Journal of Mathematical Modeling
Vol. 8, No. 3, 2020, pp. 207–218

JMM

$2n$-by-$2n$ circulant preconditioner for a kind of spatial fractional diffusion equations

Nasser Akhoundi*
School of mathematics and computer science, Damghan university, Damghan, Iran
Email(s): akhoundi@du.ac.ir

Abstract. In this paper, a $2n$-by-$2n$ circulant preconditioner is introduced for a system of linear equations arising from discretization of the spatial fractional diffusion equations (FDEs). We show that the eigenvalues of our preconditioned system are clustered around 1, even if the diffusion coefficients of FDEs are not constants. Numerical experiments are presented to demonstrate that the preconditioning technique is very efficient.

Keywords: Fractional diffusion equation, circulant matrix, skew-circulant matrix, Toeplitz matrix, Krylov subspace methods
AMS Subject Classification 2010: 65F10, 65F15.

1 Introduction

The initial-boundary value problem of the kind of spatial fractional diffusion equations (FDEs) is as follows:

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} + {}_{x_L}^{C}D_x^{(\alpha)}(d_+(x,t)\, {}_{x}^{RL}D_{x_R}^{(\alpha)}u(x,t)) + {}_{x}^{C}D_{x_R}^{(\alpha)}(d_-(x,t)\, {}_{x_L}^{RL}D_{x}^{(\alpha)}) \\ &\qquad\qquad = f(x,t), \quad (x,t) \in (x_L, x_R) \times (0,T], \qquad (1) \\ &u(x_L,t) = u_L(t), \quad u(x_R,t) = u_R(t), \quad 0 \le t \le T, \\ &u(x,0) = u^0(x), \quad x_L \le x \le x_R, \end{aligned}$$

*Corresponding author.
Received: 4 March 2020 / Revised: 24 March 2020 / Accepted: 27 March 2020.
DOI: 10.22124/jmm.2020.15908.1391

where $\frac{1}{2} < \alpha < 1$, $f(x,t)$ is the source term, and the diffusion coefficients satisfy $d_{\pm}(x,t) \geq 0$. Here, ${}_{x_L}^{C}D_x^{(\alpha)}u(x)$ and ${}_{x_L}^{C}D_x^{(\alpha)}u(x)$ denote the left- and right-sided Caputo fractional derivatives [8,11], respectively, which are defined as the following formulas:

$$\begin{aligned} {}_{x_L}^{C}D_x^{(\alpha)}u(x) &= \frac{1}{\Gamma(1-\alpha)}\int_{x_L}^{x}\frac{u'(\xi)}{(x-\xi)^{\alpha}}d\xi, \\ {}_{x}^{C}D_{x_R}^{(\alpha)}u(x) &= \frac{-1}{\Gamma(1-\alpha)}\int_{x}^{x_R}\frac{u'(\xi)}{(\xi-x)^{\alpha}}d\xi, \end{aligned}$$

where $\Gamma(.)$ is the Gamma function, while ${}_{x_L}^{RL}D_x^{(\alpha)}u(x)$ and ${}_{x}^{RL}D_{x_R}^{(\alpha)}u(x)$ denote the left- and right-sided Riemann-Liouville fractional derivatives [8,11] defined by

$$\begin{aligned} {}_{x_L}^{RL}D_x^{(\alpha)}u(x) &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_L}^{x}\frac{u(\xi)}{(x-\xi)^{\alpha}}d\xi, \\ {}_{x}^{RL}D_{x_R}^{(\alpha)}u(x) &= \frac{-1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x}^{x_R}\frac{u(\xi)}{(\xi-x)^{\alpha}}d\xi. \end{aligned}$$

In the last few decades, fractional calculus including fractional differentiation and integration has gained considerable attention and importance due to its applications in various fields of science and engineering, such as electrical and mechanical engineering, biology, physics, control theory, and data fitting; see [1,3,7,9,11].

For solving (1) numerically the finite difference discretization of (1) has been developed by Fang et al. [5]. In this paper, we use the preconditioned GMRES method as an iterative solver for solving discretized system arising from finite difference schemes (see [5]) for (1), and demonstrate the efficiency of our preconditioner by some numerical experiments.

The rest of this paper is organized as follows. In Section 2, the finite difference scheme for (1) and our preconditioner is established. Some computational remarks are given in Section 3. Numerical experiments are provided in Section 4 to demonstrate the performance of our preconditioner.

Notations are the following. Capital letters, boldface lowercase letters and regular lowercase letters denote matrices, vectors, and scalars, respectively. I_n denotes the identity matrix of order n, while J_n denotes the $n \times n$ exchange matrix $J_n = \text{antidiag}(1,1,\ldots,1)$. We denote by e_j the j'th column of identity matrix.

In this paper we use the definition of circulant and skew-circulant matrices. We say that a $n \times n$ matrix S is skew-circulant matrix if it is a

Toeplitz matrix and satisfies

$$S = \begin{pmatrix} c_0 & \omega c_{n-1} & \omega c_{n-2} & \cdots & \omega c_1 \\ c_1 & c_0 & \omega c_{n-1} & \cdots & \omega c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-1} & c_{n-2} & c_{n-3} & \cdots & c_0 \end{pmatrix}. \tag{2}$$

In the matrix S, if $\omega = 1$, then we say that matrix S is circulant matrix. All operations on skew-circulant matrices such as matrix-vector product, inversion of matrix, and etc. can be computed by fast fourier transform (FFT) via $O(n\log(n))$ operations (see [10]).

2 Preconditioning technique

The finite difference discretization of (1) was developed in [5]. Take positive integers n and m and let $\Delta x = (x_R - x_L)/(n+1)$, $\Delta t = T/m$ be the sizes of spatial grid and time step, respectively. We define a spatial and temporal partition $x_i = x_L + i\Delta x$ for $= 0, 1, \ldots, n+1$ and $t_j = j\Delta t$ for $j = 0, 1, \ldots, m$. Denoting

$$u_i^{(j)} \sim u(x_i, t_j), \quad d_{+,i}^{(j)} = d_+(x_i, t_j), \quad d_{-,i}^{(j)} = d_-(x_i, t_j), \quad f_i^{(j)} = f(x_i, t_j),$$

$$a_i^{(\alpha)} = (i+1)^{1-\alpha} - i^{1-\alpha} \qquad i = 0, 1, 2, \ldots,$$

$a_i^{(\alpha)}$ for $i = 0, 1, 2, \ldots$ are satisfied in the following relation [5]

$$1 = a_0^{(\alpha)} > a_1^{(\alpha)} > a_2^{(\alpha)} > \cdots > a_i^{(\alpha)} \to 0 \ \text{ as } \ i \to \infty. \tag{3}$$

Based on [5], in order to define the finite difference approximation of FDEs (1), we need the following definitions:

$$g_0^{(\alpha)} = \frac{a_0^{(\alpha)}}{\Gamma(2-\alpha)}, \quad g_k^{(\alpha)} = \frac{a_k^{(\alpha)} - a_{k-1}^{(\alpha)}}{\Gamma(2-\alpha)}, \quad q_k^{(\alpha)} = \frac{k^{-\alpha}}{\Gamma(1-\alpha)} - \frac{a_{k-1}^{(\alpha)}}{\Gamma(2-\alpha)}.$$

and

$$\mathbf{a}_+^{(\alpha)} = -\frac{1}{\Gamma(2-\alpha)} \begin{pmatrix} a_0^{(\alpha)} \\ a_1^{(\alpha)} \\ \vdots \\ a_{n-1}^{(\alpha)} \end{pmatrix}, \quad \mathbf{g}_+^{(\alpha)} = \begin{pmatrix} g_1^{(\alpha)} \\ g_2^{(\alpha)} \\ \vdots \\ g_n^{(\alpha)} \end{pmatrix},$$

$$\mathbf{a}_-^{(\alpha)} = J_n \mathbf{a}_+^{(\alpha)}, \qquad \mathbf{g}_-^{(\alpha)} = J \mathbf{g}_+^{(\alpha)}.$$

The vector $\mathbf{g}_+^{(\alpha)}$ has the following properties

$$\begin{aligned}\|\mathbf{g}_+^{(\alpha)}\|_1 &= \sum_{k=1}^{n}|g_k| = \frac{1}{\Gamma(2-\alpha)}\sum_{k=1}^{n}|a_k^{(\alpha)} - a_{k-1}^{(\alpha)}| \\ &= \frac{1}{\Gamma(2-\alpha)}\sum_{k=1}^{n}a_{k-1}^{(\alpha)} - a_k^{(\alpha)} = \frac{1-a_n}{\Gamma(2-\alpha)} \\ &\leq \frac{1}{\Gamma(2-\alpha)}\end{aligned} \tag{4}$$

For system of linear equations arising from finite difference of FDEs (1), we need to define lower triangular Toeplitz matrix $\tilde{G}^{(\alpha)}$ as following:

$$\tilde{G}^{(\alpha)} = \begin{pmatrix} g_0^{(\alpha)} & 0 & \cdots & 0 \\ g_1^{(\alpha)} & g_0^{(\alpha)} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ g_{n-1}^{(\alpha)} & \cdots & g_1^{(\alpha)} & g_0^{(\alpha)} \end{pmatrix} \in \mathbb{R}^{n\times n}, \tag{5}$$

the vectors,

$$\begin{aligned}\mathbf{u}^{(j)} &= \begin{pmatrix} u_1^{(j)} & u_2^{(j)} & \cdots & u_n^{(j)} \end{pmatrix}^T \in \mathbf{R}^n, \\ \mathbf{f}^{(j)} &= \begin{pmatrix} f_1^{(j)} & f_2^{(j)} & \cdots & f_n^{(j)} \end{pmatrix}^T \in \mathbf{R}^n,\end{aligned}$$

and

$$\boldsymbol{\sigma}^{(j)} = \begin{pmatrix} \sigma_1^{(j)} & \sigma_2^{(j)} & \cdots & \sigma_n^{(j)} \end{pmatrix}^T \in \mathbf{R}^n$$

where

$$\begin{aligned}\sigma_i^{(j)} &= \frac{1}{\Delta x^{2\alpha}}\left[\sum_{l=0}^{i-1} g_l^{(\alpha)} d_{+,i-l}^{(j)} q_{n-i+l+1}^{(\alpha)} u_{n+1}^{(j)} - \frac{a_{i-1}^{(\alpha)} d_{+,0}^{(j)}}{\Gamma(2-\alpha)}\left(g_0^{(\alpha)} u_0^{(j)} + q_{n+1}^{(\alpha)} u_{n+1}^{(j)}\right)\right] \\ &+ \frac{1}{\Delta x^{2\alpha}}\left[\sum_{l=0}^{n-i} g_l^{(\alpha)} d_{-,i+l}^{(j)} q_{i+l}^{(\alpha)} u_0^{(j)} - \frac{a_{n-i}^{(\alpha)} d_{-,n+1}^{(j)}}{\Gamma(2-\alpha)}\left(g_0^{(\alpha)} u_{n+1}^{(j)} + q_{n+1}^{(\alpha)} u_0^{(j)}\right)\right].\end{aligned}$$

Then the system of linear equations arising from finite difference of FDEs (1) is

$$A^{(j)}\mathbf{u}^{(j)} = \eta\mathbf{u}^{(j-1)} + \Delta x^{2\alpha}(\mathbf{f}^{(j)} - \boldsymbol{\sigma}^{(j)}), \quad j = 1, 2, \ldots, m,$$

where the coefficient matrix $A^{(j)}$ is of the form

$$A^{(j)} = \eta I_n + G_{L+}^{(\alpha)} D_+^{(j)} G_{R+}^{(\alpha)} + G_{L-}^{(\alpha)} D_-^{(j)} G_{R-}^{(\alpha)}, \tag{6}$$

in which $\eta = \frac{\Delta x^{2\alpha}}{\Delta t}$, $D_+^{(j)}$, and $D_-^{(j)}$ are diagonal matrices given by

$$D_+^{(j)} = \operatorname{diag}\left(d_{+,0}^{(j)}, d_{+,1}^{(j)}, \ldots, d_{+,n}^{(j)}\right), \quad D_-^{(j)} = \operatorname{diag}\left(d_{-,1}^{(j)}, d_{-,2}^{(j)}, \ldots, d_{-,n+1}^{(j)}\right),$$

respectively, $G_{L+}^{(\alpha)}$, $G_{R+}^{(\alpha)}$, $G_{L-}^{(\alpha)}$, and $G_{R-}^{(\alpha)}$ are non-square matrices:

$$\begin{aligned} G_{L+}^{(\alpha)} &= \begin{pmatrix} \mathbf{a}_+^{(\alpha)} & \tilde{G}^{(\alpha)} \end{pmatrix}, & G_{R+}^{(\alpha)} &= \begin{pmatrix} \mathbf{g}_+^{(\alpha)} & \tilde{G}^{(\alpha)} \end{pmatrix}^T, \\ G_{L-}^{(\alpha)} &= \begin{pmatrix} (\tilde{G}^{(\alpha)})^T & \mathbf{a}_-^{(\alpha)} \end{pmatrix}, & G_{R-}^{(\alpha)} &= \begin{pmatrix} \tilde{G}^{(\alpha)} \\ \mathbf{g}_-^{(\alpha)T} \end{pmatrix}. \end{aligned}$$

We can split matrix the matrix $A^{(j)}$ as $A^{(j)} = S^{(j)} + L^{(j)}$. Matrices $S^{(j)}$ and $L^{(j)}$ are defined as following:

$$S^{(j)} = \eta I_n + \tilde{G}^{(\alpha)} \tilde{D}_+^{(j)} \tilde{G}^{(\alpha)T} + \tilde{G}^{(\alpha)T} \tilde{D}_-^{(j)} \tilde{G}^{(\alpha)} \tag{7}$$

$$L^{(j)} = d_{+,0}^{(j)} \mathbf{a}_+^{(\alpha)} \mathbf{g}_+^{(\alpha)T} + d_{-,n+1}^{(j)} \mathbf{a}_-^{(\alpha)} \mathbf{g}_-^{(\alpha)T}, \tag{8}$$

where $\tilde{D}_\pm^{(j)} = \operatorname{diag}(d_{\pm,1}^{(j)}, d_{\pm,2}^{(j)}, \ldots, d_{\pm,n}^{(j)})$. $S^{(j)}$ is a symmetric positive definite matrix and $L^{(j)}$ is a matrix with low rank, i.e., $\operatorname{rank}(L^{(j)}) \leq 2$, and satisfies in the following relation

$$\begin{aligned} \|L^{(j)}\|_\infty &\leq |d_{+,0}^{(j)}| \|\mathbf{a}_+^{(\alpha)} \mathbf{g}_+^{(\alpha)T}\|_\infty + |d_{-,n+1}^{(j)}| \|\mathbf{a}_-^{(\alpha)} \mathbf{g}_-^{(\alpha)T}\|_\infty \\ &= |d_{+,0}^{(j)}| \|\mathbf{a}_+^{(\alpha)} \mathbf{g}_+^{(\alpha)T}\|_\infty + |d_{-,n+1}^{(j)}| \|J \mathbf{a}_+^{(\alpha)} \mathbf{g}_+^{(\alpha)T} J\|_\infty \\ &\leq |d_{+,0}^{(j)}| \|\mathbf{a}_+^{(\alpha)} \mathbf{g}_+^{(\alpha)T}\|_\infty + |d_{-,n+1}^{(j)}| \|\mathbf{a}_+^{(\alpha)} \mathbf{g}_+^{(\alpha)T}\|_\infty \\ &\leq 2 \max\{|d_{+,0}^{(j)}|, |d_{-,n+1}^{(j)}|\} \|\mathbf{a}_+^{(\alpha)}\|_\infty \|\mathbf{g}_+^{(\alpha)}\|_1 \\ &\leq \frac{2 \max\{|d_{+,0}^{(j)}|, |d_{-,n+1}^{(j)}|\}}{\Gamma(2-\alpha)}. \end{aligned} \tag{9}$$

By (9) and $\operatorname{rank}(L^{(j)}) \leq 2$, we see that $S^{(j)}$ is the dominance part of matrix $A^{(j)}$. By this assumption we construct our preconditioner based on $S^{(j)}$ and show that effectiveness of our preconditioner. First, we have the following lemma.

Lemma 1. *(See [2]) If $D = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$, $\lambda_{\min} = \min_i\{\lambda_i\}$, and $\lambda_{\max} = \max_i\{\lambda_i\}$ then $\mu^* = (\lambda_{\max} + \lambda_{min})/2$ is the optimal solution of minimization problem $\min_{\mu \in \mathbf{R}} \|D - \mu I\|_\infty$ and for every $\mathbf{x} \in \mathbf{R}^n$ we have*

$$|\mathbf{x}^T (D - \mu^* I_n) \mathbf{x}| \leq \frac{\lambda_{\max} - \lambda_{\min}}{2} \|\mathbf{x}\|_2.$$

Let $\bar{d}_{\pm}^{(j)} = (d_{\pm,max}^{(j)} + d_{\pm,min}^{(j)})/2$, with $d_{\pm,max}^{(j)} = \max\{d_{\pm,1}^{(j)}, d_{\pm,2}^{(j)}, \ldots, d_{\pm,n}^{(j)}\}$ and $d_{\pm,min} = \min\{d_{\pm,1}^{(j)}, d_{\pm,2}^{(j)}, \ldots, d_{\pm,n}^{(j)}\}$. We define our preconditioner as following:

$$P = \eta I_n + \bar{d}_{+}^{(j)} \tilde{G}^{(\alpha)} \tilde{G}^{\alpha T} + \bar{d}_{-}^{(j)} \tilde{G}^{\alpha T} \tilde{G}^{(\alpha)}. \tag{10}$$

Our preconditioner satisfies in the following theorem.

Theorem 1. *If $\lambda \in \sigma(P^{-1}S^{(j)})$, then $|1-\lambda| < k < 1$, where $k = \frac{1}{\hat{\eta}+1} < 1$ with $\hat{\eta} = \frac{\eta}{2\kappa\|\tilde{G}\|_2^2}$ and $\kappa = \max\{\frac{d_{+,max}+d_{+,min}}{2}, \frac{d_{-,max}+d_{-,min}}{2}\}$*

Proof. Let $(\lambda, \mathbf{x})$ be an eigenpair of matrix $P^{-1}S^{(j)}$ with $\|\mathbf{x}\|_2 = 1$, i.e., $P^{-1}S^{(j)}\mathbf{x} = \lambda\mathbf{x}$ or $S^{(j)}\mathbf{x} = \lambda P\mathbf{x}$. Since P and $S^{(j)}$ are symmetric positive definite matrices we have,

$$\lambda = \frac{\mathbf{x}^H S^{(j)} \mathbf{x}}{\mathbf{x}^H P \mathbf{x}}, \tag{11}$$

hence

$$\begin{aligned} |1-\lambda| &= \left| \frac{\mathbf{x}^H S^{(j)}\mathbf{x} - \mathbf{x}^H P\mathbf{x}}{\mathbf{x}^H P\mathbf{x}} \right| \\ &= \left| \frac{\mathbf{x}^H \tilde{G}^{(\alpha)}(D_+ - \bar{d}_+ I_n)\tilde{G}^{\alpha T}\mathbf{x} + \mathbf{x}^H \tilde{G}^{(\alpha)}(D_+ - \bar{d}_+ I_n)\tilde{G}^{\alpha T}\mathbf{x}}{\eta + \bar{d}_+ \mathbf{x}^H \tilde{G}^{(\alpha)}\tilde{G}^{(\alpha)T}\mathbf{x} + \bar{d}_- \mathbf{x}^H \tilde{G}^{(\alpha)T}\tilde{G}^{(\alpha)}\mathbf{x}} \right|. \end{aligned} \tag{12}$$

If we define $\mathbf{y} = \tilde{G}^{(\alpha)T}\mathbf{x}$ and $\mathbf{z} = \tilde{G}^{(\alpha)}\mathbf{x}$, then by (12) and Lemma 1 we have

$$\begin{aligned} |1-\lambda| &= \left| \frac{\mathbf{y}^H(D_+ - \bar{d}_+ I_n)\mathbf{y} + \mathbf{z}^H(D_+ - \bar{d}_+ I_n)\mathbf{z}}{\eta + \bar{d}_+ \mathbf{y}^H\mathbf{y} + \bar{d}_- \mathbf{z}^H\mathbf{z}} \right| \\ &\leq \left| \frac{\frac{d_{+,max}-d_{+,min}}{2}\|\mathbf{y}\|_2^2 + \frac{d_{-,max}-d_{-,min}}{2}\|\mathbf{z}\|_2^2}{\eta + \frac{d_{+,max}+d_{+,min}}{2}\|\mathbf{y}\|_2^2 + \frac{d_{-,max}+d_{-,min}}{2}\|\mathbf{z}\|_2^2} \right| \\ &\leq \frac{\frac{d_{+,max}+d_{+,min}}{2}\|\mathbf{y}\|_2^2 + \frac{d_{-,max}+d_{-,min}}{2}\|\mathbf{z}\|_2^2}{\eta + \frac{d_{+,max}+d_{+,min}}{2}\|\mathbf{y}\|_2^2 + \frac{d_{-,max}+d_{-,min}}{2}\|\mathbf{z}\|_2^2}. \end{aligned} \tag{13}$$

We see that

$$\begin{aligned} \frac{d_{+,max}+d_{+,min}}{2}\|\mathbf{y}\|_2^2 + \frac{d_{-,max}+d_{-,min}}{2}\|\mathbf{z}\|_2^2 &\leq \kappa(\|\mathbf{y}\|_2^2 + \|\mathbf{z}\|_2^2) \\ &= \kappa(\|\tilde{G}^{(\alpha)T}\mathbf{x}\|_2^2 + \|\tilde{G}^{(\alpha)}\mathbf{x}\|_2^2) \\ &= 2\kappa\|\tilde{G}^{(\alpha)}\|_2^2. \end{aligned} \tag{14}$$

Now, by using (14) in (13) the proof is complete □

Theorem 1 indicates that eigenvalues of $P^{-1}S^{(j)}$ are real and belong to $(1-k, 1+k)$ with $k < 1$.

3 Computational remarks

To improve the efficiency of our algorithm, we need use some techniques for computing P^{-1} in (10). If we rewrite P in (10) as follows

$$\begin{aligned} P &= \eta I_n + \bar{d}_+^{(j)} \tilde{G}^{(\alpha)} \tilde{G}^{(\alpha)T} + \bar{d}_-^{(j)} \tilde{G}^{(\alpha)T} \tilde{G}^{(\alpha)} \\ &= \eta I_n + \begin{pmatrix} \bar{d}_+^{(j)} I_n & \bar{d}_-^{(j)} I_n \end{pmatrix} \begin{pmatrix} \tilde{G}^{(\alpha)} & 0 \\ 0 & \tilde{G}^{(\alpha)T} \end{pmatrix} \begin{pmatrix} \tilde{G}^{(\alpha)T} & 0 \\ 0 & \tilde{G}^{(\alpha)} \end{pmatrix} \begin{pmatrix} I_n \\ I_n \end{pmatrix}. \end{aligned} \tag{15}$$

We know that

$$\eta I_n = \frac{\eta}{\bar{d}_+^{(j)} + \bar{d}_-^{(j)}} \begin{pmatrix} \bar{d}_+^{(j)} I_n & \bar{d}_-^{(j)} I_n \end{pmatrix} \begin{pmatrix} I_n \\ I_n \end{pmatrix}. \tag{16}$$

Hence, by using (15) and (16), we have

$$P = \begin{pmatrix} \bar{d}_+^{(j)} I_n & \bar{d}_-^{(j)} I_n \end{pmatrix} \left(\frac{\eta}{\bar{d}_+^{(j)} + \bar{d}_+^{(j)}} I_{2n} + M M^T \right) \begin{pmatrix} I_n \\ I_n \end{pmatrix}, \tag{17}$$

where

$$M = \begin{pmatrix} \tilde{G}^{(\alpha} & 0 \\ 0 & \tilde{G}^{\alpha T} \end{pmatrix}.$$

The proof of the following lemma is straightforward by applying Given's rotation in QR decomposition [6]

Lemma 2. *For the matrix $2n \times n$, $\begin{pmatrix} c_1 I_n & c_2 I_n \end{pmatrix}^T$ where c_1 and c_2 are arbitrary constants, we have*

$$\begin{pmatrix} c_1 I_n \\ c_2 I_n \end{pmatrix} = \sqrt{c_1^2 + c_2^2}\; C \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}, \tag{18}$$

where C is $2n$-by-$2n$ skew-circulant matrix defined in (2) with $\theta = -1$, and it's first column is $\frac{c_1}{\sqrt{c_1^2+c_2^2}} e_1 - \frac{c_1}{\sqrt{c_1^2+c_2^2}} e_{n+1}$.

Let C be the circulant matrix created by the first column of M (circulant approximation of M). In (18), If we using circulant approximation C instead of M then we can compute approximately P^{-1} by FFT transform in $O(n \log(n))$ operations. This approximation satisfies in the following theorem.

Theorem 2. *If C is the circulant matrix created by the first column of M, then we have*

$$\|C - M\|_\infty \le \frac{2}{\Gamma(2-\alpha)}. \tag{19}$$

Proof. We can compute the matrix $M - C$ as follows

$$M - C = \begin{pmatrix} 0 & H \\ H & H^T - H \end{pmatrix}, \tag{20}$$

where matrix H is n-by-n Toeplitz matrix and define as follows

$$H = \begin{pmatrix} 0 & g_{n-1} & g_{n-2} & \cdots & g_1 \\ 0 & 0 & g_{n-1} & \cdots & g_2 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & g_{n-1} \end{pmatrix}. \tag{21}$$

Hence, by (20) and (4) we have

$$\|M - C\|_\infty = 2\sum_{k=1}^{n-1} |g_k| \leq \frac{2}{\Gamma(2-\alpha)}, \tag{22}$$

which completes the proof. $\square$

By using Theorem 2 we substitute M in (15) by the circulant matrix C, and get the following approximation of P.

$$P \approx \bar{P} = \begin{pmatrix} \bar{d}_+^{(j)} I_n & \bar{d}_-^{(j)} I_n \end{pmatrix} \left(\frac{\eta}{\bar{d}_+^{(j)} + \bar{d}_+^{(j)}} I_{2n} + CC^T \right) \begin{pmatrix} I_n \\ I_n \end{pmatrix}, \tag{23}$$

The matrix $\bar{P}$ is a multiplication of circulant and skew circulant matrices (see Lemma 2), so $\bar{P}^{-1}$ can be computed in $O(2n\log(2n))$ operations by using FFT. Numerical experiments show the efficiency of $\bar{P}$.

4 Numerical experiments

In this section we use left preconditioned GMRES method (PGMRES, see [12]), with proposed preconditioner, to solve the spatial fractional diffusion equations (1). The stopping criterion in the numerical experiments is

$$\frac{\|r_k\|_2}{\|b\|_2} < 10^{-7},$$

where r_k is the residual vector of the linear system after k-step iterations and b is the right-hand side. For all experiments the initial guess is chosen as zero vector. All the numerical experiments are run in MATLAB on a desktop with the configuration: Intel(R) Core(TM) CPU Q9450 2.66 GHz and 8.00 GB RAM. In the following tables, "I" represents the GMRES method

Table 1: Numerical results for Example 1.

α	$n=2m$	I		P_C		P	
		Iter	CPU	Iter	CPU	Iter	CPU
0.7	2^{10}	6	0.4478	6	0.6532	5	0.3132
	2^{11}	7	2.0125	7	2.6123	5	0.9285
	2^{12}	9	9.5012	7	7.2683	5	3.8766
	2^{13}	10	42.5571	8	39.4621	6	15.9862
	2^{14}	12	2.062e2	11	2.0321e2	6	79.8243
0.9	2^{10}	35	3.0252	30	2.8932	16	1.3846
	2^{11}	52	13.9516	43	10.3192	22	4.7435
	2^{12}	79	85.7862	71	63.2415	36	29.6237
	2^{13}	127	5.5719e2	108	3.3214e2	50	1.4311e2
	2^{14}	194	3.2855e3	180	3.0129e3	74	8.2030e2

without preconditioning technique. We use P_C to denote the circulant preconditioner [4], and P the proposed preconditioner defined in (23). As for comparisons, the "Iter"denotes integer part of the average number of iteration; "CPU"denotes the total CPU time in seconds for solving the whole FDE problem; n denotes the number of spatial grid points and m denotes the number of time steps.

Example 1. In this example, we consider the FDE (1) with the source term $f(x,t)=0$. The spatial domain is $[x_L,x_R]=[0,2]$ and the time interval is $[0,T]=[0,1]$. The initial condition $u(x,0)$ is the following Guassian pulse:

$$u(x,0)=\exp(-\frac{(x-x_c)^2}{2\sigma^2}),\quad x_c=1.2,\quad \sigma=0.08, \tag{24}$$

and the diffusion coefficients $d_+(x,t)=d_-(x,t)=0.5$.

The numerical results of Example 1 are shown in Table 1. Here, the initial condition is used as the initial guess at the first step.

Example 2. We solve the FDE (1) with the spatial domain $[x_L,x_R]=[0,1]$ and time interval $[0,T]=[0,1]$. The diffusion coefficients are

$$d_+(x,t)=(1-x)^\alpha,\quad \text{and}\quad d_-(x,t)=x^\alpha,$$

respectively. The source term is

$$\begin{aligned} f(x,t) = & -e^{-t}x^2(1-x)^2+e^{-t}(\rho_1^\alpha x^{1-\alpha}+\rho_2^\alpha x^{2-\alpha}+\rho_3^\alpha x^{3-\alpha}+\rho_4^\alpha x^{4-\alpha}) \\ & +e^{-t}(\rho_1^\alpha(1-x)^{1-\alpha}+\rho_2^\alpha(1-x)^{2-\alpha}+\rho_3^\alpha(1-x)^{3-\alpha}) \\ & +e^{-t}\rho_4^\alpha(1-x)^{4-\alpha}, \end{aligned}$$

Table 2: Numerical results for Example 2.

α	$n = 2m$	I		P_C		P	
		Iter	CPU	Iter	CPU	Iter	CPU
0.7	2^{10}	7	0.55	6	0.71	5	0.032
	2^{11}	8	2.42	7	2.62	5	0.101
	2^{12}	9	10.31	7	9.27	5	0.380
	2^{13}	11	48.64	9	41.32	6	1.547
	2^{14}	13	121.12	11	97.42	6	7.660
0.9	2^{10}	35	3.092	30	2.514	16	0.119
	2^{11}	50	14.232	41	8.532	24	1.231
	2^{12}	80	9.130e1	75	7.2105e1	34	4.6530
	2^{13}	100	2.455e2	83	1.321e2	42	1.22e1
	2^{14}	> 300	-	> 300	-	78	9.55e1

where,

$$\begin{aligned}
\rho_1^\alpha &= \frac{4}{\Gamma(2-\alpha)}\left(-\frac{24}{\Gamma(5-\alpha)} + \frac{9}{\Gamma(4-\alpha)} - \frac{1}{\Gamma(3-\alpha)}\right), \\
\rho_2^\alpha &= \frac{4}{\Gamma(3-\alpha)}\left(\frac{72}{\Gamma(5-\alpha)} - \frac{18}{\Gamma(4-\alpha)} + \frac{1}{\Gamma(3-\alpha)}\right), \\
\rho_3^\alpha &= \frac{72}{\Gamma(4-\alpha)}\left(-\frac{8}{\Gamma(5-\alpha)} + \frac{1}{\Gamma(4-\alpha)}\right), \\
\rho_4^\alpha &= \left(\frac{24}{\Gamma(5-\alpha)}\right)^2.
\end{aligned}$$

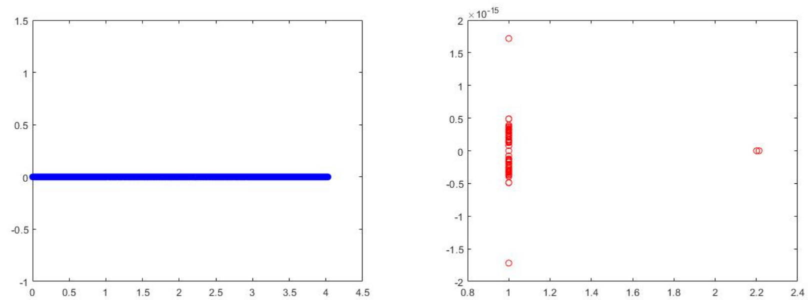
In sequel we compare eigenvalues of A and $P^{-1}A$ for both examples, the results are shown in Figures 1a and 2a. As we see the eigenvalues of preconditioned matrix $P^{-1}A^{(1)}$ are clustered around 1.

References

[1] O.P. Agrawal, *Solution for a fractional diffusion-wave equation defined in a bounded domain*, Nonlin. Dyn. **29** (2002) 145-55.

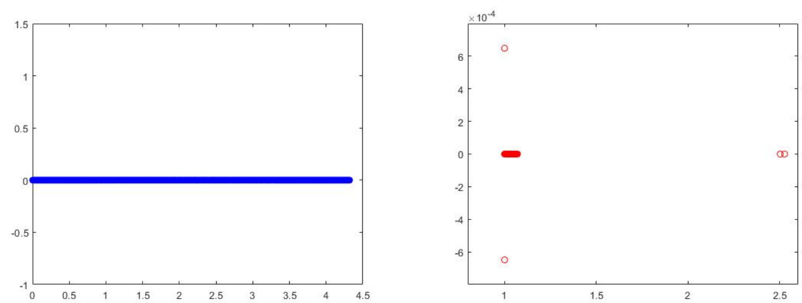
[2] Z.-Z. Bai, G.H. Golub and M.K. Ng, *Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems*, SIAM J. Matrix Anal. Appl. **24** (2003) 603-626.

[3] J.P. Bouchaud and A. Georges, *Anomalous diffusion in disordered media: statistical mechanisms, models and physical applications*, Phys. Rep. **195** (1990) 127-293.



(a) Eigenvalues of matrix $A^{(1)}$. (b) Eigenvalues of matrix $P^{-1}A^{(1)}$.

Figure 1: Eigenvalues of $A^{(1)}$ and $P^{(-1)}A^{(1)}$ for Example 1, with $\alpha = 0.9$ and $n = 2^{10}$.



(a) Eigenvalues of matrix $A^{(1)}$. (b) Eigenvalues of matrix $P^{-1}A^{(1)}$

Figure 2: Eigenvalues of $A^{(1)}$ and $P^{(-1)}A^{(1)}$ for Example 2, with $\alpha = 0.9$ and $n = 2^{10}$.

[4] M. Das, S. Amer-Yahia, G. Das and C. Yu, *MRI: Meaningful interpretations of collaborative ratings*, Proceedings of the VLDB Endowment, Vol. 4, No. 11.

[5] Z.W. Fang, M.K. Ng and H.W. Sun, *Circulant preconditioners for a kind of spatial fractional diffusion equations*, Numer. Algor. **82** (2019) 729-747.

[6] G.H. Golub and C.F. Van Loan, *Matrix computations*, Third Edition, The Johns Hopkins Press, 2013.

[7] R. Hilfer, *Applications of fractional calculus in physics*, World Scientific Publishing Company, Singapore, 2000, 87-130.

[8] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier Science Limited, 2006.

[9] R. Metzler and J. Klafter, *The random walk's guide to anomalous diffusion: a fractional dynamics approach* Phys. Rep. **339** (2000) 1-77.

[10] M.K. Ng, *Iterative methods for Toeplitz systems*, Oxford University Press, New York, NY, 2004.

[11] I. Podlubny, *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Elsevier, 1998.

[12] Y. Saad, *Preconditioned Krylov subspace methods for the numerical solution of Markov chains*, In Computations with Markov chains, pp. 49-64. Springer, Boston, MA, 1995.