Journal of Algebra and Related Topics
Vol. 7, No 1, (2019), pp 45-64

2-ABSORBING δ-PRIMARY ELEMENTS IN MULTIPLICATIVE LATTICES

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ABSTRACT. In this paper, we define a 2-absorbing δ-primary element and a weakly 2-absorbing δ-primary element in a compactly generated multiplicative lattice L. We obtain some properties of these elements. We give a characterization for 2-absorbing δ-primary elements. Also we define a δ-triple-zero and a free δ-triple-zero and prove some results on it.

1. Introduction

The concept of a 2-absorbing and weakly 2-absorbing elements which are generalizations of prime and weakly prime elements in multiplicative lattices was introduced by Jayaram, et. al. [4].

Manjarekar and Bingi [5] introduced and investigated the notions of expansions of element and δ-primary element in a multiplicative lattice. Fahid and Zhao [3] defined a 2-absorbing δ-primary ideal in a commutative ring which unifies both 2-absorbing and 2-absorbing primary ideals in one frame. This motivates us to put 2-absorbing and 2-absorbing primary elements together using expansions of elements. Also we extend the concept of 2-absorbing δ-primary ideal in a commutative ring to multiplicative lattice.

The aim of this paper is to introduce the concept of a 2-absorbing δ-primary element in a multiplicative lattice and generalize the results of Fahid and Zhao [3] to multiplicative lattices. In section 2, we recall some basic concepts in multiplicative lattices. In section 3, we introduce the notion of a 2-absorbing δ-primary element. Such

* Corresponding author MSC(2010): 06B10
Keywords: Expansion function, 2-absorbing δ-primary element, weakly 2-absorbing δ-primary element, δ-triple-zero, free δ-triple-zero. $*$Jaya Y. Nehete.

2-absorbing δ-primary elements unifies the concepts of a 2-absorbing ideal and 2-absorbing primary ideal under one frame. In section 4, we investigate some properties of 2-absorbing δ-primary elements with respect to homomorphisms. In section 5, we define weakly 2-absorbing δ-primary elements and obtain some properties of these elements. Also we define a δ-triple-zero and a free δ-triple-zero.

Throughout in this paper, L denotes a compactly generated multiplicative lattice with 1 compact in which every finite product of compact elements is compact.

2. Preliminaries

The following definitions are from Jayaram et. al. [4].

Definition 2.1. A multiplicative lattice L is a complete lattice with a commutative, associative and join distributive multiplication in which the largest element 1 acts as a multiplicative identity.

Definition 2.2. An element $a \in L$ is called compact if for $X \subseteq L$, $a \leq \bigvee X$ implies the existence of a finite number of elements $a_1, a_2, \ldots a_n \in X$ such that $a \leq a_1 \vee a_2 \vee \ldots a_n$.
The set of compact elements of L will be denoted by L_*.
A multiplicative lattice is said to compactly generated if every element of it is a join of compact elements.

Definition 2.3. An element $a \in L$ is said to be proper if $a < 1$.

Definition 2.4. A proper element $p \in L$ is called a prime element if $ab \leq p$ implies $a \leq p$ or $b \leq p$ where $a, b \in L$.

Definition 2.5. A prime element $p \in L$ is said to be minimal prime over $a \in L$, if $a \leq p$ and whenever there is a prime element $q \in L$ with $a < q \leq p$, then $q = p$.

Definition 2.6. The radical of $a \in L$ is defined as,
$\sqrt{a} = \vee\{x \in L_* | x^n \leq a \text{ for some, } n \in \mathbb{N}\}$
$= \wedge\{p \in L | p \text{ is a prime element}, a \leq p\}$.

Definition 2.7. A proper element $p \in L$ is called a primary element if $ab \leq p$ implies $a \leq p$ or $b \leq \sqrt{p}$ where $a, b \in L$.

For $a, b \in L$ we denote $(a : b) = \vee\{x \in L | bx \leq a\}$.

Definition 2.8. An element $a \in L$ is called semi primary if $\sqrt{a}$ is a prime element and is called semi prime if $\sqrt{a} = a$.

Definition 2.9. An element $a \in L$ is called p-primary if a is primary and $\sqrt{a} = p$ is a prime element of L.

Definition 2.10. A proper element $m \in L$ is said to be a maximal element if $m \nleq a$ for any other proper element $a \in L$.

Definition 2.11. An element $a \in L$ is said to be nilpotent if $a^n = 0$ for some $n \in \mathbb{N}$.

Definition 2.12. A proper element p of L is called a 2-absorbing element of L if whenever $a, b, c \in L$ and $abc \leq p$ implies that either $ab \leq p$ or $bc \leq p$ or $ac \leq p$.

Definition 2.13. A proper element p of L is called a 2-absorbing primary element of L if whenever $a, b, c \in L$ and $abc \leq p$ implies that either $ab \leq p$ or $bc \leq \sqrt{p}$ or $ac \leq \sqrt{p}$.

3. Properties of 2-absorbing δ-primary elements

The following definitions are from Manjarekar and Bingi [5].

Definition 3.1. An expansion of elements, or an expansion function, is a function $\delta : L \to L$, such that the following conditions are satisfied: (i) $a \leq \delta(a)$ for all $a \in L$ (ii) $a \leq b$ implies $\delta(a) \leq \delta(b)$ for all $a, b \in L$.

Example 3.2. (1) The identity function $\delta_0 : L \to L$,
where $\delta_0(a) = a$ for every $a \in L$, is an expansion of elements.
(2) For each element a, $\mathbf{M} : L \to L$,
where $\mathbf{M}(a) = \wedge\{m \in L | a \leq m,\ m \text{ is a maximal element }\}$, where a is a proper element of L and $\mathbf{M}(1) = 1$. Then $\mathbf{M}$ is an expansion of elements.
(3) For each element a define $\delta_1 : L \to L$ as $\delta_1(a) = \sqrt{a}$, the radical of a. Then $\delta_1(a)$ is an expansion of elements.

Definition 3.3. Given an expansion δ of elements, an element p of L is called δ-primary if $ab \leq p$ implies either $a \leq p$ or $b \leq \delta(p)$ for all $a, b \in L$.

Definition 3.4. A proper element p of L is called a 2-absorbing δ-primary element of L if whenever $a, b, c \in L$ and $abc \leq p$ implies $ab \leq p$ or $bc \leq \delta(p)$ or $ac \leq \delta(p)$.

Example 3.5. Consider the lattice L of ideals of the ring
$R = \langle Z_{60}, +_{60}, \times_{60} \rangle$. Clearly,
$L = \{(0), (1), (2), (3), (4), (5), (6), (10), (12), (15), (20), (30)\}$ is a compactly generated multiplicative lattice. Its lattice structure is shown in Figure 1.

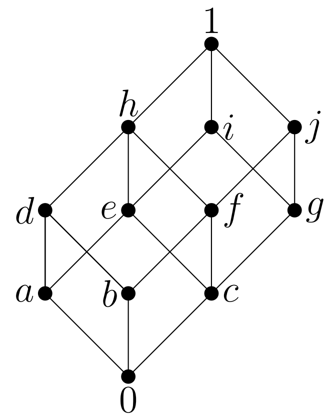

Figure 1

$L = \{0, a, b, c, d, e, f, g, h, i, j, 1\}$, where 0 denotes the (0) ideal, $a = (12)$ denotes the ideal generated by 12, $b = (20)$ denotes ideal generated by 20, $c = (30)$ denotes the ideal generated by 30, $d = (4)$ denotes the ideal generated by 4, $e = (6)$ denotes the ideal generated by 6, $f = (10)$ denotes the ideal generated by 10, $g = (15)$ denotes the ideal generated by 15, $h = (2)$ denotes the ideal generated by 2, $i = (3)$ denotes the ideal generated by 3, $j = (5)$ denotes the ideal generated by 5 and $1 = (1)$ denotes the ideal generated by 1.

(i) From Example 3.2 and multiplication table 1, here the elements $a, b, d, e, f, g, h, i, j$, where $\mathbf{M}(a) = e$, $\mathbf{M}(b) = f$, $\mathbf{M}(d) = h$, $\mathbf{M}(e) = e$, $\mathbf{M}(f) = f$, $\mathbf{M}(g) = g$, $\mathbf{M}(h) = h$, $\mathbf{M}(i) = i$, $\mathbf{M}(j) = j$, are 2-absorbing $\mathbf{M}$-primary elements. But the elements c, where $\mathbf{M}(c) = c$ is not a 2-absorbing $\mathbf{M}$-primary element. Since $dij = 0 \leq c$ but neither $di = a \leq c$ nor $ij = g \leq \mathbf{M}(c)$ nor $dj = b \leq \mathbf{M}(c)$.

(ii) The elements e, f, g, h, i, j, where $\delta_0(e) = e$, $\delta_0(f) = f$, $\delta_0(g) = g$, $\delta_0(h) = h$, $\delta_0(i) = i$, $\delta_0(j) = j$, $\delta_1(I) = I$, are 2-absorbing δ_0-primary. But the elements a, b, c, d, where $\delta_0(a) = a$, $\delta_0(b) = b$, $\delta_0(c) = c$, $\delta_0(d) = d$, are not 2-absorbing δ_0-primary. Since $fih = 0 \leq a$ but neither $fi = c \leq a$ nor $ih = e \leq \delta_0(a)$ nor $fh = b \leq \delta_0(a)$.
Since $egh = 0 \leq b$ but neither $eg = c \leq b$ nor $eh = a \leq \delta_0(b)$ nor $hg = c \leq \delta_0(b)$. Since $dij = 0 \leq c$ but neither $di = a \leq c$ nor $ij = g \leq \delta_0(c)$ nor $dj = b \leq \delta_0(c)$. Since $ghi = 0 \leq d$ but neither $hi = e \leq d$ nor $gi = g \leq \delta_0(d)$ nor $gh = c \leq \delta_0(d)$.

(iii) The elements $a, b, d, e, f, g, h, i, j$, where $\delta_1(a) = e$, $\delta_1(b) = f$, $\delta_1(d) = h$, $\delta_1(e) = e$, $\delta_1(f) = f$, $\delta_1(g) = g$, $\delta_1(h) = h$, $\delta_1(i) = i$, $\delta_1(j) = j$, are 2-absorbing δ_1-primary. But the elements c, where $\delta_1(c) = c$, is not 2-absorbing δ_1-primary. Since $dij = 0 \leq c$ but neither

Table 1. Multiplication Table

$\cdot$	0	a	b	c	d	e	f	g	h	i	j	1
0	0	0	0	0	0	0	0	0	0	0	0	0
a	0	a	0	0	a	a	0	0	a	a	0	a
b	0	0	b	0	b	0	b	0	b	0	b	b
c	0	0	0	0	0	0	0	c	0	c	c	c
d	0	a	b	0	d	a	b	0	d	a	b	d
e	0	a	0	0	a	a	0	c	a	e	c	e
f	0	0	b	0	b	0	b	c	b	c	f	f
g	0	0	0	c	0	c	c	g	c	g	g	g
h	0	a	b	0	d	a	b	c	d	e	f	h
i	0	a	0	c	a	e	c	g	e	i	g	i
j	0	0	b	c	b	c	f	g	f	g	j	j
1	0	a	b	c	d	e	f	g	h	i	j	1

$di = a \leq c$ nor $ij = g \leq \delta_1(c) = c$ nor $dj = b \leq \delta_1(c) = c$.

The following theorem gives a characterization of a 2-absorbing δ-primary element of L.

Theorem 3.6. *An element $q \in L$ is a 2-absorbing δ-primary element of L if and only if for any $a, b, c \in L_*$, $abc \leq q$ implies that either $ab \leq q$ or $ac \leq \delta(q)$ or $bc \leq \delta(q)$.*

Proof. Assume that the condition hold. Let $abc \leq q$ and $ac \nleq \delta(q)$ and $bc \nleq \delta(q)$ then there exists compact elements $x \leq a$, $y \leq b$ and $z \leq c$ such that $xyz \leq q$. Since $ac \nleq \delta(q)$ and $bc \nleq \delta(q)$, there exist compact elements $a_1 \leq a$, $b_1 \leq b$ and $c_1 \leq c$ and $c_2 \leq c$ such that $a_1c_1 \nleq \delta(q)$ and $b_1c_2 \nleq \delta(q)$. Put $c_3 = c_1 \vee c_2 \vee z$, $a_2 = a_1 \vee x$, $b_2 = b_1 \vee y$. We show that $ab \leq q$. Choose compact elements $a_\alpha \leq a$, $b_\alpha \leq b$. Then $(a_2 \vee a_\alpha)c_3(b_2 \vee b_\alpha) \leq q$, $(a_2 \vee a_\alpha)c_3 \nleq \delta(q)$ and $(b_2 \vee b_\alpha)c_3 \nleq \delta(q)$ and hence by hypothesis $(a_2 \vee a_\alpha)(b_2 \vee b_\alpha) \leq q$. So $a_\alpha b_\alpha \leq q$. Consequently, $ab \leq q$. Therefore q is a 2-absorbing δ-primary element of L.

The converse part follows from the definition.

□

Lemma 3.7. *Every prime element of L is 2-absorbing δ-primary.*

Proof. Let p be a prime element of L. Suppose that $abc \leq p$ for some $a, b, c \in L$. As p is a prime element of L, we have either (1) $ab \leq p$ or $c \leq p$, or (2) $bc \leq p$ or $a \leq p$, or (3) $ac \leq p$ or $b \leq p$.

Suppose that (1) $ab \leq p$ or $c \leq p$. If $ab \leq p$ then the proof is clear. If $c \leq p$ then $ac \leq p \leq \delta(p)$.
Thus we get either $ab \leq p$ or $bc \leq \delta(p)$ or $ac \leq \delta(p)$. Similarly, we can prove the result in the other two cases. Therefore, p is a 2-absorbing δ-primary element. □

Remark 3.8. The following example shows that the converse of Lemma 3.7 does not hold.

Example 3.9. Let L be a multiplicative lattice shown in Figure 1. Here the element d is δ_1-primary and $\mathbf{M}$-primary, $ce = 0 \leq d$ but neither $c \leq d$ nor $e \leq d$. Thus d is not a prime element.

Remark 3.10. (i) An element p is 2-absorbing δ_0-primary if and only if it is 2-absorbing.
(ii) An element p is 2-absorbing δ_1-primary if and only if it is 2-absorbing primary.

Now we establish a relation between a 2-absorbing element and a 2-absorbing δ-primary element.

Lemma 3.11. *If δ and γ are two element expansions, and $\delta(a) \leq \gamma(a)$ for each element a, then every 2-absorbing δ-primary element is also 2-absorbing γ-primary. Thus, in particular, a 2-absorbing element is a 2-absorbing δ-primary element for every element expansion δ.*

Proof. Let $p \in L$ be a 2-absorbing δ-primary element. Then $abc \leq p$ implies that either $ab \leq p$ or $bc \leq \delta(p)$ or $ac \leq \delta(p)$. As p is 2-absorbing δ-primary. $\delta(p) \leq \gamma(p)$. So p is γ-primary.
Next, suppose that p is a 2-absorbing element. By Remark 3.10(i), p is a 2-absorbing δ_0-primary element. For any element expansion δ, $p \leq \delta(p)$, so $\delta_0(p) = p \leq \delta(p)$.
Thus we get $\delta_0(p) \leq \delta(p)$ and p is δ_0-primary. Therefore p is 2-absorbing δ-primary for every δ. □

The following theorem proves that the radical of a 2-absorbing δ primary element is again a 2-absorbing δ-primary element.

Theorem 3.12. *If p is a 2-absorbing δ-primary element of L such that $\sqrt{\delta(p)} = \delta(\sqrt{p})$, then $\sqrt{p}$ is a 2-absorbing δ primary element of L.*

Proof. Let $a, b, c \in L$ be such that $abc \leq \sqrt{p}$. Then there exists a positive integer n such that $(abc)^n \leq p$. As p is a 2-absorbing δ-primary element of L we get either $a^nc^n \leq p$ or $b^nc^n \leq \delta(p)$ or $a^nb^n \leq \delta(p)$, that is either $ac \leq \sqrt{p}$ or $bc \leq \sqrt{\delta(p)} = \delta(\sqrt{p})$ or $ab \leq \sqrt{\delta(p)} = \delta(\sqrt{p})$. Therefore, $\sqrt{p}$ is a 2-absorbing δ-primary element of L. □

Lemma 3.13. *Let δ be an element expansion such that $\delta(\delta(a)) = \delta(a)$, for every element a of L. Let $\delta(q)$ be a 2-absorbing δ-primary element of L. Then $(\delta(q) : x)$ is a 2-absorbing δ-primary element of L, for all $x \not\leq \delta(q)$.*

Proof. Let $x \not\leq \delta(q)$. Let $a, b, c \in L$ be such that $abc \leq (\delta(q) : x)$. Thus $a(bc)x \leq \delta(q)$ and so either $a(bc) \leq \delta(q)$ or $ax \leq \delta(\delta(q)) = \delta(q)$ or $bcx \leq \delta(\delta(q)) = \delta(q)$. If $ax \leq \delta(q)$ or $bcx \leq \delta(q)$, then the proof is clear. If $abc \leq \delta(q)$, as $\delta(q)$ be a 2-absorbing δ-primary element of L, we get either $ab \leq \delta(q)$ or $ac \leq \delta(\delta(q)) = \delta(q)$ or $bc \leq \delta(\delta(q)) = \delta(q)$ and hence either $abx \leq \delta(q)$ or $acx \leq \delta(q)$ or $bcx \leq \delta(q)$. Thus $(\delta(q) : x)$ is a 2-absorbing δ-primary element of L. □

We prove the following characterization of a 2-absorbing δ primary element of L.

Lemma 3.14. *Let δ be an element expansion of L and p is a proper element of L. If p is a 2-absorbing δ-primary element, then for elements $x, y \in L$ with $xy \not\leq \delta(p)$, $(p : xy) \leq (\delta(p) : y) \vee (p : x)$.*

Proof. Suppose that $x, y \in L$ with $xy \not\leq \delta(p)$, let $a \leq (p : xy)$. So $axy \leq p$. If $ax \leq p$, then $a \leq (p : x)$. Assume that $ax \not\leq p$. Since p is a 2- absorbing δ-primary element, $ay \leq \delta(p)$. So $a \leq (\delta(p) : y)$. Thus $(p : xy) \leq (\delta(p) : y) \vee (p : x)$. □

Theorem 3.15. *If δ is an expansion function such that $\delta(p) \leq \delta_1(p)$ and $\delta(p)$ is a semi prime element of L for every element p, then for any 2-absorbing δ-primary element p, $\delta(p) = \delta_1(p)$.*

Proof. Let $a \leq \delta_1(p)$. Then there exists k which is the least positive integer k with $a^k \leq p$. If $k = 1$, then $a \leq p \leq \delta(p)$. If $k > 1$, then $a^{k-2}aa \leq p$. But $a^{k-1} \not\leq p$, so $a^2 \leq \delta(p)$ implies $a \leq \sqrt{\delta(p)}$. Since $\delta(p)$ is semi prime, then $a \leq \sqrt{\delta(p)} = \delta(p)$. Hence $\delta_1(p) \leq \delta(p)$ and $\delta(p) = \delta_1(p)$. □

It is known (see [1]) that for any $a \in L$, $L/a = \{b \in L | a \leq b\}$ is a multiplicative lattice with multiplication $c \circ d = cd \vee a$.

Proposition 3.16. *Let L be a multiplicative lattice and p be a 2-absorbing δ-primary element. If $a \in L$ with $a \leq p$ then p is a 2-absorbing δ-primary element of L/a.*

Proof. Let $x \circ y \circ z \leq p$, for some $x, y, z \in L/a$ then clearly $xyz \leq p$. As p is a 2-absorbing δ-primary element, we get either $xy \leq p$ or $yz \leq \delta(p)$ or $xz \leq \delta(p)$. Thus we get either $x \circ y \leq p$ or $y \circ z \leq \delta(p)$ or $x \circ z \leq \delta(p)$. Therefore p is a 2-absorbing δ-primary element of L/a. □

4. Expansions with extra properties and 2-absorbing δ-primary elements

In this section we investigate 2-absorbing δ-primary elements where δ satisfies additional conditions, and prove some results with respect to such expansions. Recall from Manjarekar and Bingi [5] that an element expansion δ is meet preserving if it satisfies:

$$\delta(a \wedge b) = \delta(a) \wedge \delta(b) \text{ for any } a, b \in L.$$

Lemma 4.1. *Let δ be a meet preserving element expansion. If $q_1, q_2, \ldots, q_n$ are 2-absorbing δ-primary elements of L, and $p = \delta(q_i)$ for all i, then $q = \bigwedge_{i=1}^{n} q_i$ is 2-absorbing δ-primary.*

Proof. Let $xyz \leq q$ and $xy \nleq q$ then, for some k, $xy \nleq q_k$. Now $xyz \leq q_k$, as each q_k is 2-absorbing δ-primary we get either $yz \leq \delta(q_k)$ or $xz \leq \delta(q_k)$, But $\delta(q) = \delta(\bigwedge_{i=1}^{n} q_i) = \bigwedge_{i=1}^{n}(\delta(q_i)) = p = \delta(q_k)$. Thus either $yz \leq \delta(q)$ or $xz \leq \delta(q)$, so q is a 2-absorbing δ-primary element. □

Next Lemma prove that the meet of a pair of distinct prime elements of L is 2-absorbing δ-primary.

Lemma 4.2. *Let δ be a meet preserving element expansion. Then the meet of a pair of distinct prime elements of L is 2-absorbing δ-primary.*

Proof. Assume that p_1 and p_2 are two distinct prime elements of L. Let $abc \leq p_1 \wedge p_2$. Since p_1 and p_2 are prime elements of L, we get either
(1) $ab \leq p_1$ or $c \leq p_1$ and $ab \leq p_2$ or $c \leq p_2$,
or (2) $bc \leq p_1$ or $a \leq p_1$ and $bc \leq p_2$ or $a \leq p_2$,
or (3) $ac \leq p_1$ or $b \leq p_1$ and $ac \leq p_2$ or $b \leq p_2$.
Suppose that (1) $ab \leq p_1$ or $c \leq p_1$ and $ab \leq p_2$ or $c \leq p_2$. If $ab \leq p_1$ and $ab \leq p_2$ then $ab \leq p_1 \wedge p_2$ and proof is done. If $c \leq p_1$ and $c \leq p_2$ then either $bc \leq p_1$ and $bc \leq p_2$ or $ac \leq p_1$ and $ac \leq p_2$. Further it implies either $bc \leq p_1 \wedge p_2 \leq \delta(p_1 \wedge p_2)$ or $ac \leq p_1 \wedge p_2 \leq \delta(p_1 \wedge p_2)$. Similarly, we can prove the result in the other two cases.
Therefore, the meet of each pair of distinct prime elements of L is 2-absorbing δ-primary. □

Definition 4.3. Let L_1 and L_2 be compactly generated multiplicative lattices with largest element compact in which every finite product of compact elements is compact. Let δ be an element expansion of L_2 and γ be an element expansion of L_1. We say that a lattice isomorphism $f : L_1 \to L_2$ is a $\gamma\delta$- lattice isomorphism if

$$\gamma(f^{-1}(a)) = f^{-1}(\delta(a)) \text{ for all } a \in L_2.$$

In particular, if f is a $\gamma\delta$-lattice isomorphism, then $f(\gamma(a)) = \delta(f(a))$ for every element of L.
In the following result, we prove that the inverse image of a 2-absorbing δ-primary element of L under the homomorphism is again a 2-absorbing δ-primary element.

Lemma 4.4. *Let L_1 and L_2 be compactly generated multiplicative lattices with largest element compact in which every finite product of compact elements is compact. Let f be a $\gamma\delta$- lattice isomorphism $f : L_1 \to L_2$. Then for any 2-absorbing δ-primary element $p \in L_2$, $f^{-1}(p)$ is a 2-absorbing γ-primary element of L_1.*

Proof. Let $a, b, c \in L$ with $abc \leq f^{-1}(p)$, So $f(abc) = f(a)f(b)f(c) \leq p$ but p is 2-absorbing δ-primary, then we get either $f(a)f(b) \leq p$ or $f(a)f(c) \leq \delta(p)$ or $f(b)f(c) \in \delta(p)$, which implies either $ab \leq f^{-1}(p)$ or $ac \leq f^{-1}(\delta(p)) = \gamma(f^{-1}(p))$ or $bc \in f^{-1}(\delta(p)) = \gamma(f^{-1}(p))$. Hence $f^{-1}(p)$ is 2-absorbing γ-primary element of L_1. □

The next result gives a characterization for a 2-absorbing δ-primary element.

Lemma 4.5. *Let L_1 and L_2 be compactly generated multiplicative lattices with largest element compact in which every finite product of compact elements is compact. Let $f : L_1 \to L_2$ be a $\delta\gamma$- lattice isomorphism. Then an element $p \in L_1$ is a 2-absorbing δ-primary element if and only if $f(p)$ is a 2-absorbing γ-primary element of L_2.*

Proof. First suppose that $f(p)$ is 2-absorbing γ-primary and we have $f^{-1}(f(p)) = p$. Then by Lemma 4.4, p is a 2-absorbing δ-primary element of L_1.
Conversely, suppose that p is a 2-absorbing δ-primary element of L_1. If $a, b, c \in L_2$ and $abc \leq f(p)$ then there exist $x, y, z \in L_1$ such that $f(x) = a$ and $f(y) = b$, and $f(z) = c$, then
$f(xyz) = f(x)f(y)f(z) = abc \leq f(p)$ implies $xyz \leq f^{-1}(f(p)) = p$, as p is a 2-absorbing δ-primary element of L_1, we get either $xy \leq p$ or $xz \leq \delta(p)$ or $yz \leq \delta(p)$. As f is an $\delta\gamma$-lattice isomorphism, then $\gamma(f(a)) = f(\delta(a))$. We get either $xy \leq p$ or $xz \leq \delta(p) = f^{-1}(\gamma(f(p)))$ or $yz \leq \delta(p) = f^{-1}(\gamma(f(p)))$ which implies that either $ab \leq f(p)$ or $ac \leq \gamma(f(p))$ or $bc \leq \gamma(f(p))$. Thus $f(p)$ is a 2-absorbing γ-primary element of L_2. □

We prove following Lemma which is helpful to prove next results, it is an extension of [4, Lemma 2].

Lemma 4.6. *Let δ be an element expansion such that $\delta(\delta(a)) = \delta(a)$, for every element $a \in L$. Let $\delta(q)$ be a 2-absorbing*

δ-primary element of L. Then
(i) If $x \leq \sqrt{\delta(q)}$ then $x^2 \leq \delta(q)$.
(ii) If $x, y \leq \sqrt{\delta(q)}$ then $xy \leq \delta(q)$.
(iii) $(\sqrt{\delta(q)})^2 \leq \delta(q)$.

Proof. (i) Let $a \leq \sqrt{\delta(q)}$ be a compact element then there exists a positive integer n such that $a^n \leq \delta(q)$. Since $\delta(q)$ is a 2-absorbing δ-primary element of L, we get $a^2 \leq \delta(q)$.

Suppose that$x \leq \sqrt{\delta(q)}$. Let $a, b \in L_*$ be such that $a \leq x$ and $b \leq x$. Since $a \leq \sqrt{\delta(q)}$ and $b \leq \sqrt{\delta(q)}$, it follows that $a^2 \leq \delta(q)$ and $b^2 \leq \delta(q)$ and so $a(a \vee b)b \leq \delta(q)$. As $\delta(q)$ is a 2-absorbing δ-primary element of L, then either $a(a \vee b) \leq \delta(q)$ or $(a \vee b)b \leq \delta(\delta(q)) = \delta(q)$ or $ab \leq \delta(\delta(q)) = \delta(q)$. Since $ab \leq a(a \vee b) = a^2b \vee ab^2 \leq \delta(q)$. As $x^2 = \vee\{ab | a, b \in L_*, a \leq x, b \leq x\}$, it follows that $x^2 \leq \delta(q)$.
(ii) Suppose that $x, y \leq \sqrt{\delta(q)}$. By (i), $x^2 \leq \delta(q)$ and $y^2 \leq \delta(q)$, so $x(x \vee y)y \leq \delta(q)$. As $\delta(q)$ is a 2-absorbing δ-primary element of L, it follows that $xy \leq \delta(q)$.
(iii) We note that $(\sqrt{\delta(q)})^2 = \vee\{ab | a, b \in L_*, a \leq \sqrt{\delta(q)}, b \leq \sqrt{\delta(q)}\}$. Now the result follows from (ii). □

The next result gives the condition for a p-primary element of L to be 2-absorbing δ-primary element of L .

Lemma 4.7. *Let δ be an element expansion such that $\delta(\delta(a)) = \delta(a)$, for every $a \in L$. Suppose that $\delta(q)$ is a p-primary element of L. Then $\delta(q)$ is a 2-absorbing δ-primary element of L if and only if $p^2 \leq \delta(q)$.*

Proof. Let $\delta(q)$ be a 2-absorbing δ-primary element of L. Since $\delta(q)$ is a p-primary element of L, $\sqrt{\delta(q)} = p$ so by Lemma 4.6(iii),
$p^2 = (\sqrt{\delta(q)})^2 \leq \delta(q)$.

Conversely, assume that $p^2 \leq \delta(q)$ and $xyz \leq \delta(q)$. If either $x \leq \delta(q)$ or $yz \leq \delta(q)$, then the proof is clear. So assume that $x \nleq \delta(q)$ and $yz \nleq \delta(q)$. Since $\delta(q)$ is a p-primary element, so we have
$xyz \leq \delta(q) \leq \sqrt{\delta(q)} = p$ and p is prime then either $x \leq p$ or $yz \leq p$. Thus either $x \leq p$ or $y \leq p$ or $z \leq p$. Hence $xy \leq p^2$ or $xz \leq p^2$. Since $p^2 \leq \delta(q)$, it follows that either $xy \leq \delta(q)$ or $xz \leq \delta(q)$, and hence $\delta(q)$ is a 2-absorbing δ-primary element of L. □

Lemma 4.8. *Let δ be an element expansion such that $\delta(\delta(a)) = \delta(a)$, for every $a \in L$. Let $\delta(q) \in L$ be such that $\sqrt{\delta(q)} = p$ is a prime element of L and $\delta(q)$ be a 2-absorbing δ-primary element of L. Assume that $\delta(q) \neq p$.*

(i) For each $x \leq p$ and $x \nleq \delta(q)$, $(\delta(q) : x)$ is a prime element of L and $p \leq (\delta(q) : x)$.
(ii) Either $(\delta(q) : x) \leq (\delta(q) : y)$ or $(\delta(q) : y) \leq (\delta(q) : x)$, for all $x, y \leq p$ and $x, y \nleq \delta(q)$.

Proof. (i) Let $x \leq p = \sqrt{\delta(q)}$ and $x \nleq \delta(q)$, then by Lemma 4.6 (iii), we have $p^2 = (\sqrt{(\delta(q)})^2 \leq \delta(q)$. Since $x \leq p$ implies that $px \leq p^2 \leq \delta(q)$ implies that $px \leq \delta(q)$. So $p \leq (\delta(q) : x)$.
Next, Let $y, z \in L$ be such that $yz \leq (\delta(q) : x)$. If either $y \leq p$ or $z \leq p$, since $p \leq (\delta(q) : x)$ then either $y \leq (\delta(q) : x)$ or $z \leq (\delta(q) : x)$. So assume that $y \nleq p$ and $z \nleq p$. If $yz \leq \delta(q)$ then
$yz \leq \delta(q) \leq \sqrt{(\delta(q)} = p$. As p is prime we get $y \leq p$ or $z \leq p$, a contradiction. So we assume that $yz \nleq \delta(q)$. Since $yz \leq (\delta(q) : x)$, it follows that $xyz \leq \delta(q)$ which implies that either $xy \leq \delta(\delta(q)) = \delta(q)$ or $]xz \leq \delta(\delta(q)) = \delta(q)$. Hence either $y \leq (\delta(q) : x)$ or $z \leq (\delta(q) : x)$. Thus $(\delta(q) : x)$ is a prime element of L.
(ii) Let $x, y \leq p$ and $x, y \nleq \delta(q)$. Choose any $z \leq L_*$ such that
$z \leq (\delta(q) : x)$ and $z \nleq (\delta(q) : y)$. By (i), $p \leq (\delta(q) : y)$. So $z \nleq p$. We show that $(\delta(q) : y) \leq (\delta(q) : x)$. Let $w \leq L_*$ and let $w \leq (\delta(q) : y)$. If $w \leq p$, then $w \leq (\delta(q) : x)$. So assume that $w \nleq p$. Since $z \leq (\delta(q) : x)$ and $w \leq (\delta(q) : y)$ then we get $z(x \vee y)w \leq \delta(q)$. As $\delta(q)$ is a 2-absorbing δ-primary element of L, either
$zy \leq zx \vee zy \leq z(x \vee y) \leq \delta(q)$ or $zw \leq \delta(\delta(q)) = \delta(q)$ or
$(x \vee y)w \leq \delta(\delta(q)) = \delta(q)$. Since $z \nleq (\delta(q) : y)$ and $z \nleq p$ and $w \nleq p$, we get $zy \nleq \delta(q)$ and $zw \nleq \delta(q)$. Thus we get $xw \leq \delta(q)$ implies that $w \leq (\delta(q) : x)$. Therefore $(\delta(q) : y) \leq (\delta(q) : x)$. □

Theorem 4.9 characterize nonradical 2-absorbing elements of L.

Theorem 4.9. *Let δ be an element expansion such that $\delta(\delta(a)) = \delta(a)$, for every $a \in L$. Let $\delta(q) \in L$ be such that $\delta(q) \neq \sqrt{\delta(q)}$ and $\sqrt{\delta(q)}$ be a prime element of L. Then the following statements are equivalent:*
(i) $\delta(q)$ is a 2-absorbing δ-primary element of L.
(ii) $(\delta(q) : x)$ is a prime element of L for each $x \leq \sqrt{\delta(q)}$ and $x \nleq \delta(q)$.

Proof. $(i) \Rightarrow (ii)$ Follows from Lemma 4.8.
$(ii) \Rightarrow (i)$ Let $xyz \leq \delta(q) \leq \sqrt{\delta(q)}$. Since $\sqrt{\delta(q)}$ is a prime element then $x \leq \sqrt{\delta(q)}$ or $y \leq \sqrt{\delta(q)}$ or $z \leq \sqrt{\delta(q)}$. Let us assume that $x \leq \sqrt{\delta(q)}$. If $x \leq \delta(q)$ then the proof is clear. Suppose that $x \nleq \delta(q)$ then by (ii), $(\delta(q) : x)$ is a prime element of L, and $yz \leq (\delta(q) : x)$, so either $y \leq (\delta(q) : x)$ or $z \leq (\delta(q) : x)$ it follows that either $xy \leq \delta(q)$ or $xz \leq \delta(q)$. Thus $\delta(q)$ is a 2-absorbing δ-primary element of L. □

Theorem 4.10. *Let δ be an element expansion such that $\delta(\delta(a)) = \delta(a)$, for every $a \in L$. Let $\delta(q)$ be a 2-absorbing δ-primary element of L. Then the following statements holds:*
(i) If $y \in L$, $x \leq \sqrt{\delta(q)}$ and $x \nleq \delta(q)$ and $xy \nleq \delta(q)$, then $(\delta(q) : xy) = (\delta(q) : x)$.
(ii) If $x \leq \sqrt{\delta(q)}$ and $x \nleq \delta(q)$ and $(\delta(q) : x) < (\delta(q) : y)$, then $(\delta(q) : ax \vee by) = (\delta(q) : x)$, for all $a, b, y \in L$ such that $ab \nleq (\delta(q) : x)$. In particular, $(\delta(q) : x \vee y) = (\delta(q) : x)$.

Proof. (i) Let $y \in L$, $x \leq \sqrt{\delta(q)}$ and $x \nleq \delta(q)$ and $xy \nleq \delta(q)$. Let $a \leq (\delta(q) : x)$ implies that $ax \leq \delta(q)$ implies that $axy \leq \delta(q)y \leq \delta(q) \wedge y \leq \delta(q)$, so we get $a \leq (\delta(q) : xy)$, Thus $(\delta(q) : x) \leq (\delta(q) : xy)$. Let $z \leq (\delta(q) : xy)$, then $xyz \leq \delta(q)$, since $x \nleq \delta(q)$, we have $yz \leq (\delta(q) : x)$, by Lemma 4.8, $(\delta(q) : x)$ is a prime element. Hence either $y \leq (\delta(q) : x)$ or $z \leq (\delta(q) : x)$, but $xy \nleq \delta(q)$, so $z \leq (\delta(q) : x)$. Thus $(\delta(q) : xy) \leq (\delta(q) : x)$. Hence $(\delta(q) : xy) = (\delta(q) : x)$.
(ii) Suppose that $x \leq \sqrt{\delta(q)}$ and $x \nleq \delta(q)$ and $(\delta(q) : x) < (\delta(q) : y)$. Let $a, b \in L$ such that $ab \nleq (\delta(q) : x)$. Let $z \leq (\delta(q) : x) \Rightarrow zx \leq \delta(q)$ $\Rightarrow zxa \leq \delta(q)a \leq \delta(q) \wedge a \leq \delta(q)$. Similarly we get $zby \leq \delta(q)$, These imply that $zax \vee zby \leq \delta(q) \Rightarrow z(ax \vee by) \leq \delta(q) \Rightarrow z \leq (\delta(q) : ax \vee by)$. Thus $(\delta(q) : x) \leq (\delta(q) : ax \vee by)$.

On the other hand, suppose that $z \leq (\delta(q) : ax \vee by) \nleq (\delta(q) : x)$ then $z \leq (\delta(q) : ax \vee by)$ and $z \nleq (\delta(q) : x)$. As $z \leq (\delta(q) : ax \vee by) \Rightarrow zax \leq zax \vee zby \leq \delta(q) \Rightarrow zax \leq \delta(q) \Rightarrow za \leq (\delta(q) : x)$ and $(\delta(q) : x)$ is prime, it follows that $z \leq (\delta(q) : x)$ or $a \leq (\delta(q) : x)$, a contradiction. Therefore $(\delta(q) : x) = (\delta(q) : ax \vee by)$. By taking $a = b = 1$ then we get $(\delta(q) : x) = (\delta(q) : x \vee y)$. □

Lemma 4.11. *Let δ be an element expansion such that $\delta(\delta(a)) = \delta(a)$, for every $a \in L$. Suppose that $\delta(q)$ is a 2-absorbing δ-primary element of L and p_1, p_2 are two distinct minimal primes over $\delta(q)$.*
Let $x_1, x_2 \in L_*$ be such that $x_1 \leq p_1$, $x_1 \nleq p_2$, $x_2 \leq p_2$, $x_2 \nleq p_1$. Then $x_1x_2 \leq \delta(q)$.

Proof. Let $x_1, x_2 \in L_*$ be such that $x_1 \leq p_1$, $x_1 \nleq p_2$, $x_2 \leq p_2$, $x_2 \nleq p_1$. By [2, Lemma 3.5], there exist $c_1, c_2 \in L_*$ and $c_1 \nleq p_1$, $c_2 \nleq p_2$ such that $c_2x_1^n \leq q \leq \delta(q)$, $c_1x_2^m \leq q \leq \delta(q)$ for some positive integers n and m. As $\delta(q)$ is a 2-absorbing δ-primary element, it follows that either $c_2x_1 \leq \delta(\delta(q)) = \delta(q)$ or $x_1 \leq \delta(q)$. If $x_1 \leq \delta(q) \leq p_2$, then $x_1 \leq p_2$, a contradiction. Therefore $c_2x_1 \leq \delta(q)$. Similarly, it can be easily shown that $c_1x_2 \leq \delta(q)$.
Now observe that $(c_1 \vee c_2)x_1x_2 \leq \delta(q)$. Since $c_1 \vee c_2 \nleq p_1$ and

$c_1 \vee c_2 \nleq p_2$, we conclude that $(c_1 \vee c_2)x_2 \nleq p_1$ and $(c_1 \vee c_2)x_1 \nleq p_2$. Hence $(c_1 \vee c_2)x_2 \nleq \delta(\delta(q)) = \delta(q)$ and $(c_1 \vee c_2)x_1 \nleq \delta(\delta(q)) = \delta(q)$ and so $x_1x_2 \leq \delta(q)$, as $\delta(q)$ is a 2-absorbing δ-primary element of L. $\square$

Lemma 4.12. *Let δ be an element expansion such that $\delta(\delta(a)) = \delta(a)$, for every $a \in L$. Let $\delta(q)$ be a 2-absorbing δ-primary element of L. Then there are at most two prime elements of L that are minimal over $\delta(q)$.*

Proof. Let $A = \{ p_i : p_i$ is prime elements of L that are minimal over $\delta(q)\}$, and assume that A has at least three elements. Let $p_1, p_2 \in A$. Then there exist $x_1, x_2 \in L_*$ such that $x_1 \leq p_1$, $x_1 \nleq p_2$, $x_2 \leq p_2$, $x_2 \nleq p_1$. Then $x_1x_2 \leq \delta(q)$, by Lemma 4.11. Now assume that there exists $p_3 \in A$ distinct from p_1 and p_2. Then we can choose $y_i \in L_*$ such that $y_i \leq p_i$, $y_i \nleq p_j$, for $i \neq j$, where $i, j = 1, 2, 3$. By Lemma 4.11, we have $y_1y_2 \leq \delta(q) \leq p_3$, as p_3 is prime this implies that either $y_1 \leq p_3$ or $y_2 \leq p_3$, a contradiction. Therefore A has at most two elements. $\square$

Theorem 4.13. *Let δ be an element expansion such that $\delta(\delta(a)) = \delta(a)$, for every $a \in L$. Let $\delta(q)$ be a 2-absorbing δ-primary element of L. Then one of the following statement hold true:*
(1) $\sqrt{\delta(q)} = p$ is a prime element of L such that $p^2 \leq \delta(q)$.
(2) $\sqrt{\delta(q)} = p_1 \wedge p_2$ and $p_1p_2 \leq \delta(q)$, where p_1 and p_2 are the only nonzero distinct prime elements of L that are minimal over $\delta(q)$.

Proof. By Lemma 4.12, we have either $\sqrt{\delta(q)} = p$ is a prime element of L or $\sqrt{\delta(q)} = p_1 \wedge p_2$, where p_1 and p_2 are the only nonzero distinct prime elements of L that are minimal over $\delta(q)$.
If $\sqrt{\delta(q)} = p$ is a prime element of L, then by Lemma 4.7, $p^2 \leq \delta(q)$, so the condition (1) holds.
Now assume $\sqrt{\delta(q)} = p_1 \wedge p_2$, where p_1 and p_2 are the only nonzero distinct prime elements of L that are minimal over $\delta(q)$. We show that $p_1p_2 \leq \delta(q)$. If $x, y \leq \sqrt{\delta(q)} = p_1 \wedge p_2$, then by Lemma 4.6,(ii), $xy \leq \delta(q)$.
If $x, y \in L_*$ be such that $x \leq p_1$, $x \nleq p_2$, $y \leq p_2$, $y \nleq p_1$. Then $xy \leq \delta(q)$, by Lemma 4.11.
If $x, y \in L_*$ be such that $x \leq \sqrt{\delta(q)}$, $y \leq p_2$, $y \nleq p_1$. Take $y_1 \in L_*$ such that $y_1 \leq p_1$, $y_1 \nleq p_2$. Then $yy_1 \leq \delta(q)$, by Lemma 4.11. Note that $x \vee y_1 \leq p_1$, $x \vee y_1 \nleq p_2$. Then $(x \vee y_1)y \leq \delta(q)$, by Lemma 4.11 and hence $xy \leq \delta(q)$. Similarly, we can show that if $y \in \sqrt{\delta(q)}$, $x \leq p_1$, $x \nleq p_2$, then $xy \leq \delta(q)$. Consequently, we get $p_1p_2 \leq \delta(q)$. So the condition (2) holds. $\square$

Lemma 4.14. *Let δ be an element expansion such that $\delta(\delta(a)) = \delta(a)$, for every $a \in L$. Let $\delta(q)$ be a 2-absorbing δ-primary element of L such that $\delta(q) \neq \sqrt{\delta(q)} = p_1 \wedge p_2$, where p_1 and p_2 are the only nonzero distinct prime elements of L that are minimal over $\delta(q)$.*
(i) For each $x \leq \sqrt{\delta(q)}$ and $x \nleq \delta(q)$, $(\delta(q) : x)$ is a prime element of L and $p_1 \leq (\delta(q) : x)$ and $p_2 \leq (\delta(q) : x)$.
(ii) Either $(\delta(q) : x) \leq (\delta(q) : y)$ or $(\delta(q) : y) \leq (\delta(q) : x)$, for all $x, y \leq \sqrt{\delta(q)}$ and $x, y \nleq (\delta(q)$.

Proof. (i) Let $x \leq \sqrt{(\delta(q)}$ and $x \nleq (\delta(q)$. Since $p_1p_2 \leq \delta(q)$, by Theorem 4.13, we have $xp_1 \leq \delta(q)$, and $xp_2 \leq \delta(q)$. Thus $p_1 \leq (\delta(q) : x)$ and $p_2 \leq (\delta(q) : x)$. Suppose $y, z \in L$, and $yz \leq (\delta(q) : x)$. So we have $xyz \leq \delta(q)$, since $\delta(q)$ is a 2-absorbing δ-primary element of L, we have either $xy \leq \delta(q)$ or $xz \leq \delta(\delta(q)) = \delta(q)$ or $yz \leq \delta(\delta(q)) = \delta(q)$. If either $y \leq p_1$ or $y \leq p_2$ or $z \leq p_1$ or $z \leq p_2$, then the proof is clear. If $y, z \nleq p_1$ or $y, z \nleq p_2$, and so $yz \nleq \delta(q)$, then we get either $xy \leq \delta(q)$ or $xz \leq \delta(\delta(q)) = \delta(q)$. Thus we get either $y \leq (\delta(q) : x)$ and $z \leq (\delta(q) : x)$. Therefore $(\delta(q) : x)$ is a prime element of L.
(ii)The proof is similar to the proof of Lemma 4.8(ii). □

We prove the following characterization.

Theorem 4.15. *Let δ be an element expansion such that $\delta(\delta(a)) = \delta(a)$, for every $a \in L$. Let $q \in L$, and let $\delta(q) \neq \sqrt{\delta(q)} = p_1 \wedge p_2$ where p_1 and p_2 are the only nonzero distinct prime elements of L that are minimal over $\delta(q)$. Then the following statements are equivalent :*
(1) $\delta(q)$ is a 2-absorbing δ-primary element of L;
(2) $p_1p_2 \leq \delta(q)$ and $(\delta(q) : x)$ is a prime element of L, for each $x \leq \sqrt{\delta(q)}$ and $x \nleq \delta(q)$;
(3) If $(\delta(q) : x)$ is proper and either $x \leq p_1$ or $x \leq p_2$, then $(\delta(q) : x)$ is a prime element of L.

Proof. (1) ⇒ (2) follows from Lemma 4.14 and Theorem 4.13.
(2) ⇒ (3) Let $x \leq p_1$ and $x \nleq p_2$, since $p_1p_2 \leq \delta(q)$, we have $xp_2 \leq \delta(q)$. Hence $(\delta(q) : x) = p_2$ is a prime element of L. Similarly, $x \leq p_2$ and $x \nleq p_1$, then $p_1x \leq \delta(q)$. Hence $(\delta(q) : x) = p_1$ is a prime element of L. If $x \leq p_1$ and $x \leq p_2$ and $x \nleq \delta(q)$ then by condition (2), $(\delta(q) : x)$ is a prime element of L, so (3) holds.
(3) ⇒ (1) Let $xyz \leq \delta(q)$, for some $x, y, z \in L$.
So $xyz \leq \delta(q) \leq \sqrt{\delta(q)} = p_1 \wedge p_2$, so $xyz \leq p_1 \wedge p_2$ then $xyz \leq p_1$ and $xyz \leq p_2$ implies that either $x \leq p_1$ or $yz \leq p_1$ and either $x \leq p_2$ or

$yz \leq p_2$. Without loss of generality, we assume that $x \leq p_1$. If $x \leq \delta(q)$ then the proof is clear. If $x \nleq \delta(q)$ then $yz \leq (\delta(q) : x)$ and by (3), $(\delta(q) : x)$ is prime then either $y \leq (\delta(q) : x)$ or $z \leq (\delta(q) : x)$, it follows that either $xy \leq \delta(q) \leq \delta(\delta(q))$ or $xz \leq \delta(q) \leq \delta(\delta(q))$. Thus $\delta(q)$ is a 2-absorbing δ-primary element of L. □

5. WEAKLY 2-ABSORBING δ-PRIMARY ELEMENTS

In this section, we define a weakly 2-absorbing δ-primary element and obtain some properties of these elements. Also we define a δ-triple-zero.

Definition 5.1. A proper element p of L is called a weakly 2-absorbing δ-primary element if, whenever, $a, b, c \in L$ and $0 \neq abc \leq p$ implies that $ab \leq p$ or $bc \leq \delta(p)$ or $ac \leq \delta(p)$.

We prove the following characterization of a weakly 2-absorbing δ-primary element, the proof of this Theorem is similar to the proof of Theorem 3.6.

Theorem 5.2. *An element $q \in L$ is a weakly 2-absorbing δ-primary element if and only if for any $a, b, c \in L_*$, $0 \neq abc \leq q$ implies that either $ab \leq q$ or $ac \leq \delta(q)$ or $bc \leq \delta(q)$.*

Lemma 5.3. *Every 2-absorbing δ-primary element of L is a weakly 2-absorbing δ-primary element of L.*

Proof. Suppose that p is a 2-absorbing δ-primary element of L. Let $0 \neq abc \leq p$. As p is a 2-absorbing δ-primary element of L, we get $ab \leq p$ or $bc \leq \delta(p)$ or $ac \leq \delta(p)$. Thus p is a weakly 2-absorbing δ-primary element of L. □

Remark 5.4. The following example shows that the converse of Lemma 5.3 does not hold.

Example 5.5. Consider the multiplicative lattice shown in Figure 1. Here the element 0 is weakly 2-absorbing δ_0-primary, δ_1-primary, $\mathbf{M}$-primary element. For $g, h, i \in L$, $ghi = 0 \leq 0$ but neither $hi = e \leq 0$ nor $gi = g \leq \delta_0(0) = 0$ nor $gh = c \leq \delta_0(0) = 0$.
$dij = 0 \leq 0$ but neither $di = a \leq 0$ nor $ij = g \leq \delta_1(0) = c$ nor $dj = b \leq \delta_1(0) = c$.
$dij = 0 \leq 0$ but neither $di = a \leq 0$ nor $ij = g \leq \mathbf{M}(0) = c$ nor $dj = b \leq \mathbf{M}(0) = c$.
Thus 0 is not 2-absorbing δ_0-primary, δ_1-primary, $\mathbf{M}$-primary element of L.

We have proved Lemmas 3.7 and 4.2 for 2-absorbing δ-primary elements. The following two results can be similarly proved for weakly 2-absorbing δ-primary elements.

Lemma 5.6. *Every weakly prime element of L is a weakly 2-absorbing δ-primary element of L.*

The proof of this Lemma is similar to the proof of Lemma 3.7.

Remark 5.7. The following example shows that the converse of Lemma 5.6 does not hold.

Example 5.8. Consider the multiplicative lattice shown in Figure 1. Here the element e is 2-absorbing δ_0-primary, δ_1-primary, $\mathbf{M}$-primary element and weakly 2-absorbing δ_0-primary, δ_1-primary, $\mathbf{M}$-primary element. For $di = a \leq e$ but neither $d \leq e$ nor $i \leq e$. Hence e is neither prime nor weakly prime element.

Lemma 5.9. *Let δ be an meet preserving element expansion. Then the meet of any two weakly prime elements of L is weakly 2-absorbing δ-primary.*

The proof is similar to the proof of Lemma 4.2.

Definition 5.10. Let p be a weakly 2-absorbing δ-primary element of L. We say that (a, b, c) is a δ-triple-zero of p if whenever $a, b, c \in L$ and $abc = 0$ then $ab \nleq p$, $bc \nleq \delta(p)$ and $ac \nleq \delta(p)$.

Remark 5.11. If q is a weakly 2-absorbing δ-primary element of L that is not a 2-absorbing δ-primary element of L, then q has a δ-triple-zero (a, b, c), for some $a, b, c \in L$.

Proof. Since q is not a 2-absorbing δ-primary element of L then there exists $a, b, c \in L$, $abc \leq q$ but $ab \nleq q$, $bc \nleq \delta(q)$, $ac \nleq \delta(q)$. As q is a weakly 2-absorbing δ-primary element of L, if $abc \neq 0$ then either $ab \leq q$ or $bc \leq \delta(q)$ or $ac \leq \delta(q)$, which is not possible. Hence $abc = 0$. Thus q has a δ-triple-zero (a, b, c). □

Theorem 5.12. *Let q be a weakly 2-absorbing δ-primary element of L and suppose that (a, b, c) is a δ-triple zero of q for some $a, b, c \in L$. Then*
(1) $abq = bcq = acq = 0$
(2) $aq^2 = bq^2 = cq^2 = 0$

Proof. (1) Suppose that $abq \neq 0$. Let x be a compact element of L such that $x \leq q$. Suppose that $0 \neq abx \leq q$. Hence

$0 \neq abc \vee abx = ab(c \vee x) \leq q$. Since $ab \nleq q$ and q is a weakly 2-absorbing δ-primary element, we have either $a(c \vee x) \leq \delta(q)$ or $b(c \vee x) \leq \delta(q)$. So we get either $ac \leq a(c \vee x) \leq \delta(q)$ or $bc \leq b(c \vee x) \leq \delta(q)$, which is a contradiction. Thus $abx = 0$, and so $abq = 0$. Similarly, $bcq = acq = 0$.
(2) Suppose that $aq^2 \neq 0$. Let x, y be compact elements of L such that $x, y \leq q$. Suppose that $0 \neq axy \leq q$. Hence $0 \neq abc \vee axy \vee acx \vee aby = a(b \vee x)(c \vee y) \leq q$. Hence from (1), $aby = acx = abc = 0$, we have $0 \neq axy = a(b \vee x)(c \vee y) \leq q$ and q is a weakly 2-absorbing δ-primary element, we have either $a(b \vee x) \leq q$ or $a(c \vee y) \leq \delta(q)$ or $(b \vee x)(c \vee y) \leq \delta(q)$. So we get either $ab \leq a(b \vee x) \leq q$ or $ac \leq a(c \vee y) \leq \delta(q)$ or $bc \leq (b \vee x)(c \vee y) \leq \delta(q)$, which is a contradiction. Thus $axy = 0$, and so $aq^2 = 0$. Similarly, we can prove that $bq^2 = cq^2 = 0$. □

The following theorem establishes a condition for a weakly 2-absorbing δ-primary element of L to be a 2-absorbing δ-primary element of L.

Theorem 5.13. *If q is a weakly 2-absorbing δ-primary element of L that is not 2-absorbing δ-primary element, then $q^3 = 0$.*

Proof. Suppose that q is a weakly 2-absorbing δ-primary element of L that is not a 2-absorbing δ-primary element, then there exists a δ-triple-zero (a, b, c) of q for some $a, b, c \in L$. Assume that $q^3 \neq 0$. Hence $xyz \neq 0$ for some compact elements $x, y, z \leq q$. By Theorem 5.12, we obtain $0 \neq (a \vee x)(b \vee y)(c \vee z) \leq q$ and q is a weakly 2-absorbing δ-primary element, we have either $(a \vee x)(b \vee y) \leq q$ or $(a \vee x)(c \vee z) \leq \delta(q)$ or $(b \vee y)(c \vee z) \leq \delta(q)$. So we get either $ab \leq (a \vee x)(b \vee y) \leq q$ or $ac \leq (a \vee x)(c \vee z) \leq \delta(q)$ or $bc \leq (b \vee y)(c \vee z) \leq \delta(q)$, which is a contradiction. Thus $xyz = 0$, and so $q^3 = 0$. □

As a consequences of Theorem 5.13, we have the following result.

Corollary 5.14. *Let q be a weakly 2-absorbing δ-primary element of L that is not 2-absorbing δ-primary element, then $q \leq \sqrt{0}$.*

Manjarekar and Chavan [6] have introduced the concept of a free triple-zero. We generalize this to free δ-triple-zero as follows.

Definition 5.15. Let q be a weakly 2-absorbing δ-primary element of L and suppose that $a_1a_2a_3 \leq q$ for some $a_1, a_2, a_3 \in L$. We say that q is a free δ-triple-zero with respect to $a_1a_2a_3$ if (a, b, c) is not a δ-triple-zero of q for any $a \leq a_1$, $b \leq a_2$, $c \leq a_3$.

Lemma 5.16. *Let q be a weakly 2-absorbing δ-primary element of L, and suppose $abd \leq q$ for some elements $a, b, d \in L$ such that (a, b, c) is not a δ-triple-zero of q for every $c \leq d$. If $ab \nleq q$ then $ad \leq \delta(q)$ or $bd \leq \delta(q)$.*

Proof. Suppose that neither $ad \leq \delta(q)$ nor $bd \leq \delta(q)$. Then $ad_1 \nleq \delta(q)$ and $bd_2 \nleq \delta(q)$ for some $d_1, d_2 \leq d$. Since (a, b, d_1) is not a δ-triple-zero of q and $abd_1 \leq q$ and $ab \nleq q$, $ad_1 \nleq \delta(q)$, we have $bd_1 \leq \delta(q)$. Since (a, b, d_2) is not a δ-triple-zero of q and $abd_2 \leq q$ and $ab \nleq q$, $bd_2 \nleq \delta(q)$, we have $ad_2 \leq \delta(q)$. Since $(a, b, d_1 \vee d_2)$ is not a δ-triple-zero of q and $ab(d_1 \vee d_2) \leq q$ and $ab \nleq q$, then we have $a(d_1 \vee d_2) \leq \delta(q)$ or $b(d_1 \vee d_2) \leq \delta(q)$. If $a(d_1 \vee d_2) \leq \delta(q)$ then $ad_1 \leq \delta(q)$ and $ad_2 \leq \delta(q)$, a contradiction. Hence $b(d_1 \vee d_2) \leq \delta(q)$. This implies $bd_1 \leq \delta(q)$ and $bd_2 \leq \delta(q)$, a contradiction. Hence $ad \leq \delta(q)$ or $bd \leq \delta(q)$. □

Corollary 5.17. *Let q be a weakly 2-absorbing δ-primary element of L, and suppose that $a_1a_2a_3 \leq q$ for some $a_1, a_2, a_3 \in L$ such that q is a free δ-triple-zero with respect to $a_1a_2a_3$. If $a \leq a_1$, $b \leq a_2$, $c \leq a_3$, then $ab \leq q$ or $ac \leq \delta(q)$ or $bc \leq \delta(q)$.*

Proof. Since q is a free δ-triple-zero with respect to $a_1a_2a_3$. It follows that (a, b, c) is not a δ-triple zero of q for any $a \leq a_1$, $b \leq a_2$, $c \leq a_3$. We have $abc \leq a_1a_2a_3 \leq q$. Since (a, b, c) is not a δ-triple-zero of q, we must have either $ab \leq q$ or $ac \leq \delta(q)$ or $bc \leq \delta(q)$, if $abc = 0$. If $abc \neq 0$ then $0 \neq abc \leq q$ implies that either $ab \leq q$ or $ac \leq \delta(q)$ or $bc \leq \delta(q)$, as q is a weakly 2-absorbing δ-primary element of L. □

Theorem 5.18. *Let q be a weakly 2-absorbing δ-primary element of L, and suppose that $0 \neq a_1a_2a_3 \leq q$ for some $a_1, a_2, a_3 \in L$ such that q is a free δ-triple-zero with respect to $a_1a_2a_3$. Then $a_1a_2 \leq q$ or $a_2a_3 \leq \delta(q)$ or $a_1a_3 \leq \delta(q)$.*

Proof. Suppose that $a_1a_2 \nleq q$. If $a_2a_3 \nleq \delta(q)$ and $a_1a_3 \nleq \delta(q)$, then there exist $q_1 \leq a_1$, $q_2 \leq a_2$ such that $q_2a_3 \nleq \delta(q)$ and $q_1a_3 \nleq \delta(q)$. Since $q_1q_2a_3 \leq q$ and $q_2a_3 \nleq \delta(q)$ and $q_1a_3 \nleq \delta(q)$, we have $q_1q_2 \leq q$, by Lemma 5.16. Since $a_1a_2 \nleq q$ we have $ab \nleq q$ for some $a \leq a_1$ and $b \leq a_2$. Since $aba_3 \leq q$ and $ab \nleq q$ then we have $aa_3 \leq \delta(q)$ or $ba_3 \leq \delta(q)$.
Case (1): Suppose that $aa_3 \leq \delta(q)$ but $ba_3 \nleq \delta(q)$. Since $q_1ba_3 \leq q$ and $ba_3 \nleq \delta(q)$ and $q_1a_3 \nleq \delta(q)$, we have $q_1b \leq q$ by Lemma 5.16. Since $(a \vee q_1)ba_3 \leq q$ and $q_1a_3 \nleq \delta(q)$, so we conclude that $(a \vee q_1)a_3 \nleq \delta(q)$. Since $(a \vee q_1)a_3 \nleq \delta(q)$ and $ba_3 \nleq \delta(q)$, we get $(a \vee q_1)b \leq q$, by Lemma 5.16. Since $(a \vee q_1)b = ab \vee q_1b \leq q$, and $q_1b \leq q$, so we get $ab \leq q$, a contradiction.

Case (2): Suppose that $ba_3 \leq \delta(q)$ but $aa_3 \nleq \delta(q)$. Since $aq_2a_3 \leq q$ and $aa_3 \nleq \delta(q)$ and $q_2a_3 \nleq \delta(q)$, we have $q_2b \leq q$ by Lemma 5.16. Since $a(b \vee q_2)a_3 \leq q$ and $q_2a_3 \nleq \delta(q)$, so we conclude that $(b \vee q_2)a_3 \nleq \delta(q)$. Since $(b \vee q_2)a_3 \nleq \delta(q)$ and $aa_3 \nleq \delta(q)$, we get $a(b \vee q_2) \leq q$, by Lemma 5.16. Since $a(b \vee q_2) = ab \vee q_2a \leq q$, and $q_2a \leq q$, so we get $ab \leq q$, a contradiction.
Case (3): Suppose that $aa_3 \leq \delta(q)$ and $ba_3 \leq \delta(q)$. Since $q_2a_3 \nleq \delta(q)$, so we conclude that $(b \vee q_2)a_3 \nleq \delta(q)$. Since $q_1(b \vee q_2)a_3 \leq q$ and $q_1a_3 \nleq \delta(q)$ and $(b \vee q_2)a_3 \nleq \delta(q)$, so $q_1(b \vee q_2) = q_1b \vee q_1q_2 \leq q$, by Lemma 5.16. Since $q_1b \vee q_1q_2 \leq q$ then we get $q_1b \leq q$. Since $q_1a_3 \nleq \delta(q)$, so we conclude that $(a \vee q_1)a_3 \nleq \delta(q)$. Since $(a \vee q_1)q_2a_3 \leq q$ and $q_2a_3 \nleq \delta(q)$ and $(a \vee q_1)a_3 \nleq \delta(q)$, so $(a \vee q_1)q_2 = aq_2 \vee q_1q_2 \leq q$, by Lemma 5.16. Since $aq_2 \vee q_1q_2 \leq q$ then we get $aq_2 \leq q$. Now since $(a \vee q_1)(b \vee q_2)a_3 \leq q$ and $(a \vee q_1)a_3 \nleq \delta(q)$, $(b \vee q_2)a_3 \nleq \delta(q)$, so $(a \vee q_1)(b \vee q_2) = ab \vee aq_2 \vee bq_1 \vee q_1q_2 \leq q$, by Lemma 5.16. So we conclude that $ab \leq q$, a contradiction. Hence $a_2a_3 \leq \delta(q)$ or $a_1a_3 \leq \delta(q)$. $\square$

Acknowledgments

The authors are thankful to the referee for his/her helpful comments which improved the paper.

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