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RESOLVABILITY IN COMPLEMENT OF THE INTERSECTION GRAPH OF ANNIHILATOR SUBMODULES OF A MODULE

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Abstract. Let R be a commutative ring and M be an R-module. The intersection graph of annihilator submodules of M, denoted by $GA(M)$, is a simple undirected graph whose vertices are the classes of elements of $Z(M) \setminus \mathrm{Ann}_R(M)$ and two distinct classes $[a]$ and $[b]$ are adjacent if and only if $\mathrm{Ann}_M(a) \cap \mathrm{Ann}_M(b) \neq 0$. In this paper, we study the diameter and girth of $\overline{GA(M)}$. Furthermore, we calculate the domination number, metric dimension, adjacency metric dimension and local metric dimension of $\overline{GA(M)}$.

1. Introduction

The intersection graph of ideals of a commutative ring was studied in [5] and rings classified with some specific properties of their intersection graphs in [2, 9, 11]. The intersection graph of submodules of a module defined and studied in [1]. As noted in [1] the intersection graph of submodules of a module, denoted by $G(M)$, is a graph whose vertices are in one to one correspondence with all non-trivial submodules of M and two distinct vertices are adjacent if and only if the corresponding submodules of M have non-zero intersection. The complement of the intersection graph of submodules of a module is considered in [3]. For more work on the intersection graph of modules see [14].

Let R be a commutative ring and M be an R-module. For $a, b \in R$, we say that $a \sim b$ whenever $\mathrm{Ann}_M(a) = \mathrm{Ann}_M(b)$. It is easy to see that

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$\sim$ is an equivalence relation on R. If $[a]$ denotes the class of a, then $[a] = \mathrm{Ann}_R(M)$ and $[a] = R \setminus Z(M)$ whenever $a \in \mathrm{Ann}_R(M)$ and $a \in R \setminus Z(M)$ respectively; the other equivalence classes form a partition of $Z(M) \setminus \mathrm{Ann}_R(M)$. The intersection graph of annihilator submodules of M studied in [10] and denoted by $GA(M)$, is a simple undirected graph whose vertices are the classes of elements of $Z(M) \setminus \mathrm{Ann}_R(M)$ and two distinct classes $[a]$ and $[b]$ are adjacent whenever $\mathrm{Ann}_M(a) \cap \mathrm{Ann}_M(b) \neq 0$. In this paper, we study the complement of the intersection graph of annihilator submodules of M which is denoted by $\overline{GA(M)}$. In Section 2, we study the diameter and girth of $\overline{GA(M)}$ and in Section 3 we calculate the domination number, metric dimension, adjacency metric dimension and local metric dimension of $\overline{GA(M)}$. More precisely; let M be a Noetherian R-module, $GA(M)$, $\overline{GA(M)}$ be connected graphs of order m and $|m - \mathrm{Ass}_R(M)| = \omega(GA(M)) = n (n \geq 2)$ we prove that

(i) $\gamma(\overline{GA(M)}) = n$

(ii) If $\overline{GA(M)}$ has k end-vertices, then
$$\dim(\overline{GA(M)}) = \dim_A(\overline{GA(M)}) = m - 2n + k - 1.$$

(iii) If $\overline{GA(M)}$ has no end-vertex, then
$$\dim(\overline{GA(M)}) = \dim_A(\overline{GA(M)}) = m - 2n.$$

(iv) $\dim_\ell(\overline{GA(M)}) = n - 1$.

Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. A graph with no edge is called null graph. For every $u, v \in V(G)$, the distance between u and v is defined as the length of a shortest path from u to v and is denoted by $\mathrm{d}(u, v)$. We write $u - v$ if $\mathrm{d}(u, v) = 1$ and $u \not- v$ otherwise. For $H \subseteq V(G)$, the induced subgraph on H, consists of H and all edges whose endpoints are contained in H. Assume that u is a vertex of G. The open neighborhood of u is defined as $N(u) = \{v \in V(G) \ : \ \mathrm{d}(u, v) = 1\}$ and the closed neighborhood of u is $N[u] = N(u) \cup \{u\}$. For distinct vertices $u, v \in V(G)$, if $N(u) = N(v)$, then u and v are non-adjacent twins. The degree of a vertex u, denoted by $\deg(u)$, is the number of edges incident to u. Also, u is called end-vertex if $\deg(u) = 1$. The diameter of G is $\mathrm{diam}(G) = \sup\{\mathrm{d}(u, v) |$ u and v are vertices of G$\}$. The girth of G, denoted by $\mathrm{gr}(G)$, is the length of a shortest cycle in G ($\mathrm{gr}(G) = \infty$ if G contains no cycles). A cycle with n vertices will be denoted by C_n. The complete graph with n vertices will be denoted by K_n. A complete bipartite graph is a graph G whose vertex set may be partitioned into two disjoint non-empty vertex sets A and B such that two distinct vertices are adjacent if and only if they are in distinct vertex sets which is denoted by $K_{|A|,|B|}$. A clique of G is a complete subgraph of G and the number of vertices in the largest clique of G, denoted by $\omega(G)$, is called the clique number of G.

A dominating set of G is a subset D of $V(G)$ such that every vertex in $V(G) \setminus D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set. Let G be a connected graph. Assume that $W = \{w_1, w_2, \ldots, w_k\}$ is an ordered subset of $V(G)$. The metric representation (local metric representation) of a vertex $u \in V(G)$ with respect to W is the vector $r(u \mid W) = (\mathrm{d}(u, w_1), \mathrm{d}(u, w_2), \ldots, \mathrm{d}(u, w_k))$. The set W called a resolving (local resolving) set for G if different vertices (adjacent vertices) of G have different representation with respect to W. The minimum cardinality of any resolving (local resolving) set of G is the metric dimension (local metric dimension) of G and is denoted by $\dim(G)$ ($\dim_\ell(G)$). The adjacency representation of a vertex $u \in V(G)$ with respect to an ordered set $W_A = \{w_1, w_2, \ldots, w_k\}$ is the vector $r(u \mid W) = (a_G(u, w_1), \ldots, a_G(u, w_k))$, where

$$a_G(u,v) = \begin{cases} 0 & u = v \\ 1 & u - v \\ 2 & u \not- v \end{cases}$$

for all $v \in V(G)$. The set W_A is an adjacency resolving set for G if the vectors $r(u \mid W_A)$ are distinct for every $u \in V(G)$. The minimum cardinality of an adjacency resolving set is the adjacency dimension of G, denoted by $\dim_A(G)$, see [7].

Throughout this paper, R is a commutative ring with non-zero identity and M is an R-module. The set of zero-divisors of M, denoted by $Z(M)$ is defined to be the set $\{r \in R \ : \ rm = 0 \text{ for some } 0 \neq m \in M\}$. For $a \in R$, $\mathrm{Ann}_M(a) = \{m \in M \ : \ am = 0\}$. A proper submodule P of M is said to be prime whenever for $r \in R$ and $m \in M$, $rm \in P$ implies that $m \in P$ or $r \in \mathrm{Ann}_R(M/P)$. Let $\mathrm{Spec}_R(M)$ denote the set of prime submodules of M and $m - \mathrm{Ass}_R(M) = \{P \in \mathrm{Spec}_R(M) : P = \mathrm{Ann}_M(a) \text{ for some } 0 \neq a \in R\}$. For notations and terminologies not given in this article, the reader is referred to [12, 13].

Theorem 1.1. [4, Theorem 5(i)] *For all $a \in R$, $aM + \mathrm{Ann}_M(a)$ is an essential submodule of M.*

Theorem 1.2. [10, Theorem 2.6] *Let M be a Noetherian R-module. Then $GA(M)$ is a disconnected graph if and only if $m - \mathrm{Ass}_R(M) = \{P_1, P_2\}$ and $P_1 \cap P_2 = 0$.*

Theorem 1.3. [6, Corollary 2.4] *Suppose that u, v are twins in a connected graph G and S resolves G. Then either u or v is in S. Moreover, if $u \in S$ and $v \notin S$, then $(S \setminus \{u\}) \cup \{v\}$ also resolves G.*

2. Diameter and Girth of $\overline{GA(M)}$

In this section, we study diameter and girth of $\overline{GA(M)}$. Note that if M is a Noetherian R-module and $m-\mathrm{Ass}_R(M) = \{\mathrm{Ann}_M(a)\}$, then $[a]$ is a universal vertex in $GA(M)$ so $\overline{GA(M)}$ is a disconnected graph. Hence, $|m-\mathrm{Ass}_R(M)| \geq 2$ whenever we assume $\overline{GA(M)}$ is a connected graph.

Lemma 2.1. *Let M be a Noetherian R-module and $\overline{GA(M)}$ be a non-empty connected graph. If $\mathrm{Ann}_M(a), \mathrm{Ann}_M(b) \in m-\mathrm{Ass}_R(M)$, then $[a]$ and $[b]$ have no common neighbors in $\overline{GA(M)}$.*

Proof. Assume that $a, b \in Z(M) \setminus \mathrm{Ann}_R(M)$ and $P_1 = \mathrm{Ann}_M(a), P_2 = \mathrm{Ann}_M(b)$ are two distinct elements of $m-\mathrm{Ass}_R(M)$. Assume in the contrary that $[a]$ and $[b]$ have a common neighbor in $\overline{GA(M)}$ such as $[x]$. Thus $P_1 \cap \mathrm{Ann}_M(x) = 0 = P_2 \cap \mathrm{Ann}_M(x)$. So $\mathrm{Ann}_M(x) \nsubseteq P_2$. Suppose that $P_1 \nsubseteq P_2$. Hence, there exist $m_1 \in \mathrm{Ann}_M(x) \setminus P_2$ and $m_2 \in P_1 \setminus P_2$ such that $xm_1 = am_2 = 0 \in P_2$. So it follows that $a, x \in \mathrm{Ann}_R(M/\mathrm{Ann}_M(b)) = \mathrm{Ann}_R(bM)$. Therefore, $bM \subseteq \mathrm{Ann}_M(a) \cap \mathrm{Ann}_M(x) = P_1 \cap \mathrm{Ann}_M(x)$. Thus $bM = 0$ and $b \in \mathrm{Ann}_R(M)$, contrary to the assumption. Hence, $P_1 \subseteq P_2$. By a similar argument one can show that $P_2 \subseteq P_1$ so $P_1 = P_2$ that is a contradiction. □

Lemma 2.2. *Let M be a Noetherian R-module and $GA(M)$, $\overline{GA(M)}$ be non-empty connected graphs. If $P_1 = \mathrm{Ann}_M(a), P_2 = \mathrm{Ann}_M(b) \in m-\mathrm{Ass}_R(M)$, then $\mathrm{d}([a], [b]) = 3$.*

Proof. Let $P_1 = \mathrm{Ann}_M(a)$ and $P_2 = \mathrm{Ann}_M(b)$ be two distinct elements of $m-\mathrm{Ass}_R(M)$. By the assumption $GA(M)$ is a connected graph so by Theorem 1.2, $P_1 \cap P_2 \neq 0$. Thus $\mathrm{d}([a], [b]) \neq 1$ also Lemma 2.1 shows that $\mathrm{d}([a], [b]) \neq 2$. Let $[x], [y]$ be two arbitrary vertices of $\overline{GA(M)}$ such that $[x] - [a]$ and $[y] - [b]$. Then $\mathrm{Ann}_M(a) \cap \mathrm{Ann}_M(x) = 0 \subseteq P_2$ so either $\mathrm{Ann}_M(a) \subseteq P_2$ or $\mathrm{Ann}_M(x) \subseteq P_2$. If $\mathrm{Ann}_M(a) \subseteq \mathrm{Ann}_M(b)$, then $[a] - [y] - [b]$ contrary to the Lemma 2.1. Thus $\mathrm{Ann}_M(x) \subseteq \mathrm{Ann}_M(b)$ and so $[x] - [y]$ which completed the proof. □

Lemma 2.3. *Let M be a Noetherian R-module and $\overline{GA(M)}$ be a non-empty connected graph. If $P = \mathrm{Ann}_M(a) \in m-\mathrm{Ass}_R(M)$, then the induced subgraph on the vertices which are adjacent to $[a]$ is empty.*

Proof. Let $P = \mathrm{Ann}_M(a) \in m - \mathrm{Ass}_R(M)$ and let $[x], [y] \in V(\overline{GA(M)})$ be such that $[x] - [a] - [y]$. Suppose on the contrary that $[x] - [y]$. Thus $\mathrm{Ann}_M(x) \cap \mathrm{Ann}_M(y) = 0 \subseteq P$. Hence, either $\mathrm{Ann}_M(x) \subseteq P$ or $\mathrm{Ann}_M(y) \subseteq P$ which contradicts to the assumption $[x] - [a] - [y]$. □

Theorem 2.4. *Let M be a Noetherian R-module and $\overline{GA(M)}$ be a non-empty connected graph. Then* $\mathrm{diam}(\overline{GA(M)}) \leq 3$.

Proof. Let $[a]$ and $[b]$ be two distinct vertices of $\overline{GA(M)}$. If $\mathrm{Ann}_M(a) \cap \mathrm{Ann}_M(b) = 0$, then $\mathrm{d}([a], [b]) = 1$. Thus assume that $\mathrm{Ann}_M(a) \cap \mathrm{Ann}_M(b) \neq 0$. If $ab \notin \mathrm{Ann}_R(M)$, then $[ab] \in V(\overline{GA(M)})$ and by connectivity of $\overline{GA(M)}$, it follows that there exists $[x] \in V(\overline{GA(M)})$ such that $\mathrm{Ann}_M(x) \cap \mathrm{Ann}_M(ab) = 0$. Hence, $\mathrm{Ann}_M(x) \cap \mathrm{Ann}_M(a) = 0$ and $\mathrm{Ann}_M(x) \cap \mathrm{Ann}_M(b) = 0$. So $\overline{GA(M)}$ has the path $[a] - [x] - [b]$ as a subgraph. Therefore, $\mathrm{d}([a], [b]) \leq 2$. Now, assume that $ab \in \mathrm{Ann}_R(M)$. If there exists $\mathrm{Ann}_M(x) \in m - \mathrm{Ass}_R(M)$ such that $\mathrm{Ann}_M(a) \subseteq \mathrm{Ann}_M(x)$ and $\mathrm{Ann}_M(b) \subseteq \mathrm{Ann}_M(x)$, then $aM \subseteq \mathrm{Ann}_M(b) \subseteq \mathrm{Ann}_M(x)$ so $aM + \mathrm{Ann}_M(a) \subseteq \mathrm{Ann}_M(x)$. By Theorem 1.1, $[x]$ is a universal vertex of $GA(M)$. Hence, $[x]$ is an isolated vertex of $\overline{GA(M)}$ that is a contradiction. Hence, $\mathrm{Ann}_M(a) \subseteq \mathrm{Ann}_M(x)$ and $\mathrm{Ann}_M(b) \subseteq \mathrm{Ann}_M(y)$ for some $\mathrm{Ann}_M(x), \mathrm{Ann}_M(y) \in m - \mathrm{Ass}_R(M)$. By Lemma 2.2, there exist $[e], [f] \in V(\overline{GA(M)})$ such that $\mathrm{Ann}_M(x) \cap \mathrm{Ann}_M(e) = 0$ and $\mathrm{Ann}_M(y) \cap \mathrm{Ann}_M(f) = 0$ so $[x] - [e] - [f] - [y]$ is a path in $\overline{GA(M)}$. Hence, $[a] - [e] - [f] - [b]$ is a subgraph of $\overline{GA(M)}$ which implies that $\mathrm{diam}(\overline{GA(M)}) \leq 3$. □

Corollary 2.5. *Let M be a Noetherian R-module and $GA(M)$ be a disconnected graph. Then $\overline{GA(M)}$ is a connected graph and* $\mathrm{diam}(\overline{GA(M)}) \leq 2$.

Theorem 2.6. *Let M be a Noetherian R-module and $\overline{GA(M)}$ be a connected graph. Then either* $\mathrm{gr}(\overline{GA(M)}) \leq 4$ *or* $\mathrm{gr}(\overline{GA(M)}) = \infty$.

Proof. Assume that $n \in \mathbb{N}$ and $C = ([a_1], \ldots, [a_n])$ is a cycle in $\overline{GA(M)}$. Suppose that $\mathrm{Ann}_M(x) = P \in m - \mathrm{Ass}_R(M)$ and $[y]$ is a vertex of $V(\overline{GA(M)})$ that is adjacent to $[x]$. Since $\mathrm{Ann}_M(a_1) \cap \mathrm{Ann}_M(a_2) = 0 \subseteq P$, either $\mathrm{Ann}_M(a_1) \subseteq P$ or $\mathrm{Ann}_M(a_2) \subseteq P$. If $\mathrm{Ann}_M(a_1) \subseteq P$ and $\mathrm{Ann}_M(a_2) \subseteq P$, then $\overline{GA(M)}$ has the cycle $[a_1] - [y] - [a_2] - [a_1]$ as a subgraph. Now, assume that $\mathrm{Ann}_M(a_1) \subseteq P$ and $\mathrm{Ann}_M(a_2) \nsubseteq P$. The fact $\mathrm{Ann}_M(a_1) \subseteq P$ implies that $[a_1]$ is adjacent to $[y]$. On the other hand, since $\mathrm{Ann}_M(a_2) \cap \mathrm{Ann}_M(a_3) = 0 \subseteq P$, $\mathrm{Ann}_M(a_3) \subseteq P$ and so $[a_3]$ is adjacent to $[y]$. Thus, $[a_1] - [a_2] - [a_3] - [y] - [a_1]$ is a cycle of length 4 in $\overline{GA(M)}$. Therefore, $gr(\overline{GA(M)}) \leq 4$.

Assume that $[y] \in V(C)$. Without loss of generality, we may assume that $[y] = [a_1]$. Thus $\mathrm{Ann}_M(a_1) \cap P = 0$. Since $\mathrm{Ann}_M(a_1) \cap \mathrm{Ann}_M(a_2) = 0$ and $\mathrm{Ann}_M(a_1) \nsubseteq P$, $\mathrm{Ann}_M(a_2) \subseteq P$. If $\mathrm{Ann}_M(a_3) \subseteq P$, then $\overline{GA(M)}$ has the cycle $[a_1] - [a_2] - [a_3] - [a_1]$ as a subgraph. If $\mathrm{Ann}_M(a_3) \nsubseteq P$ as before one can show that $[a_1] - [a_2] - [a_3] - [a_4] - [a_1]$ is a subgraph of $\overline{GA(M)}$. In the sequel, let $P = \mathrm{Ann}_M(a_1)$. Then $\mathrm{Ann}_M(a_2) \nsubseteq P$. So $\mathrm{Ann}_M(a_3) \subseteq P$ which implies that $\mathrm{Ann}_M(a_3) \cap \mathrm{Ann}_M(a_n) = 0$, hence $\overline{GA(M)}$ has the cycle $[a_1] - [a_2] - [a_3] - [a_n] - [a_1]$ as a subgraph. Therefore, either $\mathrm{gr}(\overline{GA(M)}) \leq 4$ or $\mathrm{gr}(\overline{GA(M)}) = \infty$. □

Theorem 2.7. *Let M be a Noetherian R-module and $GA(M)$ be a disconnected graph. Then $\overline{GA(M)}$ is a complete bipartite graph and $\mathrm{gr}(\overline{GA(M)}) \in \{4, \infty\}$.*

Proof. It is obvious that $\overline{GA(M)}$ is a connected graph. By Theorem 1.2, $m - \mathrm{Ass}_R(M) = \{\mathrm{Ann}_M(a) = P_1, \mathrm{Ann}_M(b) = P_2\}$ and $P_1 \cap P_2 = 0$ so $[a]$ and $[b]$ are adjacent in $\overline{GA(M)}$. Furthermore, for every vertex $[x] \in V(GA(M))$ we have either $\mathrm{Ann}_M(x) \subseteq P_1$ or $\mathrm{Ann}_M(x) \subseteq P_2$, see [8, Proposition 3.2].

Let $V_1 = \{[x] : \mathrm{Ann}_M(x) \subseteq P_1\}$ and $V_2 = \{[y] : \mathrm{Ann}_M(y) \subseteq P_2\}$. By the previous paragraph it is obvious that $V_1 \cap V_2 = N([b]) \cap N([a]) = \emptyset$ also any vertex in V_1 is adjacent to all vertices in V_2 and conversely any vertex in V_2 is adjacent to all vertices in V_1. On the other hand, Lemma 2.2 shows that the induced subgraph on $N([a])$ and $N([b])$ is empty. Thus two distinct vertices in V_1 are not adjacent and the same is true for vertices in V_2. Hence, $\overline{GA(M)}$ is a complete bipartite graph $K_{|N([a])|,|N([b])|}$. Therefore, $\mathrm{gr}(\overline{GA(M)}) = 4$ or $\mathrm{gr}(\overline{GA(M)}) = \infty$. □

3. Metric Dimension, Local Metric Dimension and Adjacency Metric Dimension of $\overline{GA(M)}$

In this section, we study domination number, metric dimension, adjacency metric dimension and local metric dimension of $\overline{GA(M)}$.

Theorem 3.1. *Let M be a Noetherian R-module and $GA(M)$ be a non-empty connected graph. If $|m - \mathrm{Ass}_R(M)| = \omega(GA(M)) = n$, then $\gamma(\overline{GA(M)}) = n$.*

Proof. Let $m - \mathrm{Ass}_R(M) = \{\mathrm{Ann}_M(a_1), \cdots, \mathrm{Ann}_M(a_n)\}$ and let $D = \{[a_1], \ldots, [a_n]\}$ we show that D is a dominating set for $\overline{GA(M)}$. By [10, Corollary 2.7], two arbitrary elements of $m - \mathrm{Ass}_R(M)$ have non-zero intersection. Thus D is a clique for $GA(M)$. Let $[x]$ be an arbitrary vertex of $\overline{GA(M)}$. If $[x]$ is adjacent to any vertices of D in $GA(M)$, then $\omega(GA(M)) \geq n+1$ which contradicts the assumption. Thus there is at least one element $\mathrm{Ann}_M(a_i)$ in $m-\mathrm{Ass}_R(M)$ with $1 \leq i \leq n$ such that $\mathrm{Ann}_M(a_i) \cap \mathrm{Ann}_M(x) = 0$. Hence, $[x]$ is adjacent to $[a_i]$ in $\overline{GA(M)}$. Therefore, all vertices out of D are adjacent in $\overline{GA(M)}$ to at least one vertex in D. So D is a dominating set for $\overline{GA(M)}$ which implies that $\gamma(\overline{GA(M)}) \leq n$.

Assume that $\overline{GA(M)}$ has a dominating set with less than n elements such as $D' = \{[b_1], \ldots, [b_t]\}(t < n)$. Thus there are two elements in D with a common neighbor in D' this contradict with Lemma 2.1. Hence, $\gamma(\overline{GA(M)}) = n$. □

Theorem 3.2. *Let M be a Noetherian R-module and let $GA(M)$ and $\overline{GA(M)}$ be non-empty connected graphs of order m. Let $|m-\mathrm{Ass}_R(M)| = \omega(GA(M)) = n (n \geq 2)$. Then the following statements are true:*

(i) *If $\overline{GA(M)}$ has k end-vertices, then $\dim(\overline{GA(M)}) = m - 2n + k - 1$.*
(ii) *If $\overline{GA(M)}$ has no end-vertex, then $\dim(\overline{GA(M)}) = m - 2n$.*

Proof. Let $m - \mathrm{Ass}_R(M) = \{\mathrm{Ann}_M(a_1), \cdots, \mathrm{Ann}_M(a_n)\}$ and $D = \{[a_1], \ldots, [a_n]\}$. Assume that $[b]$ is a vertex of $\overline{GA(M)}$ such that is not adjacent to any $[a_i]$ in $\overline{GA(M)}$. Thus $[b]$ is adjacent to $[a_i]$, for all $1 \leq i \leq n$, in $GA(M)$ and so $\omega(GA(M)) \geq n + 1$ which is a contradiction. Now, assume that $[b] \neq [a_i]$, for all $1 \leq i \leq n$, is a vertex of $\overline{GA(M)}$ we may assume that $[b] - [a_1]$. Since $\overline{GA(M)}$ is connected and $\mathrm{diam}(\overline{GA(M)}) \leq 3$, there is a path such as $[a_2] - [c] - [e] - [b]$ that $[c] \neq [a_1]$ and $[e] \neq [a_1]$, see Lemma 2.3. Hence, it follows that the end-vertices of $\overline{GA(M)}$ must belong to D. Without loss of generality, suppose that $\{[a_1], \cdots, [a_k]\}$ is the set of end-vertices of $\overline{GA(M)}$, where $1 \leq k \leq n$. Suppose that $N_{\overline{GA(M)}}([a_i]) = \{[u_{i1}], \ldots, [u_{it_i}]\}$, where $t_i \in \mathbb{N}$ for all $1 \leq i \leq n$.

(i) Consider the ordered set

$$W = \{[u_{11}], \ldots, [u_{(k-1)1}]\} \cup (\bigcup_{k+1 \leq i \leq n} (N([a_i]) \setminus \{[u_{i1}]\}))$$

of vertices of $\overline{GA(M)}$. Let $k+1 \leq i \leq n$. Then $r([u_{i1}] \mid W)$ have values 2 and 1 in its components corresponding to $[u_{i2}]$ and $[u_{j2}]$ respectively, where $k + 1 \leq j \leq n, i \neq j$; and $r([u_{k1}] \mid W)$ have value 1 in all components. Thus $[u_{i1}]$, for all $k \leq i \leq n$, have distinct representations with respect to W. Furthermore, for all $1 \leq i \leq k-1$, $r([a_i] \mid W)$ have values 1 and 2 in its components corresponding to $[u_{i1}]$ and $[u_{j1}]$, where $1 \leq j \leq k-1, j \neq i$, and $r([a_i] \mid W)$, for all $k+1 \leq i \leq n$, have values 1 and 2 in its components corresponding to $[u_{i2}]$ and $[u_{j2}]$, where $k+1 \leq j \leq n, j \neq i$; and $r([a_k] \mid W)$ have value 2 in all components. Hence, all vertices of $\overline{GA(M)}$ have different representation with respect to W and therefore it is a resolving set for $\overline{GA(M)}$. Thus $\dim(\overline{GA(M)}) \leq m - 2n + k - 1$.

On the other hand, assume that W_0 is a resolving set for $\overline{GA(M)}$. Since all vertices contained in $N([a_i])$ are twins so Theorem 1.3 implies that $|N([a_i]) \cap W_0| = |N([a_i])| - 1$, for all i with $k+1 \leq i \leq n$. Thus $|W_0| \geq m - 2n$. We may assume that $[a_1], [u_{11}] \notin W_0$ since they have distinct representations with respect to W_0 by using $|N([a_i]) \cap W_0| = |N([a_i])| - 1$. For $2 \leq i \leq k$, if $[u_{i1}], [a_i] \notin W_0$, then $r([u_{11}] \mid W_0) = r([u_{i1}] \mid W_0)$ which contradicts the fact that W_0 is a resolving set for $\overline{GA(M)}$. Hence, either $[u_{i1}] \in W_0$ or $[a_i] \in W_0$ for all i with $2 \leq i \leq k$. Thus $|W_0| \geq m-2n+k-1$ and therefore $\dim(\overline{GA(M)}) = m-2n+k-1$. The proof is completed.

(ii) Consider the ordered set

$$W = \bigcup_{1 \leq i \leq n} \big(N([a_i]) \setminus \{[u_{1i}]\}\big)$$

of vertices of $\overline{GA(M)}$. Let $1 \leq i \leq n$. Then $r([u_{i1}] \mid W)$ has values 2 and 1 in its components corresponding to $[u_{i2}]$ and $[u_{j2}]$ respectively, where $1 \leq j \leq n$ and $i \neq j$. Thus $[u_{i1}]$ have distinct representations with respect to W, for all $1 \leq i \leq n$. Also $r([a_i] \mid W)$, for all $1 \leq i \leq n$, has values 1 and 2 in its components corresponding to $[u_{i2}]$ and $[u_{j2}]$ respectively, where $1 \leq j \leq n$ and $j \neq i$. Hence, every vertex out of W has an unique representation with respect to W. Therefore, W is a resolving set for $\overline{GA(M)}$. Thus, $\dim(\overline{GA(M)}) \leq m - 2n$.

On the other hand, assume that W_0 is a resolving set for $\overline{GA(M)}$. Since all vertices contained in $N([a_i])$ are twins so Theorem 1.3 implies that $|N([a_i]) \cap W_0| = |N([a_i])| - 1$, where $1 \leq i \leq n$. So $|W_0| \geq m-2n$. Hence, W is a resolving set for $\overline{GA(M)}$ and $\dim(\overline{GA(M)}) = m-2n$. □

From the definitions of the metric and adjacency metric dimensions, it follows that $\dim(G) \leq \dim_A(G)$. This inequality and Theorem 3.2 give a lower bound for the adjacency metric dimension of $\overline{GA(M)}$.

Corollary 3.3. *Let M be a Noetherian R-module and let $GA(M)$ be a non-empty connected graph of order m. Let $|m - \mathrm{Ass}_R(M)| = \omega(GA(M)) = n$. Then the following statements are true:*

(i) *If $\overline{GA(M)}$ has k end-vertices, then $\dim_A(\overline{GA(M)}) = m - 2n + k - 1$.*

(ii) *If $\overline{GA(M)}$ has no end-vertex, then $\dim_A(\overline{GA(M)}) = m - 2n$.*

Proof. (i) By Theorem 3.2 and the inequality $\dim(G) \leq \dim_A(G)$, it follows that $\dim_A(\overline{GA(M)}) \geq m - 2n + k - 1$. On the other hand, it is easy to see that the ordered set

$$W_A = \{[u_{11}], \ldots, [u_{(k-1)1}]\} \cup (\bigcup_{k+1 \leq i \leq n} (N([a_i]) \setminus \{[u_{i1}]\}))$$

is an adjacency resolving set for $\overline{GA(M)}$. Thus $\dim_A(\overline{GA(M)}) \leq m - 2n + k - 1$ will complete the proof.

(ii) By a similar argument to that of (i) one can show that

$$W_A = \bigcup_{1 \leq i \leq n} \big(N([a_i]) \setminus \{u_{i1}\}\big)$$

is an adjacency resolving set for $\overline{GA(M)}$. Thus $\dim_A(\overline{GA(M)}) = m - 2n$ will complete the proof. □

Theorem 3.4. *Let M be a Noetherian R-module and let $GA(M)$ and $\overline{GA(M)}$ be non-empty connected graphs. If $|m{-}\mathrm{Ass}_R(M)| = \omega(GA(M)) = n$, then $\dim_\ell(\overline{GA(M)}) = n - 1$.*

Proof. Let $m - Ass_R(M) = \{P_1 = \mathrm{Ann}_M(a_1), \ldots, P_n = \mathrm{Ann}_M(a_n)\}$ and let $N([a_i]) = \{[u_{i1}], \ldots, [u_{it_i}]\}$, where $t_i \in \mathbb{N}$ for all $1 \leq i \leq n$. Set $W_\ell = \{[u_{11}], [u_{21}], \ldots, [u_{(n-1)1}]\}$. Let $[u] \in N([a_r])$ and $[v] \in N([a_s])$, where $1 \leq r, s \leq n$. If $r = s$, then the vertices $[u]$ and $[v]$ are not adjacent in $\overline{GA(M)}$ by Lemma 2.3 and so there is nothing to prove. Next if $r \neq s$, then Lemma 2.2 implies that $[u] - [v]$. In this case, $r([u] \mid W_\ell)$ has values 2 in its components corresponding to $[u_{r1}]$ and 1 in other components, while $r([v] \mid W_\ell)$ has values 2 in its components corresponding to $[u_{s1}]$ and 1 in other components. Thus $r([u] \mid W_\ell) \neq r([v] \mid W_\ell)$. Also, $r([a_i] \mid W_\ell)$ has values 1 in its components corresponding to $[u_{i1}]$ and 2 in other components, for all i with $1 \leq i \leq n-1$. Hence, $r([a_i] \mid W_\ell) \neq r([u_{ij}] \mid W_\ell)$, where $1 \leq j \leq t_i$. Finally, $r([a_n] \mid W_\ell) \neq r([u_{nj}] \mid W_\ell)$ since all components in the representation of $r([a_n] \mid W_\ell)$ are 2 and all components in the representation of $r([u_{nj}] \mid W_\ell)$ are 1, for all j with $1 \leq j \leq t_n$. By the previous arguments, every two adjacent vertices out of W_ℓ have a unique representation with respect to W_ℓ and so W_ℓ is a local resolving set for $\overline{GA(M)}$. Therefore, $\dim_\ell\big(\overline{GA(M)}\big) \leq n - 1$.

Suppose that W'_ℓ is a local resolving set for $\overline{GA(M)}$ with $|W'_\ell| < n{-}1$. Without loss of generality we may assume that $|W'_\ell| = n - 2$. Let $D = \{[a_1], \ldots, [a_n]\}$. Then the following three cases will be considered:

Case 1. $D \cap W'_\ell = \emptyset$.

In this case, there exist at least two indices $i \neq j$ with $1 \leq i, j \leq n$

such that $N([a_i]) \cap W'_\ell = N([a_j]) \cap W'_\ell = \emptyset$. Let $[u] \in N([a_i])$ and $[v] \in N([a_j])$. Then Lemma 2.3 shows that $[u]$ and $[v]$ are adjacent and $r([u] \mid W'_\ell) = r([v] \mid W'_\ell) = (1, \ldots, 1)$, which contradicts the fact that W'_ℓ is a local resolving set for $\overline{GA(M)}$.

Case 2. $|D \cap W'_\ell| = n - 2$.
Without loss of generality, we may assume that $W'_\ell = \{[a_1], \ldots, [a_{n-2}]\}$. Let $[u] \in N([a_{n-1}])$ and $[v] \in N([a_n])$. Then Lemma 2.3 shows that $[u]$ and $[v]$ are adjacent and $r([u] \mid W'_\ell) = r([v] \mid W'_\ell) = (2, \ldots, 2)$, which contradicts the fact that W'_ℓ is a local resolving set for $\overline{GA(M)}$.

Case 3. Suppose $|D \cap W'_\ell| = t \leq n - 2$.
Assume that $\{[a_1], \ldots, [a_t]\} \subset W'_\ell$ and $N([a_i]) \cap W'_\ell \neq \emptyset$, where $t + 1 \leq i \leq n - 2$. Let $[u] \in N([a_{n-1}])$ and $[v] \in N([a_n])$. Then $[u]$ and $[v]$ are adjacent and $r([u] \mid W'_\ell) = r([v] \mid W'_\ell)$ since the first t components of them are 2 and the other components are 1 and this is a contradiction. Thus $\dim_\ell(\overline{GA(M)}) \geq n-1$, which implies that $\dim_\ell\left(\overline{GA(M)}\right) = n-1$ and the proof will be completed. $\square$

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