteq N_g$. But $a_1R_e(a_2+b_2)K \subseteq N_g$ ($a_1(a_2+b_2)b_2K \subseteq N_g$) and neither $a_1K \subseteq N_g$ nor $(a_2+b_2)K \subseteq N_g$, hence $a_1(a_2+b_2) \in (N_g : M_g)$ by Lemma 2.14. Since $a_1a_2 \in (N_g : M_g)$ and $(a_1a_2 + b_1b_2) \in (N_g : M_g)$, we have $a_1b_2 \in (N_g : M_g)$. Since $(a_1+b_1)R_ea_2K \subseteq N_g$ ($(a_1+b_1)a_2K \subseteq N_g$) and neither $a_2K \subseteq N_g$ nor $(a_1+b_1)K \subseteq N_g$, we conclude $(a_1+b_1)a_2 \in (N_g : M_g)$ by Lemma 2.14. But $(a_1+b_1)a_2 = a_1a_2 + b_1a_2$, so $(a_1a_2+b_1a_2) \in (N_g : M_g)$ and since $a_1a_2 \in (N_g : M_g)$, we get $b_1a_2 \in (N_g : M_g)$. Now, since $(a_1+b_1)R_e(a_2+b_2)K \subseteq N_g$ ($(a_1+b_1)(a_2+b_2)K \subseteq N_g$) and neither $(a_1+b_1)K \subseteq N_g$ nor $(a_2+b_2)K \subseteq N_g$, we have $(a_1+b_1)(a_2+b_2) = (a_1a_2 + a_1b_2 + b_1a_2 + b_1b_2) \in (N_g : M_g)$ by Lemma 2.14. But $a_1a_2, a_1b_2, b_1a_2 \in (N_g : M_g)$, so $b_1b_2 \in (N_g : M_g)$ which is a contradiction. Consequently, $IK \subseteq N_g$ or $JK \subseteq N_g$. □

Corollary 2.17. *Let I and J be two ideals of R_e and P a g-2-absorbing R_e-submodule of M_g. If $m_g \in M_g$ such that $IJm_g \subseteq P$, then $Im_g \subseteq P$ or $Jm_g \subseteq P$ or $IJ \subseteq (P :_{R_e} M_g)$.*

Proof. Let $IJm_g \subseteq P$. Then $IJR_em_g \subseteq P$ and consequently $Im_g \subseteq IR_em_g \subseteq P$ or $Jm_g \subseteq JR_em_g \subseteq P$ or $IJ \subseteq (P : M_g)$. □

Lemma 2.18. *Let I be an ideal of R_e and N_g be a g-2-absorbing (g-strongly 2-absorbing) submodule of M_g. If $a \in R_e$, $m_g \in M_g$ and $IR_eaR_em_g \subseteq N_g$ ($Iam_g \subseteq N_g$), then $am_g \in N_g$ or $Im_g \subseteq N_g$ or $Ia \subseteq (N_g : M_g)$.*

Proof. Suppose that $am_g \notin N_g$ and $Ia \nsubseteq (N_g : M_g)$. Then there exists $b \in I$ such that $ba \notin (N_g : M_g)$. Now, $bR_eaR_em_g \subseteq N_g$ ($bam_g \in N_g$), implies that $bm_g \in N_g$, since N_g is a g-2-absorbing (g-strongly 2-absorbing) R_e-submodule of M_g. We show that $Im_g \subseteq N_g$. Let $c \in I$. Thus $(b+c)R_eaR_em_g \subseteq IR_eaR_em_g \subseteq N_g$ ($(b+c)am_g \in Iam_g \subseteq N_g$). Hence either $(b+c)m_g \in N_g$ or $(b+c)a \in (N_g : M_g)$. If $(b+c)m_g \in N_g$, then by $bm_g \in N_g$ it follows that $cm_g \in N_g$. If $(b+c)a \in (N_g : M_g)$, then $ca \notin (N_g : M_g)$, but $cR_eaR_em_g \subseteq N_g$ ($cam_g \in N_g$). Thus $cm_g \in N_g$. Hence we conclude that $Im_g \subseteq N_g$. □

Corollary 2.19. *Let N_g be a g-2-absorbing (g-strongly 2-absorbing) submodule of M_g. Then $(N_g :_{M_g} I) = \{m_g \in M_g : Im_g \subseteq N_g\}$ is a g-2-absorbing (g-strongly 2-absorbing) submodule of M_g for every ideal I of R_e.*

Proof. Let $a, b \in R_e$ and $m_g \in M_g$ be such that $aR_ebR_em_g \subseteq (N_g :_{M_g} I)$ ($abm_g \subseteq (N_g :_{M_g} I)$). Thus $IaR_ebR_em_g \subseteq N_g$ ($Iabm_g \subseteq N_g$), then

$(IaR_e)R_ebR_em_g \subseteq IaR_ebR_em_g \subseteq N_g$. Hence from Lemma 2.18 we have $(IaR_e)m_g \subseteq N_g$ or $(IaR_gb) \subseteq (N_g : M_g)$ or $bm_g \in N_g$. If $bm_g \in N_g$, then $IR_em_g \subseteq N_g$ and consequently $bm_g \in (N_g : M_g)$ and we are done. If $IaR_eb \subseteq (N_g : M_g)$, then $ab \in aR_eb \subseteq ((N_g :_{R_e} M_g) :_{R_e} I) = ((N_g :_{M_g} I) :_{R_e} M_g)$. If $(IaR_e)m_g \subseteq N_g$, then $am_g \in aR_em_g \subseteq (N_g :_{M_g} I)$. Thus $bm_g \in (N_g :_{M_g} I)$ or $am \in (N_g :_{M_g} I)$ or $ab \in ((N_g :_{M_g} I) :_{R_e} M_g)$ which complete the proof for g-2-absorbing. Since $Iabm_g \subseteq N_g$ and N_g is a g-strongly 2-absorbing, so by Lemma 2.18 we have $abm_g \in N_g$ or $Im_g \subseteq N_g$ or $Iab \subseteq (N_g : M_g)$. If $abm_g \in N_g$, then $am_g \in N_g$ or $bm_g \in N_g$ or $ab \in (N_g : M_g)$. Hence for $am_g \in N_g$ it follows that $Iam_g \subseteq IN_g \subseteq N_g$ and we have $am_g \in (N_g : M_g)$. For $bm_g \in N_g$ it follows that $Ibm_g \subseteq IN_g \subseteq N_g$ and we have $bm_g \in (N_g : M_g)$. For $ab \in (N_g : M_g)$,we have $ab \in ((N_g :_{R_e} M_g) :_{R_e} I) = ((N_g :_{M_g} I) :_{R_e} M_g)$. For $Im_g \subseteq N_g$, we have $m_g \in (N_g :_{M_g} I)$ and thus $am_g \in (N_g :_{M_g} I)$. For $Iab \subseteq (N_g : M_g)$, we have $ab \in ((N_g :_{R_e} M_g) :_{R_e} I) = ((N_g :_{M_g} I) :_{R_e} M_g)$ and $(N_g :_{M_g} I)$ is a g-strongly 2-absorbing submodule of M_g. □

Theorem 2.20. *Let N is a proper graded submodule of a graded R-module M. Let $g \in G$. Let N_g be a g-2-absorbing R_e-submodule of M_g. Then $(N_g :_{R_e} M_g)$ is a prime ideal of R_e if and only if $(N_g :_{R_e} P)$ is a prime ideal of R_e for every submodule P of M_g containing N_g.*

Proof. ($\Rightarrow$) Let I and J be ideals of R_e such that $IJ \subseteq (N_g :_{R_e} P)$. Hence $IJP \subseteq N_g$. Since N_g is a g-2-absorbing R_e-submodule of M_g it follows from Theorem 2.16 that $IP \subseteq N_g$ or $JP \subseteq N_g$ or $IJ \subseteq (N_g :_{R_e} M_g)$. For $IJ \subseteq (N_g :_{R_e} M_g)$, by assumption that $(N_g :_{R_e} M_g)$ is a prime ideal of R_e, we get $IP \subseteq IM_g \subseteq N_g$ or $JP \subseteq JM_g \subseteq N_g$. Hence $I \subseteq (N_g :_{R_e} P)$ or $J \subseteq (N_g :_{R_e} P)$ and so $(N_g :_{R_e} P)$ is a prime ideal of R_e.

($\Leftarrow$) It is clear. □

Proposition 2.21. *Let N and K be graded submodules of a graded R-module M with $K \nsubseteq N$. If N is a graded 2-absorbing (graded strongly 2-absorbing) submodule of M, then $N \cap K$ is a graded 2-absorbing (graded strongly 2-absorbing) submodule of K.*

Proof. Since N and K are graded submodules of M and $K \nsubseteq N$, $K \cap N$ is a proper graded submodule of K. Assume that N is a graded 2-absorbing (graded strongly 2-absorbing) submodule of M. Let $a_g, b_{g'} \in h(R)$ and $x_h \in h(K)$ be such that $a_gRb_{g'}Rm_h \subseteq N$ $(a_gb_{g'}m_h \in N)$. Since K is a graded submodule of M, $a_gb_{g'}K \subseteq K$ and $a_gx_h, b_{g'}x_h \in K$. Moreover, since $a_gRb_{g'}Rm_h \subseteq N \cap K \subseteq N$ $(a_gb_{g'}m_h \in N \cap K \subseteq N)$ and N is a graded 2-absorbing (graded strongly 2-absorbing) submodule of M, $a_gb_{g'}M \subseteq N$ or $a_gx_h \in N$ or $b_{g'}x_h \in N$. Thus $a_gb_{g'}K \subseteq a_gb_{g'}K \cap$

$a_g b_{g'} M \subseteq K \cap N$ or $a_g x_h \in K \cap N$ or $b_{g'} x_h \in K \cap N$. Therefore, $N \cap K$ is a graded 2-absorbing (graded strongly 2-absorbing) submodule of K. $\square$

Proposition 2.22. *Let N and K be graded submodules of a graded R-module M with $K \subseteq N$. Then N is a graded 2-absorbing (graded strongly 2-absorbing) submodule of M if and only if N/K is a graded 2-absorbing (graded strongly 2-absorbing) submodule of M/K.*

Proof. Assume that N is a graded 2-absorbing (graded strongly 2-absorbing) submodule of M, then N/K is a proper graded submodule of M/K. Let $a_g, b_{g'} \in h(R)$ and $(m_h + K) \in h(M/K)$ be such that $a_g R b_{g'} R(m_h+K) \subseteq N/K$ ($a_g b_{g'}(m_h+K) \in N/K$). Let $s, t \in R$. Hence $a_g s b_{g'} t m_h + K = a_g s b_{g'} t(m_h + K) \in N/K$. Then there exists $n \in N$ such that $a_g s b_{g'} t m_h + K = n + K$ so that $a_g s b_{g'} t m_h - n \in K \subseteq N$ and so $a_g s b_{g'} t m_h \in N$. This shows that $a_g R b_{g'} R m_h \subseteq N$. (Similarly, for the graded strongly 2-absorbing case, we get $a_g b_{g'} m_h \in N$). As a result, $a_g m_h \in N$ or $b_{g'} m_h \in N$ or $a_g b_{g'} M \subseteq N$ because N is a graded 2-absorbing (graded strongly 2-absorbing) submodule of M. Therefore, $a_g(m_h + K) \in N/K$ or $b_{g'}(m_h + K) \in N/K$ or $a_g b_{g'}(M/K) \subseteq N/K$. Hence N/K is a graded 2-absorbing (graded strongly 2-absorbing) submodule of M/K.

Conversely, assume that N/K is a graded 2-absorbing (graded strongly 2-absorbing) submodule of M/K. Then N is a proper graded submodule of M. Let $a_g, b_{g'} \in h(R)$ and $m_h \in h(M)$ be such that $a_g R b_{g'} R m_h \subseteq N$ ($a_g b_{g'} m_h \in N$). Then $a_g R b_{g'} R(m_h + K) \subseteq N/K$ ($a_g b_{g'}(m_h + K) \in N/K$). Since N/K is a graded 2-absorbing (graded strongly 2-absorbing) submodule of M/K, we obtain $a_g(m_h + K) \in N/K$ or $b_{g'}(m_h+K) \in N/K$ or $a_g b_{g'}(M/K) \subseteq N/K$. That is $a_g m_h \in N$ or $b_{g'} m_h \in N$ or $a_g b_{g'} M \subseteq N$. This implies that N is a graded 2-absorbing (graded strongly 2-absorbing) submodule of M. $\square$

Let R_1 and R_2 be two G-graded rings. Then $R = R_1 \times R_2$ becomes a G-graded ring with homogeneous elements $h(R) = \bigcup_{g \in G} R_g$, where $R_g = (R_1)_g \times (R_2)_g$ for all $g \in G$. Let M_1 be a graded R_1-module and M_2 be a graded R_2-module. Then $M = M_1 \times M_2$ is a graded $R = R_1 \times R_2$-module [14].

Theorem 2.23. *Let M_1 be a graded R_1-module, M_2 be a graded R_2-module, $R = R_1 \times R_2$ and $M = M_1 \times M_2$. Then*

(i) *N_1 is a graded 2-absorbing submodule of M_1 if and only if $N_1 \times M_2$ is a graded 2-absorbing submodule of M, and*

(ii) *N_2 is a graded 2-absorbing submodule of M_2 if and only if $M_1 \times N_2$ is a graded 2-absorbing submodule of M.*

Proof. (i) Assume that N_1 is a graded 2-absorbing submodule of M_1 and also suppose that $(a_g, b_g)R(c_h, d_h)R(m_k, m'_k) \subseteq N_1 \times M_2$ where $(a_g, b_g), (c_h, d_h) \in h(R)$ and $(m_k, m'_k) \in h(M)$. So $(a_g R_1 c_h R_1 m_k, b_g R_2 d_h R_2 m'_k) = (a_g, b_g)R(c_h, d_h)R(m_k, m'_k) \subseteq N_1 \times M_2$, i. e. $a_g R_1 c_h R_1 m_k \subseteq N_1$ and $b_g R_2 d_h R_2 m'_k \subseteq M_2$. Since N_1 is a graded 2-absorbing submodule of M_1, $a_g c_h M_1 \subseteq N_1$ or $a_g m_k \in N_1$ or $c_h m_k \in N_1$. That is $(a_g, b_g)(c_h, d_h)M = (a_g c_h M_1, b_g d_h M_2) \subseteq N_1 \times M_2$ or $(a_g, b_g)(m_k, m'_k) = (a_g m_k, b_g m'_k) \in N_1 \times M_2$ or $(c_h, d_h)(m_k, m'_k) = (c_h m_k, d_h m'_k) \in N_1 \times M_2$. Therefore $N_1 \times M_2$ is a graded 2-absorbing submodule of M. Conversely, assume $N_1 \times M_2$ is a graded 2-absorbing submodule of M. Let $a_g, b_h \in h(R_1)$ and $m_k \in h(M_1)$ be such that $a_g R_1 b_h R_1 m_k \subseteq N_1$. Let $x_g, y_h \in h(R_2)$ and $m'_k \in h(M_2)$. Then $(a_g, x_g)R(b_h, y_h)R(m_k, m'_k) = (a_g R_1 b_h R_1 m_k, x_g R_2 y_h R_2 m'_k) \subseteq N_1 \times M_2$. Since $N_1 \times M_2$ is a graded 2-absorbing submodule of M, $(a_g, x_g)(b_h, y_h)M \subseteq N_1 \times M_2$ or $(a_g, x_g)(m_k, m'_k) \in N_1 \times M_2$ or $(b_h, y_h)(m_k, m'_k) \in N_1 \times M_2$. So $(a_g b_h M_1, x_g y_h M_2) = (a_g, x_g)(b_h, y_h)M \subseteq N_1 \times M_2$ or $(a_g m_k, x_g m'_k) = (a_g, x_g)(m_k, m'_k) \in N_1 \times M_2$ or $(b_h m_k, y_h m'_k) = (b_h, y_h)(m_k, m'_k) \in N_1 \times M_2$, i. e. $a_g b_h M_1 \subseteq N_1$ or $a_g m_k \in N_1$ or $b_h m_k \in N_1$. Thus N_1 is a graded 2-absorbing submodule of M_1.
(ii) The proof is similar to that (i). $\square$

Theorem 2.24. *Let M_1 be a graded R_1-module, M_2 be a graded R_2-module, $R = R_1 \times R_2$ and $M = M_1 \times M_2$. Then*

(i) *N_1 is a graded strongly 2-absorbing submodule of M_1 if and only if $N_1 \times M_2$ is a graded strongly 2-absorbing submodule of M, and*
(ii) *N_2 is a graded strongly 2-absorbing submodule of M_2 if and only if $M_1 \times N_2$ is a graded strongly 2-absorbing submodule of M.*

Proof. (i) Assume that N_1 is a graded strongly 2-absorbing submodule of M_1 and also suppose that $(a_g, b_g)(c_h, d_h)(m_k, m'_k) \in N_1 \times M_2$ where $(a_g, b_{g'}), (c_h, d_h) \in h(R)$ and $(m_k, m'_k) \in h(M)$. Then $(a_g c_h m_k, b_g d_h m'_k) = (a_g, b_g)(c_h, d_h)(m_k, m'_k) \in N_1 \times M_2$, i. e. $a_g c_h m_k \in N_1$ and $b_g d_h m'_k \in M_2$. Since N_1 is a graded strongly 2-absorbing R_1-submodule of M_1, it follows that $a_g c_h M_1 \subseteq N_1$ or $a_g m_k \in N_1$ or $c_h m_k \in N_1$. That is $(a_g, b_g)(c_h, d_h)M = (a_g c_h M_1, b_g d_h M_2) \subseteq N_1 \times M_2$ or $(a_g, b_g)(m_k, m'_k) = (a_g m_k, b_g m'_k) \in N_1 \times M_2$ or $(c_h, d_h)(m_k, m'_k) = (c_h m_k, d_h m'_k) \in N_1 \times M_2$. Therefore $N_1 \times M_2$ is a graded strongly 2-absorbing submodule of M. Conversely, assume that $N_1 \times M_2$ is a graded strongly 2-absorbing submodule of M. Let $a_g, b_h \in h(R_1)$ and $m_k \in h(M_1)$ be such that $a_g b_h m_k \in N_1$. Let $x_g, y_h \in h(R_2)$ and $m'_k \in h(M_2)$. Then $(a_g, x_{g'})(b_h, y_{h'})(m_k, m'_k) = (a_g b_h m_k, x_g y_h m'_k) \in N_1 \times M_2$. Since $N_1 \times M_2$ is a graded strongly 2-absorbing R-submodule of M, $(a_g, x_g)(b_h, y_h)M \subseteq$

$N_1 \times M_2$ or $(a_g, x_g)(m_k, m'_k) \in N_1 \times M_2$ or $(b_h, y_h)(m_k, m'_k) \in N_1 \times M_2$. Thus $(a_g b_h M_1, x_g y_h M_2) = (a_g, x_g)(b_h, y_h)M \subseteq N_1 \times M_2$ or $(a_g m_k, x_g m'_k) = (a_g, x_g)(m_k, m'_k) \in N_1 \times M_2$ or $(b_h m_k, y_h m'_k) = (b_h, y_h)(m_k, m'_k) \in N_1 \times M_2$, i. e. $a_g b_h M_1 \subseteq N_1$ or $a_g m_k \in N_1$ or $b_h m_k \in N_1$. Hence N_1 is a graded strongly 2-absorbing submodule of M_1.
(ii) The proof is similar to that (i). □

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