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A NEW GENERALIZATION OF t-LIFTING MODULES

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Abstract. In this paper we introduce the concept of tCC-modules which is a proper generalization of (t-)lifting modules. Let M be a module over a ring R. We call M a tCC-module (related to t-coclosed submodules) provided that for every t-coclosed submodule N of M, there exists a direct summand K of M such that $M = N + K$ and $N \cap K \ll K$. We prove that a module with (D_3) property is tCC if and only if every direct summand of M is tCC. It is also shown that an amply supplemented module M is tCC if and only if M decomposed to $\overline{Z}^2(M)$ and a submodule L of M that both of them are tCC.

1. Introduction

Throughout this paper R denotes an arbitrary associative ring with identity and all modules are unitary right R-modules. A submodule N of a module M is said to be *small* in M if $N + K \neq M$ for any proper submodule K of M, and we denote it by $N \ll M$. A module M is called small if it is a small submodule of some module, equivalently, M is a small submodule of its injective hull. A submodule N of M is called *coclosed* if N/K is small in M/K, then $N = K$. Somewhere in article, we use $N \leq_{\oplus} M$ to state that N is a direct summand of M.

Recall from [7] that a module M is called *(non)cosingular*, in case ($\overline{Z}(M) = M$) $\overline{Z}(M) = 0$ where $\overline{Z}(M)$ is defined to be $\bigcap\{N \leq M \mid M/N \in \mathcal{S}\}$ (S denotes the class of all small right R-modules). From definitions we conclude that a small noncosingular module is zero.

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On the other hand, a module M is noncosingular if and only if every nonzero homomorphic image of M is non-small. In [7], $\overline{Z}^{\alpha}(M)$ is defined by $\overline{Z}^{0}(M) = M$, $\overline{Z}^{\alpha+1}(M) = \overline{Z}(\overline{Z}^{\alpha}(M))$ and $\overline{Z}^{\alpha}(M) = \bigcap_{\beta<\alpha} \overline{Z}^{\beta}(M)$ if α is a limit ordinal. Hence there is a descending chain $M = \overline{Z}^{0}(M) \supseteq \overline{Z}(M) \supseteq \overline{Z}^{2}(M) \supseteq \ldots$ of submodules of M.

A module M is called *lifting* if every submodule N of M contains a direct summand D of M such that $N/D \ll M/D$. A submodule N of M is called a *supplement* in M, if there is a submodule K of M such that $M = N + K$ and $N \cap K \ll N$. A module M is called *supplemented* in case every submodule of M has a supplement in M. A module M is *amply supplemented* if whenever $M = A + B$, then A contains a supplement of B in M. A lifting module is amply supplemented and hence supplemented. Any terminology not defined here can be found in [2] and [12].

Studying lifting modules and their generalizations from different points of view produces many nice works in this field helping researchers to determine the structure of known rings namely perfect and semiperfect rings. Among them, a famous characterization states that a ring R is (semi)perfect if and only if every (finitely generated) R-module is amply supplemented. Observing projective supplemented modules via second cosingular submodule $\overline{Z}^{2}(M)$ were returned to [1]. Amoozegar, Keskin Tütüncü and Talebi introduced a Goldie version of small notion as t-small submodules. Following [1], N is *t-small* (written $N \ll_t M$) if for every submodule K of M, $\overline{Z}^{2}(M) \subseteq N + K$ implies that $\overline{Z}^{2}(M) \subseteq K$. They also introduced an analogue for a coclosed submodules of a module. A submodule L of a module M is called *t-coclosed* in M if $L/H \ll_t M/H$ implies that $L = H$. Based on t-small notions, Amoozegar and the others in [1], called a module M a *t-lifting module* if for every submodule N of M, there exists a direct summand K of M such that $N/K \ll_t M/K$. Most of obtained results about t-lifting modules and properties of t-small and t-coclosed submodules in [1] were proved in case M is amply supplemented. To amend this objection, the authors in [11] tried to present some new observations about t-lifting modules without condition M is amply supplemented. They showed that a quasi-projective module M is t-lifting if and only if $\overline{Z}^{2}(M)$ is a noncosingular lifting direct summand of M. They also presented a decomposition for commutative t-lifting rings. By the way, it is shown that a commutative ring R is t-lifting if and only if R decomposed to two ideals R_1 and R_2 where R_1 is semisimple noncosingular and $\overline{Z}^{2}(R_2) = 0$.

We should recall a definition which is closely related to our inquiries. A module M is said to be $\oplus$-supplemented (somewhere in the literature, they are called D_{11}-modules) provided every submodule of M has a supplement which is a direct summand of M. This proper generalization of lifting modules and its primary features were given in [5]. It is well-known that a module M is lifting if and only if M is amply supplemented and every coclosed submodule of M is a direct summand of M. It is also clear by the definition of t-lifting modules that every t-coclosed submodule is a direct summand of that module. By a slight modification on this condition and in the light of works mentioned, we introduce a new generalization of t-lifting modules those whose t-coclosed submodules have supplements as direct summands. We call them tCC-modules. In this manuscript, we shall try to investigate some general properties of tCC-modules and verify their relations with some known classes of modules such as t-lifting modules, lifting modules and $\oplus$-supplemented modules. We also present a characterization of amply supplemented tCC-modules in terms of second cosingular submodule.

2. tCC-MODULES AS A PROPER GENERALIZATION OF t-LIFTING MODULES

In this section some basic properties of tCC-modules shall be studied. Before that, we need to know more about t-small submodules of a module and their attributes which can be used freely throughout this manuscript. Let M be an R-module where R is a ring. Let $K \leq M$, then we say K is t-small in M, denoted by $K \ll_t M$, if the inclusion $\overline{Z}^2(M) \subseteq K + N$ implies that $\overline{Z}^2(M) \subseteq N$. We call M, *t-small*, provided M is a t-small submodule of a module L.

The following presents a characterization for t-small submodules of amply supplemented modules.

Proposition 2.1. *Let M be an amply supplemented module and N a submodule of M. Then the following statements are equivalent:*
(1) *N is t-small in M;*
(2) *$N \cap \overline{Z}^2(M) \ll \overline{Z}^2(M)$;*
(3) *$N \cap \overline{Z}^2(M) \ll M$;*
(4) *$\overline{Z}^2(N) = 0$, i.e. $\overline{Z}(N)$ is cosingular.*

Proof. See [1, Proposition 1]. □

By Proposition 2.1, every small submodule of an amply supplemented module M and every supplement of $\overline{Z}^2(M)$ is t-small. It is clear that if N is a submodule of a noncosingular module M, then N is t-small in M if and only if N is small in M.

We shall state some properties of t-small submodules of a module which can be found in [11].

Proposition 2.2. *Let M be a module and $N, K \leq M$. Then the following holds:*

(1) If $N \subseteq K \ll_t M$, then $N \ll_t M$.

(2) $N \ll_t M$ and $K \ll_t M$ if and only if $N + K \ll_t M$.

(3) If $N \ll_t M$ and $N \subseteq K \leq_{\oplus} M$, then $N \ll_t K$.

If M is amply supplemented, then we also have:

(4) If $f : M \to T$ is an epimorphism and $N \ll_t M$, then $f(N) \ll_t T$.

(5) Let $N \leq K \leq M$. Then $N \ll_t M$ and $K/N \ll_t M/N$ if and only if $K \ll_t M$.

Proof. (1) and (2) See [11, Lemma 3].
(3) This was proved in [11, Lemma 1].
(4) It was proved in [11, Lemma 4(1)].
(5) Let $\overline{Z}^2(M) \subseteq K + L$ for some submodule L of M. Since M is amply supplemented, $\overline{Z}^2(M/N) \subseteq K/N + (L + N)/N$. Since $K/N \ll_t M/N$, we get that $\overline{Z}^2(M/N) \subseteq (L + N)/N$. Hence, $\overline{Z}^2(M) \subseteq L + N$. Now, by the assumption $N \ll_t M$, we have $\overline{Z}^2(M) \subseteq L$. It follows that $K \ll_t M$. The converse follows from (1) and (4). □

Let N be a submodule of a module M. Then N is *t-coclosed* in M, denoted by $N \leq_{tcc} M$, if $N/K \ll_t M/K$ implies that $N = K$ for $K \leq M$. We call a module M a tCC-module provided that every t-coclosed submodule of M has a supplement that is a direct summand of M.

We begin observing tCC-modules by presenting some known examples of modules having tCC-property. Before that, we should recall that a module M is *H-supplemented* in case for every submodule N of M there exists a direct summand D of M such that $M = N + X$ if and only if $M = D + X$ for every $X \leq M$.

Example 2.3. (1) It is clear that the class of tCC-modules contains the class of $\oplus$-supplemented modules. So that every H-supplemented (lifting) module is tCC.

(2) If every t-coclosed submodule of a module M is a direct summand of M, then M is tCC. Hence every t-lifting module is tCC.

(3) A module with no nonzero t-coclosed submodule is obviously tCC. For example, every module M with $\overline{Z}^2(M) = 0$ is tCC. For if, in this case every submodule of M is t-small in M. Hence M has non

nonzero t-coclosed submodule. In particular, every cosingular module is tCC.

The following introduces a tCC-module which is not lifting.

Example 2.4. Let M be the $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$. By [3, Example 10], M is not lifting. Note that $\overline{Z}^2(M) = 0$. Therefore, by Example 2.3(3), M is tCC.

The following provides another source of tCC-modules.

Example 2.5. Let R be a Dedekind domain which is not a field.

(1) By [6, Lemma 4.12], every noncosingular R-module is injective. Now, let M be an amply supplemented R-module and N a t-coclosed (noncosingular) submodule of M. Hence N is injective and clearly a direct summand of M. Therefore, any amply supplemented R-module is tCC. Especially, every Artinian $\mathbb{Z}$-module is tCC.

(2) Let M be a noncosingular R-module and N a t-coclosed submodule of M. Since M is noncosingular, the two concepts t-smallness and smallness coincide. Now it follows that N is coclosed in M. By [7, Lemma 2.3(3)], N is noncosingular. Now from [6, Lemma 4.12], N is injective. So N is a direct summand of M. Now, it is obvious that M is tCC. Especially for any indexed set I, the $\mathbb{Z}$-modules $\mathbb{Q}^{(I)}$ and $\mathbb{Z}_{p^\infty}^{(I)}$ are tCC.

By Example 2.3, every t-lifting module is tCC. Note that the converse may not hold. The following introduces some tCC-modules which are not t-lifting endorsing that the class of tCC-modules contains properly the class of t-lifting modules.

Example 2.6. (1) Let R be an incomplete rank one discrete valuation ring with quotient field K. Consider the R-module $M = K \oplus K$. By [5, Lemma A.5], the module M is $\oplus$-supplemented and tCC. By [5, Lemma A.5], M is not amply supplemented. Hence M can not be lifting. Note also that being M noncosingular implies that M is not t-lifting.

(2) Let M be the $\mathbb{Z}$-module $\mathbb{Q}$. By [2, Example 20.12], M is not supplemented and hence M is not lifting. As M is noncosingular, it can not be t-lifting. Note also that, by Example 2.5(2), M is tCC.

Recall from [12] that a ring R is a right V-ring (in honor of Villamayor) in case every simple right R-module is injective. It is known that R is a right V-ring if and only if for every right R-module M, we have $Rad(M) = 0$. A module M is called NS provided every noncosingular submodule of M is a direct summand of M([8]). In [11, Proposition 7], it is shown that a noncosingular t-lifting (lifting) module is NS (see also [8, Proposition 2.8]).

We show in the following that tCC-modules over right V-rings are precisely the semisimple ones.

Proposition 2.7. *Let R be a right V-ring and M a right R-module. Then the following statements are equivalent:*
(1) Every submodule of M is t-lifting;
(2) Every submodule of M is tCC;
(3) Every submodule of M is $\oplus$-supplemented;
(4) Every submodule of M is NS;
(5) M is semisimple.

Proof. (1) $\Rightarrow$ (2) It is clear by definitions.

(2) $\Rightarrow$ (3) Let L be a submodule of M and N an arbitrary submodule of L. We show N is t-coclosed in L. To verify this assertion, suppose $N/K \ll_t L/K$ for a submodule K of L contained in N. Being R a right V-ring implies L/K is noncosingular, so that $N/K \ll L/K$. As N/K is itself noncosingular, we conclude that $N = K$. Now by (2), N has a supplement in L which is a direct summand.

(3) $\Rightarrow$ (4) Let $N \leq L \leq M$. Note that N is noncosingular. By (3), there is a direct summand K of L such that $L = N + K$ and $N \cap K \ll K$. As $Rad(L) = 0$, we have $N \cap K = 0$, which completes the proof.

(4) $\Rightarrow$ (5) This follows from the fact that over R, every right R-module is noncosingular.

(5) $\Rightarrow$ (1) It can be easily verified. $\square$

It is an immediate consequence of last proposition that a non-semisimple right V-ring R is not tCC. For instance, let $R = \prod_{i=1}^{\infty} K_i$ where $K_i = K$ for each $i \in \mathbb{N}$ is a field. Then it is well-known that R is a von Neumann regular (V-ring) ring which is not semisimple. So R as an R-module is not tCC.

Recall that a submodule N of a module M is *fully invariant* in M provided for every endomorphism f of M we have $f(N) \subseteq N$. We say that M is a *duo* module if every submodule of M is fully invariant in M. The following verifies relations between tCC-modules with NS modules and t-lifting modules.

Proposition 2.8. *Consider the following conditions for an amply supplemented module M:*

(1) *M is NS;*
(2) *M is t-lifting;*
(3) *M is tCC.*

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3). *In addition, if M is a duo module, then they are equivalent.*

Proof. (1) $\Leftrightarrow$ (2) This follows from the fact that for an amply supplemented module, a submodule N is noncosingular if and only if N is t-coclosed ([1, Proposition 2]).

(2) $\Rightarrow$ (3) It is straightforward.

(3) $\Rightarrow$ (1) Let $K \leq M$ be noncosingular. Then by [1, Proposition 2], K is t-coclosed. By the way, there is a decomposition $M = N \oplus N'$ such that $M = N + K$ and $N \cap K \ll N$. Since K is a noncosingular submodule of M and $N \cap K \ll N$, we conclude that $N \cap K \ll K$. Being M a duo module implies $K = (N \cap K) \oplus (N' \cap K)$. Accordingly, we have $K = N' \cap K$ and $K \subseteq N'$. It follows that $M = N \oplus K$ which completes the proof. □

Recall from [9] that a module M has *C^*-property* provided that every submodule N of M contains a direct summand D of M such that N/D is cosingular. Let R be a ring. Then every right R-module satisfies C^* if and only if every right R-module is a direct sum of an injective right R-module and a cosingular right R-module (see [9, Theorem 2.9]). Recall also from [2] that a ring R is right Harada in case every injective right R-module is lifting. It follows from [2, 28.10] that R is right Harada if and only if every right R-module is decomposed to an injective right R-module and a small right R-module. So, over a right Harada ring every right R-module satisfies C^*.

As an application of Proposition 2.8, we introduce some rings over which every right R-module is tCC.

Example 2.9. (1) Let R be a right perfect ring such that every right R-module satisfies C^*. Then by [8, Example 2.3(2)], every right R-module is NS and hence every right R-module is tCC.

(2) Let R be a right Harada-ring (QF-ring). Then every right R-module satisfies C^*. Hence by (1), every right R-module is tCC. As an example, every left (right) R-module over the ring of all upper triangular 2×2 matrices with entries from a field K is tCC.

Proposition 2.10. *The following are equivalent for an indecomposable module M:*

(1) *M is tCC;*

(2) *Every t-coclosed submodule of M is small in M or $M \leq_{tcc} M$.*

Proof. (1) $\Rightarrow$ (2) Let N be a t-coclosed submodule of M. Then by assumption (1) there is a summand K of M such that $M = N + K$ and $N \cap K \ll K$. Then $K = 0$ or $K = M$. In first case $N = M$ and the second one implies $N \ll M$.

(2) $\Rightarrow$ (1) Let N be an arbitrary t-coclosed submodule of M. If N is just M or $N \ll M$, then in both cases the zero submodule is a supplement of N. $\square$

By [1, Lemma 1(2)], for an amply supplemented module M the condition $M \leq_{tcc} M$ implies $\overline{Z}(M) = M$. So we have the following result.

Corollary 2.11. *Let M be an amply supplemented indecomposable module. Then M is tCC if and only if M is noncosingular or each t-coclosed submodule is small in M.*

3. DECOMPOSITION OF tCC-MODULES

In this section we shall verify decomposition of tCC-modules. In module theory, presenting a decomposition of each new concept, in fact, help us more to characterize that definition. So that, we may try to do this. By the way as a main result, we show an amply supplemented module is tCC if and only if $M = \overline{Z}^2(M) \oplus L$ where $\overline{Z}^2(M)$ is noncosingular tCC and L is a tCC-module with $\overline{Z}^2(L) = 0$.

We shall start this section by studying on direct summands of projective tCC-modules.

Proposition 3.1. *Let M_i $(1 \leq i \leq n)$ be any finite collection of relatively projective modules. If the module $M = \bigoplus_{i=1}^n M_i$ is tCC, then M_i is tCC for each $1 \leq i \leq n$.*

Proof. We only prove M_1 is tCC. Let $A \leq_{tcc} M_1$. If $A/L \ll_t M/L$ for some $L \leq A$, then $A/L \ll_t M_1/L$ as M_1/L is a direct summand of M/L (see Proposition 2.2(3)). Hence $A = L$ implying that A is a t-coclosed submodule of M. Now by assumption, there exists $B \leq M$ such that $M = A + B$, the submodule B is a direct summand of M and $A \cap B \ll B$. Since $M = A + B = M_1 + B$, by [5, Lemma 4.47], there exists $B_1 \leq B$ such that $M = M_1 \oplus B_1$. Thus $B = B_1 \oplus (M_1 \cap B)$. Note that $M_1 = A + (M_1 \cap B)$ and $M_1 \cap B$ is a direct summand of M_1. Therefore $A \cap (M_1 \cap B) = A \cap B \ll B$. Hence M_1 is tCC. $\square$

Corollary 3.2. *Every direct summand of a projective tCC-module is tCC.*

Let M be a module. We need the following definition:

(D_3) If M_1 and M_2 are direct summands of M with $M = M_1 + M_2$, then $M_1 \cap M_2$ is also a direct summand of M.

By [5, Lemma 4.6 and Proposition 4.38], every quasi-projective module has (D_3).

Definition 3.3. We call a module M *completely* tCC if every direct summand of M is tCC.

It follows from [11, Proposition 6], a t-lifting module is completely tCC.

Theorem 3.4. *Let M be a module with (D_3) property. Then M is tCC if and only if M is completely tCC. In particular, if M is quasi-projective then M is tCC if and only if M is completely tCC.*

Proof. Sufficiency is clear. Conversely, assume that M is tCC, K a direct summand of M and $A \leq_{tcc} K$. We show A has a supplement in K that is a direct summand of K. It is easy to verify that A is a t-coclosed submodule of M. Being M a tCC-module implies that there exists a direct summand B of M such that $M = A + B$ and $A \cap B \ll B$. Then $K = A + (K \cap B)$. Furthermore $K \cap B$ is a direct summand of M because M has (D_3). So $K \cap B \leq_{\oplus} K$. Then $A \cap (K \cap B) = A \cap B \ll K \cap B$, since $K \cap B \leq_{\oplus} B$. □

We next present a characterization of tCC-modules in terms of second cosingular submodule. It can be an analogue of [11, Theorem 1] for tCC-modules. Note that in [11, Theorem 1] M is not necessarily amply supplemented. Other difference with same case is that here M need not be quasi-projective.

Theorem 3.5. *Let M be an amply supplemented module. Then M is tCC if and only if $M = \overline{Z}^2(M) \oplus L$ where $\overline{Z}^2(M)$ and L are tCC.*

Proof. Let M be a tCC-module. By [1, Corollary 1], the submodule $\overline{Z}^2(M)$ is t-coclosed. So, there exist submodules L and L' of M such that $M = L \oplus L' = \overline{Z}^2(M) + L$ and $\overline{Z}^2(M) \cap L \ll L$. Now $\overline{Z}^2(M) = \overline{Z}^2(L) \oplus \overline{Z}^2(L')$. Note that $\overline{Z}^2(M) \cap L = \overline{Z}^2(L) \ll L$. Since $L \leq_{\oplus} M$ and M is amply supplemented, $\overline{Z}^2(L)$ is noncosingular by [7, Corollary 3.4]. Hence $\overline{Z}^2(L) = 0$. It follows that $M = L \oplus \overline{Z}^2(M)$. Next, we prove that $\overline{Z}^2(M)$ and L are tCC. Now, let K be a t-coclosed (noncosingular) submodule of $\overline{Z}^2(M)$ (note that $\overline{Z}^2(M)$ is amply supplemented). Since M is tCC, there is a decomposition $M = D \oplus D'$ such that $K + D = M$ and $K \cap D \ll D$. By modularity, $K + (D \cap \overline{Z}^2(M)) = \overline{Z}^2(M)$. So, $K + \overline{Z}^2(D) = \overline{Z}^2(M)$. Suppose that $(K \cap \overline{Z}^2(D)) + T = \overline{Z}^2(D)$ for a submodule T of $\overline{Z}^2(D)$. Now, $(K \cap \overline{Z}^2(D)) + T + \overline{Z}^2(D') = \overline{Z}^2(M)$. Since $(K \cap \overline{Z}^2(D)) \subseteq (K \cap D) \ll D$, we have $K \cap \overline{Z}^2(D) \ll M$. Since $\overline{Z}^2(M)$ is a direct summand of M, we get that $K \cap \overline{Z}^2(D) \ll \overline{Z}^2(M)$. Hence, $T + \overline{Z}^2(D') = \overline{Z}^2(M)$. Because K is noncosingular, $\frac{K}{K \cap D} \cong \frac{M}{D} \cong$

D' is noncosingular. It follows that $\overline{Z}^2(D') = D'$. So, $T + D' = \overline{Z}^2(M)$. By modularity, $T = \overline{Z}^2(D)$. Therefore, $K \cap \overline{Z}^2(D) \ll \overline{Z}^2(D)$. This shows that $\overline{Z}^2(D)$ is a supplement of K in $\overline{Z}^2(M)$ (we should note that $\overline{Z}^2(D)$ is a direct summand of $\overline{Z}^2(M)$). This proves that $\overline{Z}^2(M)$ is tCC. Clearly L is tCC, since L contains no nonzero t-coclosed (noncosingular) submodule.

Conversely, let $M = \overline{Z}^2(M) \oplus L$ such that $\overline{Z}^2(M)$ and L are tCC. Let K be a t-coclosed (noncosingular) submodule of M. Since M is amply supplemented, $K \subseteq \overline{Z}^2(M)$ by [7, Corollary 3.4]. Since $\overline{Z}^2(M)$ is tCC, there exists a decomposition $D \oplus D' = \overline{Z}^2(M)$ such that $K + D = \overline{Z}^2(M)$ and $K \cap D \ll D$. We are going to prove that K has a supplement in M which is a direct summand of M. Since $M = \overline{Z}^2(M) \oplus L$, we have $M = K + D + L$. Now $M = (D \oplus D') \oplus L$. First, we show that $D' \cap (D + L) = 0$. To prove it, let $d' = d + l \in D' \cap (D + L)$ where $d \in D$, $l \in L$ and $d' \in D'$. Then $d' - d = l \in (\overline{Z}^2(M) \cap L) = 0$. It follows that $l = 0$ and $d' = d \in (D \cap D') = 0$. We next show that $K \cap (D + L) \ll (D + L)$. To prove that, we may show that $K \cap (D + L) = K \cap D$. Let $k = d + l \in K \cap (D + L)$ such that $k \in K$, $l \in L$ and $d \in D$. So, $k - d = l \in (\overline{Z}^2(M) \cap L) = 0$. Hence, $k = d \in K \cap D$. This yields that $K \cap (D + L) = K \cap D$. Since $K \cap D \ll D$, it follows that $K \cap D \ll D + L$. To sum up, $D + L$ (which is a direct summand of M) is a supplement of K in M. □

Proposition 3.6. *Let M be an amply supplemented module. Then M is completely tCC if and only if $M = \overline{Z}^2(M) \oplus L$ for some submodule L of M such that $\overline{Z}^2(M)$ and L are completely tCC.*

Proof. Assume that M is completely tCC. Then M is tCC, so that by Theorem 3.5, $M = \overline{Z}^2(M) \oplus L$ for some submodule L of M. Since all direct summands of $\overline{Z}^2(M)$ and L are also direct summands of M, $\overline{Z}^2(M)$ and L are completely tCC. Conversely, let $M = \overline{Z}^2(M) \oplus L$ for some submodule L of M with L and $\overline{Z}^2(M)$ completely tCC. By Theorem 3.5, M is tCC. Suppose $M = D \oplus D'$. Then $M = \overline{Z}^2(M) \oplus L$ and $\overline{Z}^2(M) = \overline{Z}^2(D) \oplus \overline{Z}^2(D')$ implies $D = \overline{Z}^2(D) \oplus T$ and $D' = \overline{Z}^2(D') \oplus T'$ for some submodules T of D and T' of D'. By the way, $T \oplus T' \cong L$ and L is completely tCC implies both T and T' are tCC. Being $\overline{Z}^2(M)$ completely tCC implies both $\overline{Z}^2(D)$ and $\overline{Z}^2(D')$ are tCC. Hence D and D' are tCC by Theorem 3.5. □

Let R be a ring and M be an R-module. Recall from [10], M is called FI-*t-lifting* provided that every fully invariant submodule N of M contains a direct summand D of M such that $N/D \ll_t M/D$. Let M be amply supplemented. Then by [10, Theorem 2.26], M is FI-t-lifting if and only if every fully invariant t-coclosed submodule of M is a direct summand of M. It is clear that every t-lifting module is FI-t-lifting. On the other hand, every duo FI-t-lifting module is t-lifting.

Proposition 3.7. *Every amply supplemented tCC-module is FI-t-lifting.*

Proof. Let M be an amply supplemented tCC-module and N be a fully invariant t-coclosed submodule of M. Since M is tCC, there is a direct summand K of M such that $M = N + K$ and $N \cap K \ll K$. Set $M = K \oplus K'$. Since N is fully invariant, $N = (N \cap K) \oplus (N \cap K')$. Then $M = (N \cap K) + (N \cap K') + K = (N \cap K') + K$. By modular law $K' = N \cap K'$. So $K' \subseteq N$. Since $N \cap K \leq_{tcc} N$, it is noncosingular by [1, Proposition 2]. But $N \cap K$ is cosingular, since $N \cap K \ll K$. Hence $N \cap K = 0$. It follows that $N = N \cap K'$, which implies that $N \subseteq K'$. Therefore, $N = K'$ is a direct summand of M. The result follows from [10, Theorem 2.26]. □

We shall present an amply supplemented tCC-module which is FI-t-lifting.

Example 3.8. Let F be a field and R denote the ring

$$R = \left\{ \begin{bmatrix} a & x & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & y \\ 0 & 0 & 0 & a \end{bmatrix} \mid a, b, x, y \in F \right\}.$$

The Jacobson radical J of R consists of all matrices in R with a zero diagonal, and $R/J \cong FxF$. Therefore J is nonzero and hence R is not a right V-ring. It follows from [4, 16.19(4)] that R is a QF-ring. Hence every right R-module is tCC by Example 2.9. Therefore, every right R-module is FI-t-lifting by Proposition 3.7.

A module M is said to have the summand sum property, (SSP), in case the sum of any two direct summand of M is a direct summand of M. Also M is said to have the strong summand sum property, $(SSSP)$, if the sum of any family of direct summands of M is a direct summand of M.

Corollary 3.9. *Let M be an amply supplemented tCC-module. Then M has the $(SSSP)$ for fully invariant direct summands which contained in $\overline{Z}^2(M)$.*

Proof. Let D_i for every $i \in I$ be a fully invariant direct summand of M with $D_i \subseteq \overline{Z}^2(M)$. By [1, Porposition 2], each D_i is t-coclosed. Also by [1, Corollary 2], $\sum_{i\in I} D_i$ is a t-coclosed submodule of M. Clearly $\sum_{i\in I} D_i$ is fully invariant. So $\sum_{i\in I} D_i$ is a direct summand of M by Proposition 3.7. □

In the sequel, we study some conditions which under the two concepts "t-lifting" and tCC" coincide.

Proposition 3.10. (1) *Let M be an amply supplemented module. Then M is t-lifting if and only if M is tCC and every supplement of a direct summand K of M with M/K noncosingular, is a direct summand of M.*

(2) *Let M be an amply supplemented tCC-module. If every noncosingular direct summand K of M is M/K-projective, then M is t-lifting.*

Proof. (1) Let M be t-lifting. Clearly M is tCC. Now let L be a supplement of a direct summand K of M with M/K noncosingular. So $M = L + K$ and $L \cap K \ll L$. Consider the module $L/(L \cap K)$. Then $L/(L \cap K) \cong M/K$ is noncosingular. Since the class of noncosingular modules is closed under small covers by [7, Proposition 2.4], L is noncosingular and hence t-coclosed in M by [1, Proposition 2]. Being M a t-lifting module implies that $L \leq_\oplus M$. For the converse, let L be a t-coclosed submodule of M. Then there is a direct summand S of M such that $M = L + S$ and $L \cap S \ll S$. Set $M = S \oplus S'$. Then $M/S \cong L/L \cap S$ is noncosingular. Now we prove L is a supplement of S in M. Since $L \cap S \ll M$ and L is a coclosed submodule of M, then $L \cap S \ll L$. So by assumption, L is a direct summand of M. Now the result follows from [1, Theorem 1].

(2) By (1), it suffices to show that every supplement L of a direct summand K' of M with M/K' noncosingular, is a direct summand of M. Now, $K \oplus K' = M$, $L + K' = M$ and $L \cap K' \ll L$ such that K is noncosingular. By assumption, K is K'-projective. Since $M = K \oplus K' = L + K'$, by [5, Lemma 4.47] there is a submodule S of L such that $M = S \oplus K'$. By modular law, $L = S$ is a direct summand of M. □

As an immediate consequence of Proposition 3.10(2), we have the following.

Corollary 3.11. *Let M be an amply supplemented tCC-module such that whenever $M = M_1 \oplus M_2$, then M_1 and M_2 are relatively projective. Then M is t-lifting.*

Note that the condition "amply supplemented" in last result is not necessary. In fact there are some projective t-lifting modules which are not amply supplemented. To be sure, consider the $\mathbb{Z}$-module $M = \mathbb{Z}$ which is a cosingular module. Hence every submodule of M is t-small in M resulting M is t-lifting and also tCC while M is not amply supplemented due to M is not semisimple and has no nonzero small submodule.

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