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A NEW GAUSSIAN FIBONACCI MATRICES AND ITS APPLICATIONS

B. PRASAD *

Abstract. In this paper, we introduced a new Gaussian Fibonacci matrix, G^n whose elements are Gaussian Fibonacci numbers and we developed a new coding and decoding method followed from this Gaussian Fibonacci matrix, G^n. We established the relations between the code matrix elements, error detection and correction for this coding theory. Correction ability of this method is 93.33%.

1. Introduction

The Fibonacci numbers are defined by the recurrence relation:

$$F_n = F_{n-1} + F_{n-2} \text{ for } n > 2 \tag{1.1}$$

with initial seeds

$$F_1 = F_2 = 1. \tag{1.2}$$

The Fibonacci numbers, F_n and golden mean,

$$\tau = \lim_{n \longrightarrow \infty} \frac{F_n}{F_{n-1}} = \frac{1+\sqrt{5}}{2} \tag{1.3}$$

have appeared in arts, sciences, high energy physics, information and coding theory [3-7].
The Gaussian Fibonacci numbers [1] are defined by the recurrence relation:

$$G_n = F_n + iF_{n-1} \text{ for } n \geqslant 2 \tag{1.4}$$

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*Corresponding author .

with initial seeds

$$G_0 = i, G_1 = 1 \tag{1.5}$$

where i is the imaginary unit which satisfies $i^2 = -1$.
We defined a new Gaussian Fibonacci matrix, G

$$G = \begin{pmatrix} G_2 & G_1 \\ G_1 & G_0 \end{pmatrix} = \begin{pmatrix} 1+i & 1 \\ 1 & i \end{pmatrix} \tag{1.6}$$

and

$$G^2 = (1+2i)\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = (1+2i)\begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix},$$

$$G^3 = (1+2i)\begin{pmatrix} G_3 & G_2 \\ G_2 & G_1 \end{pmatrix} = (1+2i)[\begin{pmatrix} F_3 & F_2 \\ F_2 & F_1 \end{pmatrix} + i\begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix}].$$

In general,

$$G^{2k} = (1+2i)^k \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix}$$

where $k = 1, 2, 3, \cdots$. and $DetG^{2k} = (1+2i)^k(-1)^k$.

$$G^{2k+1} = (1+2i)^k[\begin{pmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{pmatrix} + i\begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix}]$$

where $k = 0, 1, 2, 3, \cdots$. and $DetG = -(2-i)$, $DetG^{2k+1} = (1+2i)^k(2-i)$ where $k = 1, 2, 3, \cdots$.

2. Gaussian Fibonacci coding and decoding method

In this paper, we introduced a new coding theory, Gaussian Fibonacci coding and decoding, which is the extension of Fibonacci coding and decoding method and it is applicable for a complex plane also. In this method, we represent the message in the form of nonsingular square matrix, M of order 2 and we represent the Gaussian matrix, G^n of order 2 as coding matrix and its inverse matrix $(G^n)^{-1}$ as a decoding matrix. We represent a transformation $M \times G^n = E$ as Gaussian Fibonacci coding and a transformation $E \times (G^n)^{-1} = M$ as Gaussian Fibonacci decoding. We represent the matrix, E as code matrix.

Note

The code matrix, E is defined by the following formula $E = M \times G^n$.
According to the matrix theory [2] we have

$$Det\ E = Det(M \times G^n) = Det\ M \times Det\ G^n. \tag{2.1}$$

3. Main results

Case I

When $n = 2k$.

We can write the code matrix, E and the initial message, M as the following

$$E = M\times(G^{2k}) = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} (1+2i)^k \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix} = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix}$$

and

$$M = E \times (G^{2k})^{-1} = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix} \frac{1}{(1+2i)^k(-1)^k} \begin{pmatrix} F_{k-1} & -F_k \\ -F_k & F_{k+1} \end{pmatrix}$$
$$= \frac{1}{(1+2i)^k(-1)^k} \begin{pmatrix} e_1F_{k-1} - e_2F_k & -e_1F_k + e_2F_{k+1} \\ e_3F_{k-1} - e_4F_k & -e_3F_k + e_4F_{k+1} \end{pmatrix}.$$

Since $m_1,\ m_2,\ m_3,\ m_4$ are positive integers, we have

$$m_1 = \frac{e_1F_{k-1} - e_2F_k}{(1+2i)^k(-1)^k} > 0, \tag{3.1}$$

$$m_2 = \frac{-e_1F_k + e_2F_{k+1}}{(1+2i)^k(-1)^k} > 0, \tag{3.2}$$

$$m_3 = \frac{e_3F_{k-1} - e_4F_k}{(1+2i)^k(-1)^k} > 0, \tag{3.3}$$

$$m_4 = \frac{-e_3F_k + e_4F_{k+1}}{(1+2i)^k(-1)^k} > 0. \tag{3.4}$$

From (3.1) and (3.2) we get

$$\frac{F_k}{F_{k-1}} < \frac{e_1}{e_2} < \frac{F_{k+1}}{F_k}. \tag{3.5}$$

From (3.3) and (3.4) we get

$$\frac{F_k}{F_{k-1}} < \frac{e_3}{e_4} < \frac{F_{k+1}}{F_k}. \tag{3.6}$$

Therefore, for large value of k we get

$$\frac{e_1}{e_2} \approx \mu,\ \frac{e_3}{e_4} \approx \mu \text{ where } \mu = \frac{1+\sqrt{5}}{2}. \tag{3.7}$$

Case II

When $n = 2k + 1$.

We will get the same result as above in (3.7).

The main aim of the coding theory are the detection and correction of errors arising in the code message, E under influence of noise in the communication channel. The most important idea is using the property of determinant of the matrix as the check criterion of the transmitted message, E. Let the initial message, M is given by

$$M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \tag{3.8}$$

where all elements m_1, m_2, m_3, m_4 of the matrix, M are positive integers.
Now determinant of M is

$$Det\ M = m_1 m_4 - m_2 m_3 \tag{3.9}$$

and the code message, E

$$E = (M \times (G^n)). \tag{3.10}$$

So,

$$Det\ E = Det\ (M \times (G^n)) = Det\ M \times Det\ (G^n). \tag{3.11}$$

This shows that the determinant of the initial message, M is connected with the determinant of the code message, E by the relation (3.11). The value of the determinant of the message, E depends on the number n is even or an odd. The essence of the method consists that the sender calculates the determinant of the initial message, M represented in the matrix form (3.8) and sends it to the channel after the code message, E (3.10). The receiver calculates the determinant of the code message, E (3.11) and compares the determinant of the initial message of M (3.9) received from the channel. If this comparison corresponds to (3.11) it means that the code message, E (3.10) is correct and the receiver can decode the code message, E (3.10) otherwise the code message, E (3.10) is not correct. Error detection is the first step in communication of messages.
The possibility of restoration of the code message, E can be done by using the property of the Gaussian Fibonacci G^n matrix. For selecting $n = 4$, Gaussian Fibonacci G^n matrix will be

$$G^4 = (1+2i)^2 \begin{pmatrix} F_3 & F_2 \\ F_2 & F_1 \end{pmatrix} = (1+2i)^2 \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}. \tag{3.12}$$

Then the Gaussian Fibonacci coding of the message (3.8) consists of the multiplication of the initial matrix (3.12) that is

$$M \times G^4 = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} (1+2i)^2 \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} =$$

$$(1+2i)^2 \begin{pmatrix} 2m_1 + m_2 & m_1 + m_2 \\ 2m_3 + m_4 & m_3 + m_4 \end{pmatrix} = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix} = E \tag{3.13}$$

where $e_1 = (1+2i)^2(2m_1+m_2)$, $e_2 = (1+2i)^2(m_1+m_2)$, $e_3 = (1+2i)^2(2m_3+m_4)$, $e_4 = (1+2i)^2(m_3+m_4)$.
After constructing the code matrix, E we calculate the determinant of the initial matrix, M (3.8). The determinant is sent to the communication channel after the code message, $E = e_1, e_2, e_3, e_4$. Assume that the communication channel has the special means for the error detection in each of elements e_1, e_2, e_3, e_4 of the code message, E. Assume that the first element e_1 of E is received with the error. Then, we can represent the code message in the matrix form

$$E' = \begin{pmatrix} x & e_2 \\ e_3 & e_4 \end{pmatrix} \tag{3.14}$$

where x is the destroyed element of the code message, E but the rest matrix entries must be correct and equal to the following:

$$e_2 = (1+2i)^2(m_1+m_2); e_3 = (1+2i)^2(2m_3+m_4);$$

$$e_4 = (1+2i)^2(m_3+m_4). \tag{3.15}$$

Then, according to the properties of the Gaussian Fibonacci coding method, we can write the following equation for calculation of x

$$\begin{aligned} xe_4 - e_2e_3 = \\ x(1+2i)^2(m_3+m_4) - \\ (1+2i)^2(m_1+m_2)(1+2i)^2(2m_3+m_4) = \\ (1+2i)^2(m_1m_4 - m_2m_3). \end{aligned} \tag{3.16}$$

From (3.16), we get

$$x = (1+2i)^2(2m_1+m_2). \tag{3.17}$$

Comparing the calculated value (3.17) with the entry e_1 of the code matrix, E given with (3.13) we conclude that $x = e_1$. Thus, we have restored the code message, E using the property of determinant of the Gaussian Fibonacci G^n matrix. But in the real situation usually we do not know what element of the code message is destroyed. In this case, we suppose different hypotheses about the possible destroyed elements and then we test these hypotheses. However, we have one more condition for the elements of the code matrix, E that all its elements are integers. Our first hypothesis is that we have the case of single error in the code matrix, E received from the communication channel. It is clear that there are four variants of the single errors in the code matrix, E:

$$(a)\begin{pmatrix} x & e_2 \\ e_3 & e_4 \end{pmatrix}(b)\begin{pmatrix} e_1 & y \\ e_3 & e_4 \end{pmatrix}(c)\begin{pmatrix} e_1 & e_2 \\ z & e_4 \end{pmatrix}(d)\begin{pmatrix} e_1 & e_2 \\ e_3 & t \end{pmatrix} \quad (3.18)$$

where x, y, z, t are destroyed elements. In this case we can check different hypotheses (3.18). For checking the hypothesis (a), (b), (c), (d) we can write the following algebraic equations based on the checking relation (3.11):

$$xe_4 - e_2e_3 = Det\ G^n Det\ M(\text{a possible single error is in the element } e_1), \quad (3.19)$$

$$e_1e_4 - ye_3 = Det\ G^n Det\ M(\text{a possible single error is in the element } e_2), \quad (3.20)$$

$$e_1e_4 - e_2z = Det\ G^n Det\ M(\text{a possible single error is in the element } e_3), \quad (3.21)$$

$$e_1t - e_2e_3 = Det\ G^n Det\ M(\text{a possible single error is in the element } e_4). \quad (3.22)$$

It follows from (3.19)-(3.22) four variants for calculation of the possible single errors.

$$x = \frac{Det\ G^n Det\ M + e_2e_3}{e_4}, \quad (3.23)$$

$$y = \frac{-Det\ G^n Det\ M + e_1e_4}{e_3}, \quad (3.24)$$

$$z = \frac{-Det\ G^n Det\ M + e_1e_4}{e_2}, \quad (3.25)$$

$$t = \frac{Det\ G^n Det\ M + e_2e_3}{e_1}. \quad (3.26)$$

The formula (3.23)-(3.26) give four possible variants of single error but we have to choice the correct variant only among the cases of the integer solutions x, y, z, t; besides, we have to choice such solutions, which satisfies to the additional checking relations (3.11). If calculations by formulas (3.23)-(3.26) do not give an integer result we have to conclude that our hypothesis about single error is incorrect or we have error in the checking element Det M. For the latter case we can use the approximate equalities (3.7) for checking a correctness of the code matrix, E. By analogy we can check all hypotheses of double error in the code matrix. As example let us consider the following case of double errors in the code matrix, E

$$\begin{pmatrix} x & y \\ e_3 & e_4 \end{pmatrix} \tag{3.27}$$

where x, y are the destroyed elements of the code message. Using the first checking relation (3.11) we can write the following algebraic equation for the matrix (3.27):

$$xe_4 - ye_3 = Det\ G^n Det\ M. \tag{3.28}$$

However, according to the second checking relation (3.7) there is the following relation between x and y :

$$x \approx \mu y. \tag{3.29}$$

It is important to emphasize that (3.28) is Diophantine one. As the Diophantine equation (3.28) has many solutions we have to choice such solutions x, y which satisfy to the checking relation (3.29). By analogy one may prove that using checking relations (3.7), (3.11) by means of solution of the Diophantine equation similar to (3.28) we can correct all possible double errors in the code matrix. However, we can show by using such approach there is a possibility to correct all possible triple errors in the code matrix E, for example $\begin{pmatrix} x & y \\ z & e_4 \end{pmatrix}$ etc. where x, y, z are destroyed elements. Thus, our method of error correction is based on the verification of different hypotheses about errors in the code matrix by using the checking relations (3.7), (3.11) and by using the fact that the elements of the code matrix are integers. If all our solutions do not bring to integer solutions it means that the checking element Det M is erroneous or we have the case of fourfold error in the code matrix, E and we have to reject the code matrix, E as defective and not correctable. Our method allows to correct 14 cases among $({}^4C_1 +^4 C_2 +^4 C_3 +^4 C_4) = 2^4 - 1 = 15$ cases. It means that correction ability of the method is $\frac{14}{15} = 0.9333 = 93.33\%$.

4. Conclusion

The Gaussian Fibonacci coding method is based on matrix approach which possess many peculiarities and advantages in comparison to classical (algebraic) coding method. The use of matrix theory for designing new error-correction codes is the first peculiarity of the Gaussian Fibonacci coding method. The large information units, in particular matrix elements, are objects of detection and correction of errors in the Gaussian Fibonacci coding method. There is no theoretical restrictions for the value of the numbers that can be matrix elements whereas in algebraic coding theory there are very small information elements, bits and their combinations are the objects of detection and correction. Gaussian Fibonacci coding method has very high correction ability in comparison to classical (algebraic) coding method. The Gaussian Fibonacci coding method is the main application of the G^n matrix. The Gaussian Fibonacci coding method reduces to matrix multiplication, a well-known algebraic operation, which is realized very well in modern computers. The main practical peculiarity of this method is that large information units, in particular, matrix elements, are objects of detection and correction of errors. The elements of the initial matrix, M and therefore the elements of the code matrix, E can be the numbers of unlimited value. This means that theoretically the Gaussian Fibonacci coding method allows to correct the numbers of unlimited value. The correction ability of this method is 93.33% that exceeds essentially all well-known correcting codes.

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Bandhu Prasad
Department of Mathematics, Kandi Raj College, Kandi, India.
Email: bandhu_iit@rediffmail.com