Journal of Algebra and Related Topics
Vol. 8, No 1, (2020), pp 15-26

PULLBACK OF LIE ALGEBRA AND LIE GROUP BUNDLES, AND THEIR HOMOTOPY INVARIANCE

K. AJAYKUMAR*, B. S. KIRANAGI AND R. RANGARAJAN

Abstract. We study the pullback Lie algebra (group) bundle of a Lie algebra (group) bundle and show that the Lie algebra bundle of the pullback of a Lie group bundle $\mathfrak{G}$ is isomorphic to the pullback of the Lie algebra bundle of $\mathfrak{G}$. Then, using the notion of Lie connection on a Lie algebra bundle, we show that the pullbacks of a Lie algebra bundle ξ over a smooth manifold M with respect to two smooth homotopic functions $f_0, f_1 : N \to M$ are isomorphic to Lie algebra bundles over N.

1. Introduction

Pullback of a Lie algebra bundle over a topological space was studied in [9]. In this paper we extend the result to smooth Lie algebra bundles with admissible norm.

We then show that for any Lie group bundle there exists a pullback Lie group bundle unique upto isomorphism. We establish an isomorphism between the Lie algebra bundle of the pullback of a Lie group bundle $\mathfrak{G}$, and the pullback of the Lie algebra bundle of Lie group bundle $\mathfrak{G}$.

The importance of pullback bundles lies in promoting a bundle morphism T between bundles over different base spaces to a bundle morphism $\tilde{T}$ between bundles over a common base space. We shall show the existence and uniqueness of such $\tilde{T}$.

MSC(2010): Primary: 55R25; Secondary: 22E99, 57R22.
Keywords: Lie algebra and Lie group bundles, pullback of bundles, principal bundles, homotopy invariance.
Received: 30 July 2019, Accepted: 24 July 2020.
*Corresponding author .

Finally using the notion of connection on smooth Lie algebra bundles [1] we show that the pullbacks of a Lie algebra bundle ξ over a smooth manifold M with respect to two smooth homotopic functions $f_0, f_1 : N \to M$ are isomorphic Lie algebra bundles over N.

Notations and Terminology: All base spaces are assumed to be paracompact smooth manifolds. We denote the pullback of a bundle $\xi = (\xi, \pi, M)$ by $f^*\xi = (f^*\xi, f^*\pi, N)$ with respect to smooth map $f : N \to M$, where N and M are smooth manifolds. All underlying fields are of characteristic zero.

2. Preliminary definitions

Definition 2.1 ([11])**.** Let G denote any topological group. **A principal** $G-$**bundle** is a fiber bundle $\pi : P \to X$ together with a continuous right action $P \times G \to P$ such that G preserves the fibers of P (i.e. if $y \in P_x$ then $yg \in P_x$ for all $g \in G$) and acts freely and transitively on them in such a way that for each $x \in X$ and $y \in P_x$, the map $G \to P_x$ sending g to yg is a homeomorphism. In particular each fiber of the bundle is homeomorphic to the group G itself. Frequently, one requires the base space X to be Hausdorff and possibly paracompact.

Definition 2.2 ([2])**.** A real (complex) **smooth vector bundle** is a locally trivial smooth family of vector spaces. That is a surjective smooth map $p : E \to M$ of a smooth manifold E onto smooth manifold M, such that:

(1) for each $x \in M$, $E_x = p^{-1}(x)$ has a finite dimensional real(complex) vector space structure.
(2) for each $x \in M$, there is a neighbourhood U of x in M, a positive integer k and a diffeomorphism $h : p^{-1}(U) \to U \times \mathbb{R}^k$ such that $h|_{E_x} : E_x \to x \times \mathbb{R}^k$ is a vectorspace isomorphism.

Definition 2.3. Let $p : E \to M$ and $p' : E' \to N$ be smooth vector bundles. A smooth map $\tilde{f} : E \to E'$ is said to be a **smooth vector bundle homomorphism** if $\tilde{f}$ induces a map $f : M \to N$ such that the diagram,

$$\begin{array}{ccc} E & \xrightarrow{\tilde{f}} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ M & \xrightarrow{f} & N \end{array}$$

commutes and the restriction $\tilde{f} : E_x \to E'_{f(x)}$ is linear for all $x \in M$.

Definition 2.4. Let $p : E \to M$ and $p' : E' \to M$ be smooth vector bundles. A smooth vector-bundle homomorphism $\tilde{f} : E \to E'$ is said to be an **isomorphism of smooth vector bundles** if

(1) $p' \circ \tilde{f} = p$
(2) $\tilde{f}$ is a diffeomorphism, such that its restriction to each fibre is a vectorspace isomorphism.

If such an isomorphism exists then E and E' are said to be **isomorphic smooth vector bundles**.

Definition 2.5 ([2])**.** A **smooth weak (normed) Lie algebra bundle** (ξ, π, M) is a smooth vector bundle together with a smooth morphism $\Theta : \xi \oplus \xi \to \xi$ inducing a (normed) Lie algebra structure on each fibre ξ_x.

Definition 2.6 ([2])**.** A weak Lie algebra bundle, $\xi = (\xi, \pi, M)$ is said to have an **admissible norm** if there is a continuous map $\| \; \| : \xi \to \mathbb{R}$ such that it induces a norm on each fiber which satisfies

$$\|[x, y]\| \leq \|x\| \|y\|$$

Definition 2.7 ([7])**.** A locally trivial smooth (normed) Lie algebra bundle, for short a **smooth Lie algebra bundle**, is a smooth vector bundle $\xi = (\xi, \pi, M)$ whose standard fibre is a (normed) Lie algebra say L, in which each fibre is a (normed) Lie algebra such that for each x in M there is an open set U in M containing x and a diffeomorphism $\varphi : U \times L \to \pi^{-1}(U)$ such that for each x in U, $\varphi_x : \{x\} \times L \to \pi^{-1}(x)$ is a (normed) Lie algebra isomorphism.

Remark 2.8. Every locally trivial smooth Lie algebra bundle is a weak Lie algebra bundle. But the converse is not true in general [8].

Definition 2.9 ([2])**.** A Lie (topological) group bundle is a smooth (topological) fibre bundle $(\mathcal{G}, \pi, M)$ in which each fibre $\mathcal{G}_m = \pi^{-1}(m)$ and the fibre type F, has a Lie (topological) group structure and for which there is an atlas $\{\phi_i : U_i \times F \to \mathcal{G}_{U_i}\}$ such that each $\phi_{i,m} : F \to \mathcal{G}_m$, $m \in U_i$ is an isomorphism of Lie (topological) groups.

Definition 2.10 ([2])**.** Let $\mathcal{G} = (\mathcal{G}, \pi, M)$ be a Lie group bundle with local trivialization $\{(U, \phi)\}$. Then for the identity section $\hat{e} : M \to \mathcal{G}$, defined by $\hat{e}(m) = e_m$, the identity element in the Lie group $\mathcal{G}_m$, we see that

$$\mathcal{L}(\mathcal{G}) = \bigcup_{m \in M} T_{e_m}(\mathcal{G}_m)$$

forms a smooth fibre bundle over M, where $T_{e_m}(\mathcal{G}_m)$ is the tangent space of $\mathcal{G}_m$ at e_m. Since each fibre $\mathcal{G}_m$ is a Lie group isomorphic to the standard fibre G, it is clear that $T_{e_m}(\mathcal{G}_m)$ is a Lie algebra isomorphic to $\mathfrak{g} = T_e G$. Therefore $\mathcal{L}(\mathcal{G}) = \cup_{m \in M} T_{e_m}(\mathcal{G}_m)$ forms a smooth Lie algebra bundle over M with local trivialization $(U, d\phi)$ given by

$$d\phi : U \times T_e G \to \cup_{m \in U} T_{e_m}(\mathcal{G}_m).$$

such that

$$d\phi_m : \{m\} \times T_e G \to T_{e_m}(\mathcal{G}_m)$$

is a Lie algebra isomorphism induced by the differential of ϕ_m. We call $\mathcal{L}(\mathcal{G})$ the **Lie algebra bundle of the Lie group bundle** $\mathcal{G}$.

Definition 2.11. Let ξ, η be two (smooth) Lie algebra bundles over the same base space M then a (smooth) vector bundle morphism $f : \xi \to \eta$ is said to be a **Lie algebra bundle morphism** if for each x in M, $f_x : \xi_x \to \eta_x$ is Lie algebra homomorphism.

3. Pullback of Lie algebra and Lie group bundles

Theorem 3.1. *Let $f : N \to M$ be a smooth map and $\mathfrak{G} = (\mathfrak{G}, \pi, M)$, a Lie group bundle. Then there exists a Lie group bundle $f^*\mathfrak{G} = (f^*\mathfrak{G}, f^*\pi, N)$ and a unique Lie group bundle morphism $\tilde{f} : f^*\mathfrak{G} \to \mathfrak{G}$ such that for each n in N the fibre $(f^*\mathfrak{G})_n$ of $f^*\mathfrak{G}$ is isomorphic to the fibre $\mathfrak{G}_{f(n)}$ of $\mathfrak{G}$ under $\tilde{f}$. Further such a Lie group bundle is unique upto isomorphism.*

Proof. The set $f^*\mathfrak{G} = \{(n, g) \in N \times \mathfrak{G} |\ f(n) = \pi(g)\}$ is a smooth submanifold of $N \times \mathfrak{G}$, by the implicit function theorem for maps, and $f^*\pi : f^*\mathfrak{G} \to N$, $(f^*\pi)(n, g) = n$ is a smooth surjective map. It is then clear that each fibre $(f^*\mathfrak{G})_n$ of $f^*\mathfrak{G}$ carries Lie group structure isomorphic to the fibre $\mathfrak{G}_{f(n)}$ of $\mathfrak{G}$.

Suppose $\{(U, \phi)\}$ is a local trivialization of $\mathfrak{G} = (\mathfrak{G}, \pi, M)$. For n in N, let U be the neighbourhood of $f(n)$ and $\phi : \pi^{-1}(U) \to U \times G$ is given by $\phi(g) = (\phi_1(g), \phi_2(g))$. Define

$$f^*\phi : (f^*\pi)^{-1}(f^{-1}(U)) \to f^{-1}(U) \times G$$

by

$$f^*\phi(n, g) = (n, \phi_2(g)).$$

Then

$$(f^*\phi)_n : (f^*\pi)^{-1}(n) \to n \times G$$

is a Lie group isomorphism and $\{(f^{-1}(U), f^*\phi)\}$ forms a local trivialization for $(f^*\mathfrak{G}, f^*\pi, N)$. Thus $f^*\mathfrak{G} = (f^*\mathfrak{G}, f^*\pi, N)$ is a Lie group bundle. We call $f^*\mathfrak{G}$ the pullback of Lie group bundle $\mathfrak{G}$ with respect to smooth map $f : N \to M$.

Now define $\tilde{f} : f^*\mathfrak{G} \to \mathfrak{G}$, by $\tilde{f}(n, g) = g$. Then clearly $\tilde{f}$ is a Lie group bundle morphism such that $\tilde{f}((f^*\mathfrak{G})_n) = \mathfrak{G}_{f(n)}$. □

Theorem 3.2. *Given a smooth (normed) Lie algebra bundle $\xi = (\xi, \pi, M)$ and a smooth map $f : N \to M$. Then there exists a smooth (normed) pullback Lie algebra bundle $f^*\xi = (f^*\xi, f^*\pi, N)$ such that for each n in N the fibre $f^*\xi_n$ of $f^*\xi$ is isomorphic to the fibre $\xi_{f(n)}$ of ξ and such a Lie algebra bundle is unique upto isomorphism.*

Proof. From the methods of [9] and arguments in the proof of Theorem 3.1, it follows that there exists a unique (upto isomorphism) smooth pullback Lie algebra bundle $f^*\xi = (f^*\xi, f^*\pi, N)$ such that for each n in N the fibre $f^*\xi_n$ of $f^*\xi$ is isomorphic to the fibre $\xi_{f(n)}$ of ξ.

Further let $\|.\| : \xi \to \mathbb{R}$ be the map on ξ which induces norm on each fiber. Let $\|.\|^*$ be the map defined on $f^*\xi$ as

$$\|(n, g)\| = \|g\|$$

Then the diagram

$$\begin{array}{ccc} f^*\xi & \xrightarrow{\;i\;} & N \times \xi \\ \Big\downarrow{\scriptstyle \|\cdot\|^*} & & \Big\downarrow{\scriptstyle \pi'} \\ \mathbb{R} & \xleftarrow[\|\cdot\|]{} & \xi \end{array}$$

commutes, where i is the inclusion map and π' is the projection map onto the second component. Hence $\|.\|^*$ is a continuous map from $f^*\xi$ to $\mathbb{R}$ which induces norm on each fiber of the pullback of ξ. Thus $f^*(\xi)$ is a Lie algebra bundle with admissible norm. Hence the theorem.

□

We now show in the succeeding theorems that the pullback of Lie group bundles promote a bundle morphism T between Lie group bundles over different base manifolds to bundle morphism $\tilde{T}$ between bundles over a common base manifold.

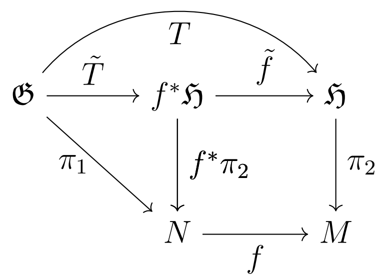
Theorem 3.3. *Let $\mathfrak{G} = (\mathfrak{G}, \pi_1, N)$ and $\mathfrak{H} = (\mathfrak{H}, \pi_2, M)$ be Lie group bundles and $T : \mathfrak{G} \to \mathfrak{H}$ be a Lie group bundle morphism descending to a smooth map $f : N \to M$. Then there is a unique Lie group bundle morphism $\tilde{T} : \mathfrak{G} \to f^*\mathfrak{H}$ such that $\tilde{f} \circ \tilde{T} = T$ where $\tilde{f} : f^*\mathfrak{H} \to \mathfrak{H}$ is the Lie group bundle morphism induced by f.*

Proof. Uniqueness of such $\tilde{T}$ follows from the uniqueness of $\tilde{f}$.

Define $\tilde{T} : \mathfrak{G} \to N \times \mathfrak{H}$ by

$$\tilde{T}(g) = (\pi_1(g), T(g)), \quad \text{for all } g \in \mathfrak{G}$$

Then $\tilde{T} : \mathfrak{G} \to f^*\mathfrak{H}$ is a Lie group bundle morphism as $T : \mathfrak{G} \to \mathfrak{H}$ is a Lie group bundle morphism and $\pi_2 \circ T = f \circ \pi_1$. Thus we have the commutative diagram,



where $\tilde{f} : f^*\mathfrak{H} \to \mathfrak{H}$, $\tilde{f}(n, h) = h$ is a Lie group bundle morphism.

We see that $\tilde{f} \circ \tilde{T} = T$ as

$$(\tilde{f} \circ \tilde{T})(g) = \tilde{f}(\tilde{T}(g)) = \tilde{f}(\pi_1(g), T(g)) = T(g)$$

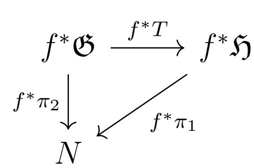
□

Theorem 3.4. *Let $\mathfrak{G} = (\mathfrak{G}, \pi_1, M)$ and $\mathfrak{H} = (\mathfrak{H}, \pi_2, M)$ be Lie group bundles and $T : \mathfrak{G} \to \mathfrak{H}$ be a Lie group bundle morphism. If $f : N \to M$ is a smooth map, then there is a unique Lie group bundle morphism $f^*T : f^*\mathfrak{G} \to f^*\mathfrak{H}$ over N such that $f^*T|_n = T|_{f(n)}$ set theoretically.*

Proof. Define $f^*T : f^*\mathfrak{G} \to f^*\mathfrak{H}$ by,

$$f^*T((n, g)) = (n, T(g)) \text{ for all } (n, g) \in f^*\mathfrak{G}$$

Then f^*T is smooth since $T : \mathfrak{G} \to \mathfrak{H}$ is smooth. Also we see that the following diagram commutes



as

$$\begin{aligned}(f^*\pi_1 \circ f^*T)((n, g)) &= f^*\pi_1(f^*T((n, g))) \\ &= f^*\pi_1((n, T(g)) = n \\ &= f^*\pi_2((n, g)), \quad \text{for all } (n, g) \in f^*\mathfrak{G}.\end{aligned}$$

f^*T is a Lie group bundle morphism , let $(n,g_1),(n,g_2) \in (f^*\mathfrak{G})_n$,

$$\begin{aligned} f^*T((n,g_1)(n,g_2)) &= f^*T(n, g_1 \cdot g_2) \\ &= (n, T(g_1 \cdot g_2)) \\ &= (n, T(g_1)) \cdot (n, T(g_2)) \\ &= f^*T(g_1) \cdot f^*T(g_2) = f^*T((n,g_1)) \cdot f^*T((n,g_2)) \end{aligned}$$

Thus $f^*T : f^*\mathfrak{G} \to f^*\mathfrak{H}$ is a Lie group bundle morphism and obviously $f^*T|_n = T|_{f(n)}$. □

Relation between $\mathcal{L}(f^*\mathfrak{G})$ and $f^*\mathcal{L}(\mathfrak{G})$:

Theorem 3.5. *Let $\mathfrak{G} = (\mathfrak{G}, \pi, M)$ be a Lie group bundle with identity section and $\mathcal{L}(\mathfrak{G})$ be its Lie algebra bundle over M. Let $f : N \to M$ be a smooth map. Let $f^*\mathfrak{G}$ be the pullback of the Lie group bundle $\mathfrak{G}$. Then $\mathcal{L}(f^*\mathfrak{G})$, the Lie algebra bundle of $f^*\mathfrak{G}$ is isomorphic to $f^*\mathcal{L}(\mathfrak{G})$, the pullback of $\mathcal{L}(\mathfrak{G})$ over N*

Proof. Let $\pi_{\mathcal{L}} : \mathcal{L}(\mathfrak{G}) \to M$ and $f^*_{\mathcal{L}}\pi : \mathcal{L}(f^*\mathfrak{G}) \to N$ be the corresponding smooth projections. Now, define $\tilde{f} : f^*\mathfrak{G} \to \mathfrak{G}$ by $\tilde{f}((n,g)) = g$. Then $\tilde{f}$ is a smooth Lie group bundle morphism such that, $\tilde{f}_n = \tilde{f}|_{(f^*\mathfrak{G})_n} : (f^*\mathfrak{G})_n \to (\mathfrak{G})_{f(n)}$ is a Lie group isomorphism.

Thus, the differential $d\tilde{f}_n : T_{e_n}((f^*\mathfrak{G})_n) \to T_{e_{f(n)}}((\mathfrak{G})_{f(n)})$ is a Lie algebra isomorphism, for every $n \in N$, where e_n is the identity element in the fiber $f^*\mathfrak{G}_n$ and $e_{f(n)}$ is the identity in $\mathfrak{G}_{f(n)}$. Now define a map $\Theta : \mathcal{L}(f^*\mathfrak{G}) \to f^*\mathcal{L}(\mathfrak{G})$ by

$$\Theta(l) = (f^*\pi_{\mathcal{L}}(l), d\tilde{f}|_{\pi_{\mathcal{L}}}(l))$$

Then Θ is a smooth vector bundle morphism and preserves Lie bracket fibre-wise as for each $n \in N$

$$\begin{aligned} \Theta[l_n, l'_n] &= (n, d\tilde{f}_n[l_n, l'_n]) \\ &= (n, [d\tilde{f}_n(l_n), d\tilde{f}_n(l'_n)]) \\ &= [(n, d\tilde{f}_n(l_n), (n, d\tilde{f}_n(l'_n)] \\ &= [\Theta(l_n), \Theta(l'_n)] \end{aligned}$$

Hence the theorem. □

Remark 3.6. Adjoint bundle associated to a principal bundle is a natural example of smooth Lie algebra bundle [5, 7]. One of the earliest examples of adjoint bundles can be found in [6]. It is easy to see that the following result holds good for adjoint bundles: *Let $f : N \to M$ be a smooth map. Let $\pi : P \to M$ be a principal G-bundle and let $f^*\pi : f^*P \to N$ be the pullback of P. If $\xi = adP$ is the adjoint bundle* *associated with P, then the pullback $f^*\xi$ is isomorphic to the adjoint bundle associated with f^*P.*

4. Homotopy invariance of pullbacks

Here we show that the pullbacks of a Lie algebra bundle ξ over a smooth manifold M with respect to two smooth homotopic functions $f_0, f_1 : N \to M$ are isomorphic Lie algebra bundles over N using the notion of Lie connection [1, 4] on ξ.

Definition 4.1 (Lie Connection)**.** Let $\xi = (\xi, \pi, M, \Theta)$ be a smooth Lie algebra bundle. A Lie connection [4] is a linear map

$$\nabla : \Gamma(TM) \to \mathrm{End}(\Gamma(\xi)),\ \ X \mapsto \nabla_X$$

satisfying the following conditions:

(1) $\nabla_X(aA) = (X.a)A + a\nabla_X A$ for all $X \in \Gamma(TM)$, $a \in C^\infty(M, \mathbb{R})$, $A \in \Gamma(\xi)$

(2) $\nabla_{(aX+bY)}A = a\nabla_X A + b\nabla_Y A$ for all $X, Y \in \Gamma(TM)$, $a, b \in C^\infty(M, \mathbb{R})$, $A \in \Gamma(\xi)$

(3) $\nabla_X[A, B] = [\nabla_X A, B] + [A, \nabla_X B]$ for all $X \in \Gamma(TM)$, $A, B \in \Gamma(\xi)$

Remark 4.2. The notion of Lie Ehresmann connection is studied in [1]. It is shown that it is equivalent to the notion of Lie conncetion on (normed) smooth Lie algebra bundles and that a smooth Lie algebra bundle over a paracompact manifold admits Lie Ehresmann connection.

Throughout this section we fix a smooth Lie algebra bundle $\pi : \xi \to M$ and a Lie connection ∇ on ξ.

Definition 4.3 (Parallel transport)**.** Let $f : \mathbb{R} \to M$ be a smooth path. A Lie algebra isomorphism $\varphi_t : \xi_m \to \xi_{f(t)}$ is called the parallel transport along the path f from $f(0) = m$ to $f(t)$. A section $s \in \Gamma(\xi)$ is called parallel along f if

$$\varphi_t(s(m)) = s(f(t))$$

Such a parallel transport can be obtained from an injective Lie algebra homomorphism $\xi_m \to \Gamma(f^*\xi)$. We have from [10, Lemma 1.32] that the map $v \mapsto s^v$, where $s^v \in \Gamma(f^*\xi)$ such that $s^v(0) = v$ and the covariant derivatie $\frac{\nabla}{dt}s^v = 0$ is an injective vector space homomorphism, as the Lie connection ∇ on ξ is also a Linear connection on ξ, as a vector bundle.

Further for $v_1, v_2 \in \xi_m$, we have $\frac{\nabla}{dt}[s^{v_1}, s^{v_2}] = [\frac{\nabla}{dt}s^{v_1}, s^{v_2}] + [s^{v_1}, \frac{\nabla}{dt}s^{v_2}] = 0$ and $[s^{v_1}, s^{v_2}](0) = [s^{v_1}(0), s^{v_2}(0)] = [v_1, v_2]$. Therefore $[v_1, v_2] \mapsto [s^{v_1}, s^{v_2}]$ (by injectivity) and hence the map is an injective Lie algebra homomorphism. Thus we have

Proposition 4.4 (Parallel transport)**.** *Let $f : \mathbb{R} \to M$ be a smooth path and let $f(0) = m$. Then there is a unique Lie bundle morphism*

$$\varphi : \mathbb{R} \times \xi_m \to \xi \tag{4.1}$$

which induces $f : \mathbb{R} \to M$ and restricts to Lie algebra isomorphism in the fibres such that $f_0 = f|_{\{0\}\times\xi_m} : \{0\} \times \xi_m \to \xi_m$ is the identity map.

4.1. **Bundles over $M \times \mathbb{R}$.** First let us recall the definition of cartesian product of two vector bundles.

Definition 4.5 ([3])**.** Suppose $\pi : \xi \to M$ and $\pi' : \xi' \to M'$ are two vector bundles with standard fibre F and F' respectively. Then the cartesian product of the bundles ξ and ξ' denoted by $\xi \times \xi'$ is the tuple $(\xi \times \xi', \pi \times \pi', M \times M', F \oplus F')$. That is, the vector bundle

$$\pi \times \pi' : \xi \times \xi' \to M \times M'$$

whose standard fibre is $F \oplus F'$, the direct sum of the vector spaces F and F'.

Now let $\pi : \zeta \to M \times \mathbb{R}$ be a Lie algebra bundle with standard fibre L. Let $i_0 : M \to M \times \mathbb{R}$ be defined by $i_0(m) = (m, 0)$ and $\zeta_0 = (i_0)^*(\zeta)$ be the pullback of ζ with respect to i_0. Then we can form the bundle

$$\zeta_0 \times \mathbb{R} \tag{4.2}$$

the cartesian product of the bundle ζ_0 with the bundle $i : \mathbb{R} \to \mathbb{R}$. Clearly the bundle $\zeta_0 \times \mathbb{R}$ is a Lie algebra bundle over $M \times \mathbb{R}$ with standard fibre L (in fact it is $L \times \{0\} \cong L$).

Lemma 4.6. *With the hypothesis and notations above, there is a Lie algebra bundle isomorphism*

$$\varphi : \zeta_0 \times \mathbb{R} \to \zeta \tag{4.3}$$

Proof. Let ∇_ζ be a Lie connection on ζ. For each $m \in M$ define

$$g_m : \mathbb{R} \to M \times \mathbb{R}$$

by

$$g_m(t) = (m, t)$$

Clearly each g_m is a smooth path in $M \times \mathbb{R}$ such that $g_m(0) = (m, 0) = i_0(m)$. Then by Proposition 4.4 we have a Lie bundle morphism inducing $g_m : \mathbb{R} \to M \times \mathbb{R}$ say

$$\varphi : \zeta_{(m,o)} \times \mathbb{R} \to \zeta$$

such that it restricts to Lie algebra isomorphism on each fibre,

$$\varphi_{m,t} : \zeta_{(m,0)} \times \{t\} \to \zeta_{g_m(t)} = \zeta_{(m,t)} \tag{4.4}$$

But we know that $(\zeta_0)_m$ is isomorphic to $\zeta_{(m,0)} = \zeta_{i_0(m)}$. So we have a Lie algebra isomorphism

$$\varphi_{m,t} : (\zeta_0)_m \times \{t\} \to \zeta_{(m,t)} \tag{4.5}$$

extending to a Lie bundle morphism

$$\varphi : \zeta_0 \times \mathbb{R} \to \zeta$$

over $M \times \mathbb{R}$. □

Theorem 4.7 (Homotopy invariance)**.** *Let $f_0 : N \to M$ and $f_1 : N \to M$ be smooth homotopic maps. Then the pullbacks $f_0^*\xi$ and $f_1^*\xi$ of a smooth Lie algebra bundle $\pi : \xi \to M$ are isomorphic over N.*

Proof. Let $h : N \times \mathbb{R} \to M$ be a homotopy connecting f_0 and f_1 i.e. $h|_0 : N \times \{0\} \to M$ and $h|_1 : N \times \{1\} \to M$ are equal to f_0 and f_1 respectively. Let $i_0 : N \times N \times \mathbb{R}$ and $i_1 : N \to N \times \mathbb{R}$ be defined by $i_0(x) = (x, 0)$ and $i_1(x) = (x, 1)$. Then $h \circ i_0 = f_0$ and $h \circ i_1 = f_1$. So we have

$$f_0^*\xi = i_0^*(h^*\xi)$$

and

$$f_1^*\xi = i_1^*(h^*\xi)$$

By applying Lemma 4.6 to $h^*\xi$ corresponding to i_0, we see that $h^*\xi$ is isomorphic to $i_0^*(h^*\xi) \times \mathbb{R}$. Similarly Lemma 4.6 applied to $h^*\xi$ corresponding to i_1, we see that $h^*\xi$ is isomorphic to $i_1^*(h^*\xi) \times \mathbb{R}$.

Therefore $i_0^*(h^*\xi) \times \mathbb{R} \cong i_1^*(h^*\xi) \times \mathbb{R}$ and hence we have

$$i_0^*(h^*\xi) \cong i_1^*(h^*\xi)$$

that is

$$f_0^*\xi \cong f_1^*\xi$$

□

Corollary 4.8. *Let $f_0 : N \to M$ and $f_1 : N \to M$ be smooth homotopic maps. Then the pullbacks $f_0^*\mathfrak{G}$ and $f_1^*\mathfrak{G}$ of the Lie group bundle $\pi : \mathfrak{G} \to M$ with simply connected fibres, are isomorphic over N.*

Proof. Let ξ be the Lie algebra bundle $\mathcal{L}(\mathfrak{G})$ of $\mathfrak{G}$ over M. Then by the above Theorem 4.7, we have

$$f_0^*\xi \cong f_1^*\xi$$

But we know by Theorem 3.5,

$$\mathcal{L}(f_0^*\mathfrak{G}) \cong f_0^*\xi$$

and

$$\mathcal{L}(f_1^*\mathfrak{G}) \cong f_1^*\xi$$

Therefore the Lie algebra bundles $\mathcal{L}(f_0^*\mathfrak{G})$ and $\mathcal{L}(f_1^*\mathfrak{G})$ are isomorphic. But since the fibres of $\mathfrak{G}$ are simply connected, so are the fibres of $f_0^*\mathfrak{G}$ and $f_1^*\mathfrak{G}$. Therefore by uniqueness in [2, Theorem 3], we have

$$f_0^*\mathfrak{G} \cong f_1^*\mathfrak{G}$$

□

Corollary 4.9. *Every Lie algebra bundle over a contractible base space is trivial.*

Proof. Let us fix m_0 in M, the base space of Lie algebra bundle ξ. Since M is contractible the identity map of M is homotopic to the constant map $f : M \to M$, $f(m) = m_0$. Therefore by Theorem 4.7 it follows that $i^*\xi \cong f^*\xi$. That is

$$\xi \cong M \times \xi_{m_0}$$

□

In the same way we have

Corollary 4.10. *Every Lie group bundle over a contractible base space is trivial.*

Acknowledgments

The authors would like to thank the referee for careful reading and providing valuable suggestions.

References

1. K. Ajaykumar and B. S. Kiranagi, *Connections on smooth Lie algebra bundles*, Indian J. Pure Appl. Math. **50** (2019), 891–901.
2. A. Douady and M. Lazard, *Espaces fibres en algebres de Lie et en groupes*, Invent. Math. **1** (1966), 133–151.
3. W. Greub, S. Halperin and R. Vanstone, *Connections, curvature, and cohomology. Vol. II: Lie groups, principal bundles, and characteristic classes*, Academic Press, New York-London, 1973.
4. H. Gündoğan, *Lie algebras of smooth sections.* Diploma thesis. Technische Universität Darmstadt, 2007.
5. Josef Janyška, *Higher order Utiyama-like theorem*, Rep. Math. Phys., (1) **58** (2006), 93–118.
6. B. S. Kiranagi, *Lie algebra fibre bundles.* Doctorat 3^e Cycle thesis. Universite de Paris sud XIII, Orsay, Paris, 1976.
7. B. S. Kiranagi, *Lie algebra bundles*, Bull. Sci. Math. (2), (1) **102** (1978), 57–62.

8. B. S. Kiranagi, *Lie algebra bundles and Lie rings*, Proc. Nat. Acad. Sci. India Sect. A, (1) **54** (1984), 38–44.
9. B. S. Kiranagi, R. Rajendra and R. Kumar, *On pull back Lie algebra bundles*, Adv. Stud. Contemp. Math., (4) **22** (2012), 521–524.
10. E. Lerman, *Connections and Curvature Notes*, Lecture Notes, 2013.
11. N. Steenrod, *The topology of fibre bundles*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, 1999.

K. Ajaykumar
School of PPSH, Chinmaya Vishwavidyapeeth, Adi Sankara Nilayam, Ernakulam, India.
Email: ajay.kmr627@gmail.com

B. S. Kiranagi
Department of Mathematics, University of Mysore, Manasagangotri, Mysuru, India.
Email: bskiranagi@gmail.com

R. Rangarajan
Department of Mathematics, University of Mysore, Manasagangotri, Mysuru, India.
Email: rajra63@gmail.com