Journal of Mathematical Modeling
Vol. 8, No. 3, 2020, pp. 257–278

JMM

Ulam stabilities for nonlinear fractional integro–differential equations with constant coefficient via Pachpatte's inequality

Shivaji Ramchandra Tate[†] [*] **, Hambirrao Tatyasaheb Dinde**[‡]

[†] *Department of Mathematics, Kisan Veer Mahavidyalaya, Wai, India*
[‡] *Department of Mathematics, Karmaveer Bhaurao Patil College, Urun–Islampur, India*

Email(s): tateshivaji@gmail.com, drhtdmaths@gmail.com

Abstract. In this article, we study some existence, uniqueness and Ulam type stability results for a class of boundary value problem for nonlinear fractional integro–differential equations with positive constant coefficient involving the Caputo fractional derivative. The main tools used in our analysis is based on Banach contraction principle, Schaefer's fixed point theorem and Pachpatte's integral inequality. Finally, results are illustrated with suitable example.

Keywords: Boundary value conditions, Caputo's fractional derivative, fixed point, integral inequality, Stability.
AMS Subject Classification 2010: 26A33, 45J05, 34K10, 45M10.

1 Introduction

The fractional calculus is an old branch of mathematics, but even then it is still new. This is an old branch because this branch was born in the time when Newton and Leibniz had introduced the concept of differential calculus. On September 30, 1695, Leibniz and L'Hospital discussed the

[*] Corresponding author.
Received: 6 March 2020 / Revised: 26 March 2020 / Accepted: 3 May 2020.
DOI: 10.22124/jmm.2020.15923.1392

 http://jmm.guilan.ac.ir

derivative of one half order. This was believed to be the moment of starting of fractional calculus. Since then many mathematicians have contributed to the basic concept of fractional calculus, see [1,4,15,17,19,31,34,35,42] and the references therein.

Fractional differential equations have been proved to be valuable tools in the modeling of many phenomena in various fields such as control theory, signal processing, rheology, fractals, chaotic dynamics, modelling, bioengineering and biomedical applications and so on, for example, see [12,24] and the references therein. Recently, many researchers studied the fractional differential and integro-differential equations and obtained many interesting existence and uniqueness results, for detail see [3,6,10,11,25–29,39] and the references therein.

Recently, many mathematicians have been attracted by the research field of the stability problems of fractional differential equations and fractional integro-differential equations. This research field started from the speech given at Wisconsin University by Ulam in 1940. In this speech, Ulam [32,33] asked the question about the stability of the functional equation. Hyers [7] was the first who gave the answer of this question in Banach space. Rassias [20] studied the Ulam-Hyers stability of linear and nonlinear mapping. Jung [8,9] established Ulam-Hyers stability for more general mapping on restricted domain. In 1993, Obloza [16] made the first study of Ulam-Hyers stability for linear differential equations. Later many researchers studied the Ulam type stability, for detail see [2,5,21–23,36–38,40,41].

In [30], Tate et al. studied the existence, uniqueness and various types of Ulam stability of the following nonlinear Caputo fractional integro–differential equations of order α $(0 < \alpha \leq 1)$:

$$^{c}D^{\alpha}y(t) = \lambda y(t) + f\left(t, y(t), \int_0^t h(t,s)y(s)ds\right),\ t \in J := [0,T],\ T > 0,$$

$$y(0) + g(y) = y_0 \in \mathbf{R},$$

where $f : J \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and $g : C(J, \mathbf{R}) \to \mathbf{R}$ are continuous functions.

The above results motivate us and therefore, in this paper, we obtain the existence, uniqueness and various types of Ulam stability for the following boundary value problems (BVP for short) for nonlinear fractional integro–differential equations with constant coefficient $\lambda > 0$ of the type:

$$^{c}D^{\alpha}y(t) = \lambda y(t) + f\left(t, y(t), \int_0^t h(t,s)y(s)ds\right),\ t \in J := [0,T],\ T > 0, \quad (1)$$

$$ay(0) + by(T) = c, \quad (2)$$

where ${}^cD^\alpha(0 < \alpha \leq 1)$ denotes the Caputo fractional derivative, $f : J \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is a given continuous function, and a, b, c are real constants with $a + b \neq 0$.

The rest of the paper is organized as follows. In Section 2, some definitions, notations and basic results are given. Section 3 is devoted to study the existence, uniqueness and stability of the problem (1)-(2). Illustrative example is given in the last section.

2 Preliminaries

In this section, we introduce some definitions, notations and results which are useful for further discussion. For $T > 0$ and $J = [0, T]$, $C(J, \mathbf{R})$ denotes the Banach space of all continuous functions from J into $\mathbf{R}$ with the norm

$$||y||_\infty = \sup\{||y(t)|| : t \in J\}.$$

Suppose $L^1(J)$ denotes the space of Lebesgue-integrable functions $y : J \to \mathbf{R}$ with the norm

$$||y||_{L^1} = \int_0^T |y(t)|\, dt.$$

Definition 1 ([19]). The Riemann–Liouville fractional integral of a function $h \in L^1([0, T], \mathbf{R}_+)$ of order $\alpha \in \mathbf{R}_+$ is defined by

$$I^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2 ([12]). The Caputo fractional derivative of order $\alpha > 0$ of a function $h \in L^1([0, T], \mathbf{R}_+)$ is defined as

$${}^cD^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} h^{(n)}(s)\, ds,\ n-1 < \alpha < n,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of the real number α.

Lemma 1 ([12]). *Let $\alpha > 0$ and $n = [\alpha] + 1$, then*

$$I^\alpha({}^cD^\alpha f(t)) = f(t) - \sum_{i=0}^{n-1} \frac{f^k(0)}{k!} t^k,$$

where $f^k(t)$ is the usual derivative of $f(t)$ of order k.

Lemma 2 ([19]). *Let $\alpha > 0$. Then the fractional differential equation*

$$^cD^\alpha h(t) = 0,$$

has a solution $h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$, where c_i, $i=0,1,2,\ldots,n-1$ are constant and $n = [\alpha] + 1$.

The following Pachpatte's inequality plays an important role in obtaining our main results.

Theorem 1 ([18], p. 39). *Let $u(t)$, $f(t)$ and $q(t)$ be nonnegative continuous functions defined on $\mathbf{R}_+$, and $n(t)$ be a positive and nondecreasing continuous function defined on $\mathbf{R}_+$ for which the inequality*

$$u(t) \leq n(t) + \int_0^t f(s)\left[u(s) + \int_0^s q(\tau)u(\tau)\, d\tau\right] ds,$$

holds for $t \in \mathbf{R}_+$. Then

$$u(t) \leq n(t)\left[1 + \int_0^t f(s) exp\left(\int_0^s [f(\tau) + q(\tau)]\, d\tau\right) ds\right],$$

for $t \in \mathbf{R}_+$.

The following definitions are useful in the study of stability results.

Definition 3 ([5, 23]). The equation (1) is Ulam-Hyers stable if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C^1(J, \mathbf{R})$ satisfying the inequality

$$\left\|{}^cD^\alpha z(t) - \lambda z(t) - f\left(t, z(t), \int_0^t h(t,s)z(s)ds\right)\right\| \leq \epsilon, \ t \in J,$$

there exists a solution $y \in C^1(J, \mathbf{R})$ of equation (1) with

$$||z(t) - y(t)|| \leq c_f \epsilon, \ t \in J.$$

Definition 4 ([5, 23]). The equation (1) is generalized Ulam-Hyers stable if there exists $\psi_f \in C(\mathbf{R}_+, \mathbf{R}_+)$, $\psi_f(0) = 0$, such that for each solution $z \in C^1(J, \mathbf{R})$ satisfying the inequality

$$\left\|{}^cD^\alpha z(t) - \lambda z(t) - f\left(t, z(t), \int_0^t h(t,s)z(s)ds\right)\right\| \leq \epsilon, \ t \in J,$$

there exists a solution $y \in C^1(J, \mathbf{R})$ of equation (1) with

$$||z(t) - y(t)|| \leq \psi_f(\epsilon), \ t \in J.$$

Definition 5 ([5,23]). The equation (1) is Ulam-Hyers-Rassias stable with respect to $\varphi \in C(J, \mathbf{R}_+)$ if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C^1(J, \mathbf{R})$ satisfying the inequality

$$\left|\left| {}^cD^\alpha z(t) - \lambda z(t) - f\left(t, z(t), \int_0^t h(t,s)z(s)ds\right)\right|\right| \leq \epsilon\varphi(t),\ t \in J,$$

there exists a solution $y \in C^1(J, \mathbf{R})$ of equation (1) with

$$||z(t) - y(t)|| \leq c_f \epsilon \varphi(t),\ t \in J.$$

Definition 6 ([5,23]). The equation (1) is generalized Ulam-Hyers-Rassias stable with respect to $\varphi \in C(J, \mathbf{R}_+)$ if there exists a real number $c_{f,\varphi} > 0$ such that for each solution $z \in C^1(J, \mathbf{R})$ satisfying the inequality

$$\left|\left| {}^cD^\alpha z(t) - \lambda z(t) - f\left(t, z(t), \int_0^t h(t,s)z(s)ds\right)\right|\right| \leq \varphi(t),\ t \in J,$$

there exists a solution $y \in C^1(J, \mathbf{R})$ of equation (1) with

$$||z(t) - y(t)|| \leq c_{f,\varphi} \varphi(t),\ t \in J.$$

Remark 1 ([5,23]). A function $z \in C^1(J, \mathbf{R})$ satisfies the inequality

$$\left|\left| {}^cD^\alpha z(t) - \lambda z(t) - f\left(t, z(t), \int_0^t h(t,s)z(s)ds\right)\right|\right| \leq \epsilon,\ t \in J,$$

if and only if there exists a function $g \in C(J, \mathbf{R})$ (which depends on solution y) such that
(i) $||g(t)|| \leq \epsilon,\ \forall t \in J$;
(ii) ${}^cD^\alpha z(t) = \lambda z(t) + f\left(t, z(t), \int_0^t h(t,s)z(s)ds\right) + g(t),\ t \in J.$

Remark 2. *Clearly,*
(i) Definition 3 implies Definition 4.
(ii) Definition 5 implies Definition 6.

Remark 3. A solution satisfying the inequality

$$\left|\left| {}^cD^\alpha z(t) - \lambda z(t) - f\left(t, z(t), \int_0^t h(t,s)z(s)ds\right)\right|\right| \leq \epsilon,\ t \in J,$$

is called an fractional $\epsilon-$solution of the nonlinear fractional integro–differential equation (1).

3 Existence and Ulam-Hyers stability of the boundary value problem

In this section we obtain existence, uniqueness and stability results for the problem (1)-(2). Now we introduce the following set of conditions:

(H_1) There exists a constant $L > 0$ such that
$||f(t,x,y) - f(t,\bar{x},\bar{y})|| \leq L(||x-\bar{x}|| + ||y-\bar{y}||)$, for each $t \in J$ and $x, y, \bar{x}, \bar{y} \in \mathbf{R}$.

(H_2) The function $f : J \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is continuous.

(H_3) There exists a constant $a_f > 0$ such that $||f(t,x,y)|| \leq a_f(1 + ||x|| + ||y||)$, for each $t \in J$ and $x, y \in \mathbf{R}$.

Lemma 3 ([13]). *Let $0 < \alpha < 1$ and let $h : J \to \mathbf{R}$ be continuous. A function y is a solution of the fractional integral equation*

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds,$$

if and only if y is a solution of the initial value problem for the fractional differential equation

$$^cD^\alpha y(t) = h(t),\ t \in J = [0,T],\ T > 0,$$

$$y(0) = y_0.$$

Lemma 4 ([3]). *Let $0 < \alpha < 1$ and let $h : J \to \mathbf{R}$ be continuous. A function y is a solution of the fractional integral equation*

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds - \frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} h(s)\, ds - c\right],$$

if and only if y is a solution of the fractional BVP

$$^cD^\alpha y(t) = h(t),\ t \in J = [0,T],\ T > 0,$$

$$ay(0) + by(T) = c.$$

As a consequence of Lemma 3 and Lemma 4 and [14], we have the following result which is useful in our main results.

Lemma 5. *If $f : J \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is a continuous function, then the problem (1)-(2) is equivalent to the following integral equation*

$$y(t) = \tilde{A} + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, y(s), \int_0^s h(t,\tau) y(\tau) d\tau\right) ds, \qquad (3)$$

for $t \in J$, and

$$\tilde{A} = \frac{1}{a+b}\left[c - \frac{b\lambda}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} y(s)\, ds - \frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f\left(s, y(s), \int_0^s h(t,\tau) y(\tau) d\tau\right) ds\right].$$

Theorem 2. *Assume that (H_1) holds. If*

$$\left[\frac{(\lambda + L)T^{\alpha} + L h_T T^{\alpha+1}}{\Gamma(\alpha+1)}\right]\left(1 + \frac{|b|}{|a+b|}\right) < 1, \qquad (4)$$

where $h_T = sup\{|h(t,s)|\,|0 \le s \le t \le T\}$, then the BVP (1)-(2) has a unique solution on J.

Proof. Transform the problem (1)-(2) into a fixed point problem. Consider the operator $F : C(J, \mathbf{R}) \to C(J, \mathbf{R})$ defined by

$$\begin{aligned} F(y)(t) = & \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, y(s), \int_0^s h(t,\tau) y(\tau) d\tau\right) ds \\ & - \frac{1}{a+b}\left[\frac{b\lambda}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} y(s)\, ds \right. \\ & \left. + \frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f\left(s, y(s), \int_0^s h(t,\tau) y(\tau) d\tau\right) ds - c\right]. \end{aligned} \qquad (5)$$

Let $x, y \in C(J, \mathbf{R})$. Then for each $t \in J$, we have

$$||F(x)(t) - F(y)(t)||$$

$$\le \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ||x(s) - y(s)||\, ds$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\bigg\|f\left(s, x(s), \int_0^s h(t,\tau)x(\tau)d\tau\right)\\
&-f\left(s, y(s), \int_0^s h(t,\tau)y(\tau)d\tau\right)\bigg\| ds\\
&+\frac{|b|\,\lambda}{\Gamma(\alpha)\,|a+b|}\int_0^T (T-s)^{\alpha-1}\,||x(s)-y(s)||\;ds\\
&+\frac{|b|}{\Gamma(\alpha)\,|a+b|}\int_0^T (T-s)^{\alpha-1}\bigg\|f\left(s, x(s), \int_0^s h(t,\tau)x(\tau)d\tau\right)\\
&-f\left(s, y(s), \int_0^s h(t,\tau)y(\tau)d\tau\right)\bigg\| ds\\
\leq\;& \frac{(\lambda+L)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,||x(s)-y(s)||\;ds\\
&+\frac{Lh_T}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(\int_0^s ||x(\tau)-y(\tau)||\,d\tau\right)ds\\
&+\frac{|b|\,(\lambda+L)}{\Gamma(\alpha)\,|a+b|}\int_0^T (T-s)^{\alpha-1}\,||x(s)-y(s)||\;ds\\
&+\frac{|b|\,Lh_T}{\Gamma(\alpha)\,|a+b|}\int_0^T (T-s)^{\alpha-1}\left(\int_0^s ||x(\tau)-y(\tau)||\,d\tau\right)ds\\
\leq\;& \left[\frac{(\lambda+L)T^{\alpha}+Lh_T T^{\alpha+1}}{\Gamma(\alpha+1)}\right]\left(1+\frac{|b|}{|a+b|}\right)||x-y||_{\infty}.
\end{aligned}
$$

Thus

$$
||F(x)-F(y)||_{\infty} \leq \left[\frac{(\lambda+L)T^{\alpha}+Lh_T T^{\alpha+1}}{\Gamma(\alpha+1)}\right]\left(1+\frac{|b|}{|a+b|}\right)||x-y||_{\infty}.
$$

Thus, F is a contraction due to the inequality (4).

As a consequence of Banach contraction principle, it is deduced that F has a unique fixed point which is just the unique solution of the problem (1)-(2). □

The second result is based on Schaefer's fixed point theorem.

Theorem 3. *Assume that (H_2) and (H_3) hold. Then the BVP* (1)-(2) *has at least one solution on J.*

Proof. Let the operator F be defined as in (5). We complete the proof in the following four steps.

Step 1: F is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C(J, \mathbf{R})$. Then for each $t \in J$, we have

$$
\begin{aligned}
&||F(y_n)(t) - F(y)(t)|| \\
&\quad \leq \frac{\lambda T^{\alpha}}{\Gamma(\alpha+1)} \left|\left|y_n(s) - y(s)\right|\right|_{\infty} \\
&\quad + \frac{T^{\alpha}}{\Gamma(\alpha+1)} \left|\left| f\left(s, y_n(s), \int_0^s h(t,\tau) y_n(\tau) d\tau\right)\right.\right. \\
&\quad \left.\left. - f\left(s, y(s), \int_0^s h(t,\tau) y(\tau) d\tau\right)\right|\right|_{\infty} \\
&\quad + \frac{|b|\,\lambda T^{\alpha}}{\Gamma(\alpha+1)\,|a+b|} \left|\left|y_n(s) - y(s)\right|\right|_{\infty} \\
&\quad + \frac{|b|\,T^{\alpha}}{\Gamma(\alpha+1)\,|a+b|} \left|\left| f\left(s, y_n(s), \int_0^s h(t,\tau) y_n(\tau) d\tau\right)\right.\right. \\
&\quad \left.\left. - f\left(s, y(s), \int_0^s h(t,\tau) y(\tau) d\tau\right)\right|\right|_{\infty} \\
&\quad \leq \frac{\lambda}{\Gamma(\alpha+1)} \left(1 + \frac{|b|}{|a+b|}\right) T^{\alpha} \left|\left|y_n(s) - y(s)\right|\right|_{\infty} \\
&\quad + \frac{1}{\Gamma(\alpha+1)} \left(1 + \frac{|b|}{|a+b|}\right) T^{\alpha} \left|\left| f\left(s, y_n(s), \int_0^s h(t,\tau) y_n(\tau) d\tau\right)\right.\right. \\
&\quad \left.\left. - f\left(s, y(s), \int_0^s h(t,\tau) y(\tau) d\tau\right)\right|\right|_{\infty}
\end{aligned}
$$

Since f is a continuous function and $y_n \to y$, we have

$$||F(y_n)(t) - F(y)(t)||_{\infty} \to 0,$$

as $n \to \infty$. Consequently, F is continuous.

Step 2: F maps bounded sets into bounded sets in $C(J, \mathbf{R})$.

We need to show that for any $\mu^* > 0$, there exists a positive constant l such that for each $y \in B_{\mu^*} = \{y \in C(J, \mathbf{R}) : ||y||_{\infty} \leq \mu^*\}$, we have $||F(y)||_{\infty} \leq l$.

By condition (H_3), we have for each $t \in [0, T]$,

$$||F(y)|| \leq \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \, ||y(s)|| \, ds$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|\left|f\left(s, y(s), \int_0^s h(t,\tau)y(\tau)d\tau\right)\right|\right| ds \\
&+\frac{|b|\,\lambda}{|a+b|\,\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\,||y(s)||\, ds \\
&+\frac{|b|}{|a+b|\,\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\left|\left|f\left(s, y(s), \int_0^s h(t,\tau)y(\tau)d\tau\right)\right|\right| ds \\
&+\frac{|c|}{|a+b|} \\
\leq\ &\frac{\lambda\mu^*}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\, ds \\
&+\frac{a_f(1+||y||+\int_0^s |h(t,\tau)|\,||y||\,d\tau)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\, ds \\
&+\frac{|b|\,\lambda\mu^*}{|a+b|\,\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\, ds \\
&+\frac{a_f(1+||y||+\int_0^s |h(t,\tau)|\,||y||\,d\tau)\,|b|}{|a+b|\,\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\, ds + \frac{|c|}{|a+b|} \\
\leq\ &\frac{\lambda\mu^* T^\alpha}{\Gamma(\alpha+1)}\left[1+\frac{|b|}{|a+b|}\right] + \frac{a_f(1+\mu^*+\mu^* h_T T)T^\alpha}{\Gamma(\alpha+1)}\left[1+\frac{|b|}{|a+b|}\right] \\
&+\frac{|c|}{|a+b|}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
||F(y)||_\infty \leq\ &\left[\frac{\lambda\mu^* T^\alpha}{\Gamma(\alpha+1)} + \frac{a_f(1+\mu^*+\mu^* h_T T)T^\alpha}{\Gamma(\alpha+1)}\right]\left[1+\frac{|b|}{|a+b|}\right] \\
&+\frac{|c|}{|a+b|} := l.
\end{aligned}
$$

Step 3: F maps bounded sets into equicontinuous sets of $C(J, \mathbf{R})$. Let $t_1, t_2 \in (0, T]$, $t_1 < t_2$, B_{μ^*} be a bounded set in $C(J, \mathbf{R})$ as in step 2, and let $y \in B_{\mu^*}$. Then

$$
\begin{aligned}
&||F(y)(t_1) - F(y)(t_2)|| \\
&\leq \frac{\lambda}{\Gamma(\alpha)}\int_0^{t_1}\{(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}\}||y(s)||\,ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\{(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}\}||f\left(s, y(s), \int_0^s h(t,\tau)y(\tau)d\tau\right)||\,ds
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\lambda}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}||y(s)||\,ds\\
&-\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}||f\left(s,y(s),\int_0^s h(t,\tau)y(\tau)d\tau\right)||\,ds\\
\leq\ &\frac{\lambda||y(s)||}{\Gamma(\alpha)}\int_0^{t_1}\{(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\}\,ds\\
&+\frac{a_f(1+||y(s)||+\int_0^s|h(t,\tau)|||y(\tau)||d\tau)}{\Gamma(\alpha)}\int_0^{t_1}\{(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\}\,ds\\
&+\frac{\lambda||y(s)||}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\,ds\\
&+\frac{a_f(1+||y(s)||+\int_0^s|h(t,\tau)|||y(\tau)||d\tau)}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\,ds\\
\leq\ &\frac{(\lambda\mu^*+a_f(1+\mu^*+\mu^*h_T T))}{\Gamma(\alpha+1)}\{2(t_2-t_1)^\alpha+(t_1^\alpha-t_2^\alpha)\}.
\end{aligned}
$$

As $t_1\to t_2$, the right-hand side of the above inequality tends to zero. As a consequence of steps 1 to 3 together with the Arzelà-Ascoli theorem, we can conclude that $F: C(J,\mathbf{R})\to C(J,\mathbf{R})$ is continuous and completely continuous.

Step 4: A priori bounds. Now it remains to show that the set

$$\mathcal{E}=\{y\in C(J,\mathbf{R}): y=\beta F(y),\ \text{for some } \beta\in(0,1)\},$$

is bounded.

Let $y\in\mathcal{E}$, then $y=\beta F(y)$, for some $\beta\in(0,1)$. Thus, for each $t\in J$, we have

$$
\begin{aligned}
y(t)\ =\ \beta\Bigg\{&\frac{\lambda}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)\,ds\\
&+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f\left(s,y(s),\int_0^s h(t,\tau)y(\tau)d\tau\right)ds\\
&-\frac{1}{a+b}\left[\frac{b\lambda}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}y(s)\,ds\right.\\
&\left.+\frac{b}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}f\left(s,y(s),\int_0^s h(t,\tau)y(\tau)d\tau\right)ds-c\right]\Bigg\}. \quad (6)
\end{aligned}
$$

From condition (H_3), for each $t \in J$,we have

$$\begin{aligned}
||F(y)(t)|| &\le \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \, ||y(s)|| \, ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\| f\left(s, y(s), \int_0^s h(t,\tau) y(\tau) d\tau\right) \right\| ds \\
&+ \frac{|b| \, \lambda}{|a+b| \, \Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \, ||y(s)|| \, ds \\
&+ \frac{|b|}{|a+b| \, \Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \left\| f\left(s, y(s), \int_0^s h(t,\tau) y(\tau) d\tau\right) \right\| ds \\
&+ \frac{|c|}{|a+b|} \\
&\le \frac{\lambda \mu^*}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \, ds + \frac{a_f(1+\mu^*+\mu^* h_T T)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\
&+ \frac{|b| \, \lambda \mu^*}{|a+b| \, \Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} ds \\
&+ \frac{a_f(1+\mu^*+\mu^* h_T T) \, |b|}{|a+b| \, \Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} ds + \frac{|c|}{|a+b|} \\
&\le \frac{\lambda \mu^* T^\alpha}{\Gamma(\alpha+1)} \left[1 + \frac{|b|}{|a+b|}\right] + \frac{a_f(1+\mu^*+\mu^* h_T T) T^\alpha}{\Gamma(\alpha+1)} \left[1 + \frac{|b|}{|a+b|}\right] \\
&+ \frac{|c|}{|a+b|}.
\end{aligned}$$

Thus for every $t \in J$, we have

$$\begin{aligned}
||F(y)||_\infty \le &\left[\frac{\lambda \mu^* T^\alpha}{\Gamma(\alpha+1)} + \frac{a_f(1+\mu^*+\mu^* h_T T) T^\alpha}{\Gamma(\alpha+1)}\right] \left[1 + \frac{|b|}{|a+b|}\right] \\
&+ \frac{|c|}{|a+b|} := R.
\end{aligned}$$

This shows that the set $\mathcal{E}$ is bounded. Now applying Schaefer's fixed point theorem, we deduce that F has a fixed point which is a solution of the problem (1)-(2). □

Theorem 4. *Assume that (H_1) and inequality (4) hold. Then the BVP (1)-(2) is Ulam-Hyers stable.*

Proof. Let $\epsilon > 0$ and let $z \in C^1(J, \mathbf{R})$ be a function which satisfies the inequality:

$$\left|\left|{}^cD^\alpha z(t) - \lambda z(t) - f\left(t, z(t), \int_0^s h(t,\tau)z(\tau)d\tau\right)\right|\right| \leq \epsilon, \text{ for any } t \in J \quad (7)$$

and let $y \in C(J, \mathbf{R})$ be the unique solution of the following Cauchy problem

$$\begin{aligned} {}^cD^\alpha y(t) &= \lambda y(t) + f\left(t, y(t), \int_0^s h(t,\tau)y(\tau)d\tau\right), \ t \in J; \ 0 < \alpha \leq 1 \\ y(0) &= z(0), \ y(T) = z(T). \end{aligned}$$

Using Lemma 5, we obtain

$$\begin{aligned} y(t) &= \tilde{A}_y + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, y(s), \int_0^s h(t,\tau)y(\tau)d\tau\right) ds, \end{aligned}$$

and

$$\begin{aligned} \tilde{A}_y &= \frac{1}{a+b}\Bigg[c - \frac{b\lambda}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} y(s)\, ds \\ &- \frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f\left(s, y(s), \int_0^s h(t,\tau)y(\tau)d\tau\right) ds\Bigg]. \end{aligned}$$

If $y(T) = z(T)$ and $y(0) = z(0)$ then we find

$$\begin{aligned} \left|\left|\tilde{A}_y - \tilde{A}_z\right|\right| &\leq \frac{|b|\,\lambda}{\Gamma(\alpha)\,|a+b|} \int_0^T (T-s)^{\alpha-1} \,||y(s) - z(s)||\, ds \\ &+ \frac{|b|}{\Gamma(\alpha)\,|a+b|} \int_0^T (T-s)^{\alpha-1} L(||y(s) - z(s)|| \\ &+ \int_0^s |h(t,\tau)|\, ||y(\tau) - z(\tau)||\, d\tau) ds \\ &\leq \frac{|b|\,(\lambda + L)}{\Gamma(\alpha)\,|a+b|} \int_0^T (T-s)^{\alpha-1} \,||y(s) - z(s)||\, ds \\ &+ \frac{|b|\, L h_T}{\Gamma(\alpha)\,|a+b|} \int_0^T (T-s)^{\alpha-1} (\int_0^s ||y(\tau) - z(\tau)||\, d\tau)\, ds \\ &\leq \frac{|b|\,(\lambda + L)}{\Gamma(\alpha)\,|a+b|} \int_0^T (T-s)^{\alpha-1} \,||y(s) - z(s)||\, ds \end{aligned}$$

$$+\frac{|b|\,Lh_T T}{\Gamma(\alpha)\,|a+b|}\int_0^T (T-s)^{\alpha-1}\,||y(s)-z(s)||\;ds$$
$$=\left[\frac{|b|\,(\lambda+L)}{|a+b|}+\frac{|b|\,Lh_T T}{|a+b|}\right]I^{\alpha}\,||y(T)-z(T)||=0$$

Thus

$$\tilde{A}_y=\tilde{A}_z.$$

Then we have

$$y(t)\;=\tilde{A}_z+\frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f\left(s,y(s),\int_0^s h(t,\tau)y(\tau)d\tau\right)ds,$$

on integration of the inequality (7), we obtain

$$\left|\left|z(t)-\tilde{A}_z-\frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}z(s)\,ds\right.\right.$$
$$\left.\left.-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f\left(s,z(s),\int_0^s h(t,\tau)z(\tau)d\tau\right)ds\right|\right|$$
$$\leq\frac{\epsilon t^{\alpha}}{\Gamma(\alpha+1)}. \tag{8}$$

For any $t\in J$ we have

$$||z(t)-y(t)||\leq\left|\left|z(t)-\tilde{A}_z-\frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}z(s)\,ds\right.\right.$$
$$\left.\left.-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f\left(s,z(s),\int_0^s h(t,\tau)z(\tau)d\tau\right)ds\right|\right|$$
$$+\frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,||z(s)-y(s)||\;ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|\left|f\left(s,z(s),\int_0^s h(t,\tau)z(\tau)d\tau\right)\right.\right.$$
$$\left.\left.-f\left(s,y(s),\int_0^s h(t,\tau)y(\tau)d\tau\right)\right|\right|ds.$$

Using inequality (8) and condition (H_1), we obtain

$$||z(t)-y(t)||\;\leq\;\frac{\epsilon t^{\alpha}}{\Gamma(\alpha+1)}+\frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,||z(s)-y(s)||\;ds$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} L\Big(||z(s)-y(s)|| \\
&+\int_0^s |h(t,\tau)|\, ||z(\tau)-y(\tau)||\, d\tau\Big)\, ds \\
\le\ & \frac{\epsilon t^\alpha}{\Gamma(\alpha+1)} + \frac{(\lambda+L)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\, ||z(s)-y(s)||\ ds \\
&+\frac{Lh_T}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big(\int_0^s ||z(\tau)-y(\tau)||\, d\tau\Big)\, ds, \\
\le\ & \frac{\epsilon t^\alpha}{\Gamma(\alpha+1)} + \int_0^t \frac{(\lambda+L)}{\Gamma(\alpha)}(T-s)^{\alpha-1}\Big[||z(s)-y(s)|| \\
&+\int_0^s \frac{Lh_T}{(\lambda+L)}\, ||z(\tau)-y(\tau)||\, d\tau\Big]\, ds. \qquad (9)
\end{aligned}
$$

Applying Pachpatte's inequality given in the Theorem 1 to the inequality (9) with $u(t) = ||z(t) - y(t)||$,

$$
n(t) = \frac{\epsilon t^\alpha}{\Gamma(\alpha+1)}, \quad f(s) = \frac{(\lambda+L)}{\Gamma(\alpha)}(T-s)^{\alpha-1}, \quad q(\tau) = \frac{Lh_T}{(\lambda+L)},
$$

we obtain

$$
\begin{aligned}
||z(t)-y(t)|| \ \le\ & \frac{\epsilon t^\alpha}{\Gamma(\alpha+1)}\Big[1+\int_0^t \frac{(\lambda+L)}{\Gamma(\alpha)}(T-s)^{\alpha-1} \\
&\times \exp\Big(\int_0^s \Big\{\frac{(\lambda+L)}{\Gamma(\alpha)}(T-\tau)^{\alpha-1} + \frac{Lh_T}{(\lambda+L)}\Big\} d\tau\Big) ds\Big] \\
\le\ & C\epsilon,
\end{aligned}
$$

for all $t \in J$, where

$$
\begin{aligned}
C \ =\ & \frac{T^\alpha}{\Gamma(\alpha+1)}\Big[1+\int_0^T \frac{(\lambda+L)}{\Gamma(\alpha)}(T-s)^{\alpha-1} \\
&\times \exp\Big(\int_0^s \Big\{\frac{(\lambda+L)}{\Gamma(\alpha)}(T-\tau)^{\alpha-1} + \frac{Lh_T}{(\lambda+L)}\Big\} d\tau\Big) ds\Big].
\end{aligned}
$$

We conclude the problem (1)-(2) is Ulam-Hyers stable. □

Corollary 1. *If f in the problem (1)-(2) satisfies the condition (H_1) and the inequality (4) holds, then the problem (1)-(2) is generalized Ulam-Hyers stable.*

Theorem 5. *Assume that (H_1) and inequality (4) hold. Further suppose there exists an increasing function $\varphi \in C(J, \mathbf{R}_+)$ and there exists $\kappa_\varphi > 0$ such that for any $t \in J$*

$$
I^\alpha \varphi(t) \le \kappa_\varphi \varphi(t)
$$

are satisfied. Then the BVP (1)-(2) is Ulam-Hyers-Rassias stable.

Proof. Let $z \in C^1(J, \mathbf{R})$ be satisfies the following inequality:

$$\left|\left|{}^cD^\alpha z(t) - \lambda z(t) - f\left(t, z(t), \int_0^s h(t,\tau)z(\tau)d\tau\right)\right|\right| \leq \epsilon\varphi(t), \tag{10}$$

for any $t \in J,\ \epsilon > 0$. Let $y \in C(J, \mathbf{R})$ be the unique solution of the following Cauchy problem

$$\begin{aligned} {}^cD^\alpha y(t) &= \lambda y(t) + f\left(t, y(t), \int_0^s h(t,\tau)y(\tau)d\tau\right),\ t \in J;\ 0 < \alpha \leq 1, \\ y(0) &= z(0), \quad y(T) = z(T). \end{aligned}$$

By Lemma 5, we have

$$\begin{aligned} y(t) &= \tilde{A}_z + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, y(s), \int_0^s h(t,\tau)y(\tau)d\tau\right) ds, \end{aligned}$$

and

$$\begin{aligned} \tilde{A}_z &= \frac{1}{a+b}\Bigg[c - \frac{b\lambda}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} z(s)\, ds \\ &\quad - \frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f\left(s, z(s), \int_0^s h(t,\tau)z(\tau)d\tau\right) ds\Bigg]. \end{aligned}$$

Integrating both sides of inequality (10), we obtain

$$\begin{aligned} &\Bigg|\Bigg| z(t) - \tilde{A}_z - \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s)\, ds \\ &\qquad - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, z(s), \int_0^s h(t,\tau)z(\tau)d\tau\right) ds \Bigg|\Bigg| \\ &\qquad \leq \frac{\epsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\varphi(t)ds = \epsilon I^\alpha \varphi(t) \leq \epsilon\kappa_\varphi \varphi(t). \end{aligned} \tag{11}$$

On the other hand, we have

$$||z(t) - y(t)|| \leq \Bigg|\Bigg| z(t) - \tilde{A}_z - \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s)\, ds$$

$$
\begin{aligned}
&- \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, z(s), \int_0^s h(t,\tau) z(\tau) d\tau\right) ds \Bigg\| \\
&+ \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|z(s) - y(s)\| \, ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Bigg\| f\left(s, z(s), \int_0^s h(t,\tau) z(\tau) d\tau\right) \\
&- f\left(s, y(s) \int_0^s h(t,\tau) y(\tau) d\tau\right) \Bigg\| \, ds.
\end{aligned}
$$

Using inequality (11) and condition (H_1), we obtain

$$
\begin{aligned}
\|z(t) - y(t)\| &\leq \epsilon \kappa_\varphi \varphi(t) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|z(s) - y(s)\| \, ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} L \bigg(\|z(s) - y(s)\| \\
&\quad + \int_0^s |h(t,\tau)| \, \|z(\tau) - y(\tau)\| \, d\tau \bigg) ds \\
&\leq \epsilon \kappa_\varphi \varphi(t) + \frac{(\lambda + L)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|z(s) - y(s)\| \, ds \\
&\quad + \frac{L h_T}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\int_0^s \|z(\tau) - y(\tau)\| \, d\tau \right) ds \\
&\leq \epsilon \kappa_\varphi \varphi(t) + \int_0^t \frac{(\lambda + L)}{\Gamma(\alpha)} (T-s)^{\alpha-1} \bigg[\|z(s) - y(s)\| \\
&\quad + \int_0^s \frac{L h_T}{(\lambda + L)} \|z(\tau) - y(\tau)\| \, d\tau \bigg] ds.
\end{aligned}
$$

By applying Pachpatte's inequality given in the Theorem 1 with $u(t) = \|z(t) - y(t)\|$,

$$
n(t) = \epsilon \kappa_\varphi \varphi(t), \; f(s) = \frac{(\lambda + L)}{\Gamma(\alpha)} (T-s)^{\alpha-1}, \; q(\tau) = \frac{L h_T}{(\lambda + L)},
$$

we obtain

$$
\begin{aligned}
\|z(t) - y(t)\| &\leq \epsilon \kappa_\varphi \varphi(t) \bigg[1 + \int_0^t \frac{(\lambda + L)}{\Gamma(\alpha)} (T-s)^{\alpha-1} \\
&\quad \times \exp\left(\int_0^s \left\{ \frac{(\lambda + L)}{\Gamma(\alpha)} (T-\tau)^{\alpha-1} + \frac{L h_T}{(\lambda + L)} \right\} d\tau \right) ds \bigg] \\
&\leq C \epsilon \varphi(t),
\end{aligned}
$$

for all $t \in J$, where

$$C = \kappa_\varphi\Big[1 + \int_0^T \frac{(\lambda+L)}{\Gamma(\alpha)}(T-s)^{\alpha-1} \times \exp\Big(\int_0^s \Big\{\frac{(\lambda+L)}{\Gamma(\alpha)}(T-\tau)^{\alpha-1} + \frac{Lh_T}{(\lambda+L)}\Big\}d\tau\Big)ds\Big].$$

The proof is complete. □

Corollary 2. *Under the assumptions of Theorem 5, the problem (1)-(2) is generalized Ulam-Hyers-Rassias stable.*

4 Examples

In this section, we illustrate our main results with the help of following example.

Example 1. Consider

$$^cD^{\frac{1}{2}}x(t) = \frac{1}{10}x(t) + \frac{x(t)+1}{t^2+9} + \frac{1}{9}\int_0^t \frac{x(s)}{(2+t)^2}ds, \ t \in [0,1] \tag{12}$$

$$x(0) + x(1) = 0. \tag{13}$$

Define

$$f(t, x(t), Hx(t)) = \frac{x(t)+1}{t^2+9} + \frac{1}{9}Hx(t), \ t \in [0,1],$$

$\alpha = \frac{1}{2}$, $\lambda = \frac{1}{10}$, where

$$Hx(t) = \int_0^t \frac{1}{(2+t)^2}x(s)ds.$$

Clearly, the function f is continuous. For any $x_1, x_2 \in \mathbf{R}$ and $t \in [0,1]$

$$||f(t,x_1,Hx_1) - f(t,x_2,Hx_2)|| \le \frac{1}{9}\Big[\ ||x_1 - x_2|| + ||Hx_1 - Hx_2||\Big] \tag{14}$$

Hence condition (H_1) is satisfied with $L = \frac{1}{9}$.

As $h_T = \frac{1}{4}, a = b = T = 1, c = 0$ and $\alpha = \frac{1}{2}$ we have

$$\Big[\frac{(\lambda+L)T^\alpha + Lh_T T^{\alpha+1}}{\Gamma(\alpha+1)}\Big]\Big(1 + \frac{|b|}{|a+b|}\Big) = \Big[\frac{(\frac{1}{10}+\frac{1}{9}) + \frac{1}{36}}{\Gamma(\frac{1}{2}+1)}\Big]\Big(1+\frac{1}{2}\Big) = \frac{43}{60\sqrt{\pi}} < 1.$$

So all conditions of Theorem 2 hold. Thus Theorem 2 implies that the problem (12)-(13) has a unique solution on [0,1]. Moverover, Theorem 4 implies that the problem (12)-(13) is Ulam-Hyers stable.

Acknowledgements

The authors are grateful to the anonymous referees for their valuable comments and which helped us to improve our results.

References

[1] S. Abbas, M. Benchohra and G.M. N'Guérékata, *Topics in fractional differential equations*, Springer-Verlag, New York, 2012.

[2] S. Abbas, M. Benchohra, *On the generalized Ulam-Hyers-Rassias stability for darboux problem for partial fractional implicit differential equations*, Appl. Math. E-Notes. **14** (2014) 20–28.

[3] R. P. Agarwal, M. Benchohra and S. Hamani, *A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions*, Acta Appl. Math. **109** (2010) 973–1033.

[4] D. Baleanu, K. Diethelm, E. Scalas and J. J. Trujillo, *Fractional calculus models and numerical methods*, World Scientific Publishing, New York, 2012.

[5] M. Benchohra and S. Bouriahi, *Existence and stability results for nonlinear boundary value problem for implicit differential equations of fractional order*, Moroccan J. Pure Appl. Anal. **1**(1) (2015) 22–37.

[6] M. Benchohra and J.E. Lazreg, *Nonlinear fractional implicit differential equations*, Commun. Appl. Anal. **17** (2013) 471-482.

[7] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. **27** (1941) 222–224.

[8] S.M. Jung, *On the Hyers–Ulam stability of the functional equations that have the quadratic property*, J. Math. Anal. Appl. **222** (1998) 126–137.

[9] S.M. Jung, *Hyers–Ulam stability of linear differential equations of first order*, Appl. Math. Lett. **19** (2006) 854–858.

[10] V.V. Kharat, *On existence and uniqueness of fractional integro-differential equations with an integral fractional boundary condition,* Malaya J. Mat. **6**(3) (2018) 485–491

[11] V.V. Kharat, D.B. Dhaigude and D.R. Hasabe, *On nonlinear mixed fractional integro-differential inclusion with four–point nonlocal Riemann–Liouville integral boundary conditions,* Indian J Pure Appl Math. **50** (2019) 937-951.

[12] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics studies, 204. Elsevier Science B. V., Amsterdam, 2006.

[13] A.A. Kilbas and S.A. Marzan, *Nonlinear differential equations with the Caputo fractional derivative in the space of continuously differentiable functions*, Differential Equations. **41**(1) (2005) 84–89.

[14] L. Lv, J. Wang and W. Wei, *Existence and uniqueness results for fractional differential euations with boundary value conditions*, Opuscula Math. **31**(4) (2011) 629–643.

[15] K.S. Miller and B. Ross, *An introduction to the fractional calculus and differential equations,* John Wiley, New York, 1993.

[16] M. Obloza, *Hyers stability of the linear differential equation*, Rocznik Nauk.-Dydakt. Prace Mat. **13** (1993) 259–270.

[17] M.D. Ortigueira, *Fractional calculus for scientists and engineers,* Lecture Notes in Electrical Engineering, 84. Springer, Dordrecht, 2011.

[18] B. Pachpatte, *Inequalities for differential and integral equations,* Academic Press, New York, 1998.

[19] I. Podlubny, *Fractional differential equations*, Academic Press, New York, 1999.

[20] T.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978) 297–300.

[21] J.M. Rassias, *Functional equations, difference inequalities and Ulam stability notions*, (F.U.N), Inc., New York, 2010.

[22] T.M. Rassias and J. Brzdek, *Functional equations in mathematical analysis*, Springer, New York, 2012.

[23] I.A. Rus, *Ulam stabilities of ordinary differential equations in a Banach space*, Carpathian J. Math. **26** (2010) 103–107.

[24] V.E. Tarasov, *Fractional dynamics: applications of fractional calculus to dynamics of particles, fields and media*, Springer, Heidelberg; Higher Education Press, Beijing, 2010.

[25] S. Tate and H.T. Dinde, *Some theorems on Cauchy problem for nonlinear fractional differential equations with positive constant coefficient*, Mediterr. J. Math. **14** (2017) 72.

[26] S. Tate and H.T. Dinde, *Existence and uniqueness results for nonlinear implicit fractional differential equations with non local Conditions*, Palest. J. Math. **9**(1) (2020) 212–219.

[27] S. Tate and H.T. Dinde, *Boundary value problems for nonlinear implicit fractional differential equations*, J. Nonlinear Anal. Appl. **2019**(2) (2019) 29–40.

[28] S. Tate, V.V. Kharat and H.T. Dinde, *On nonlinear mixed fractional integro–differential equations with positive constant coefficient*, Filomat. **33**(17) (2019) 5623–5638.

[29] S. Tate, V.V. Kharat and H.T. Dinde, *On nonlinear fractional integro–differential equations with positive constant coefficient* , Mediterr. J. Math. **16**(2) (2019) p. 41.

[30] S. Tate, V.V. Kharat and H.T. Dinde, *A nonlocal Cauchy problem for nonlinear fractional integro–differential equations with positive constant coefficient*, J. Math. Model. **7**(1) (2019) 133-151.

[31] G.S. Teodoro, J.A.T. Machado and E. Capelas de Oliveira, *A review of definitions of fractional derivatives and other operators*, J. Comput. Phys. **388** (2019) 195–208.

[32] S. M. Ulam, *Problems in modern mathematics*, John Wiley and Sons, New York, USA, 1940.

[33] S.M. Ulam, *A collection of mathematical problems*, Interscience, New York, 1960.

[34] J. Vanterler da C. Sousa and E. Capelas de Oliveira, *On the ψ–Hilfer fractional derivative,* Commun. Nonlinear Sci. Numer. Simul. **60**(2018) 72–91.

[35] J. Vanterler da C. Sousa and E. Capelas de Oliveira, *Leibniz type rule:ψ–Hilfer fractional operator,* Commun. Nonlinear Sci. Numer. Simul. **77** (2019) 305–311.

[36] J. Vanterler da C. Sousa and E. Capelas de Oliveira, *On the stability of a hyperbolic fractional partial differential equation*, Differ Equ Dyn Syst. (2019). https://doi.org/10.1007/s12591-019-00499-3

[37] J. Vanterler da C. Sousa, E. Capelas de Oliveira, *Ulam–Hyers stability of a nonlinear fractional Volterra integro–differential equation*, Appl. Math. Lett. **81** (2018) 50–56.

[38] J. Vanterler da C. Sousa, K.D. Kucche and E. Capelas de Oliveira, *Stability of ψ–Hilfer impulsive fractional differential equation*, Appl. Math. Lett. **88** (2019) 73–90.

[39] J. Vanterler da C. Sousa, E. Capelas de Oliveira and K.D. Kucche, *On the fractional functional differential equation with abstract Volterra operator*, Bull Braz Math Soc. New Series **50** (2019) 803–822.

[40] J. Wang, M. Feckan and Y. Zhou, *Ulam's type stability of impulsive ordinary differential equations*, J. Math. Anal. Appl. **395** (2012) 258–264.

[41] J. Wang, L. Lv and Y. Zhou, *Ulam stability and data dependence for fractional differential equations with Caputo derivative*, Electron. J. Qual. Theory Differ. Equ. **63** (2011) 1–10.

[42] S.S. Zeid, *Approximation methods for solving fractional equations*, Chaos Solitons Fractals **125** (2019) 171–193.