Journal of Algebra and Related Topics
Vol.7, No 1, (2019), pp 35-44

THE k-th SPECTRAL MOMENT OF SIGNED COMPLETE GRAPHS

S. DALVANDI, F. HEYDARI *, AND M. MAGHASEDI

ABSTRACT. Let $\Gamma = (G, \sigma)$ be a signed graph, where G is the underlying simple graph with at least one edge and $\sigma : E(G) \longrightarrow \{-, +\}$ is the sign function on the edges of G. In this paper, we study the k-th spectral moment of (K_n, σ), for a signature σ. Also, we obtain the number of negative cycles in a signed complete graph whose negative edges induce the disjoint union of two distinct complete bipartite graphs.

1. INTRODUCTION

Let $G = (V(G), E(G))$ be a simple graph with the vertex set $V(G)$ and the edge set $E(G)$ where $|E(G)| \geq 1$. The *order* of G is defined to be $|V(G)|$ and if the degree of a vertex u of G is 1, then u is called a *pendant vertex*. A *paw graph* is a triangle with a pendant vertex. Let K_n denote the complete graph of order n. A complete bipartite graph with parts of sizes r and s is denoted by $K_{r,s}$.

A *signed graph* Γ is an ordered pair (G, σ), where $G = (V(G), E(G))$ is a simple graph (called the *underlying graph*), and let $\sigma : E(G) \longrightarrow \{-, +\}$ be a mapping defined on the edge set of G. Signed graphs were introduced by Harary [6] in connection with the study of theory of social balance in social psychology proposed by Heider [7]. Let (K_n, H^-) be a signed complete graph whose negative edges induce a subgraph H. If H is the disjoint union of two graphs G_1 and G_2, then we denote (K_n, H^-) by $(K_n, G_1^- \cup G_2^-)$. A *walk of length k* in a signed graph Γ is a sequence

MSC(2010): Primary: 05C22; Secondary: 05C50.
Keywords: Signed graph, Complete bipartite graph, Negative cycle.
Received: 25 June 2019, Accepted: 17 September 2019.
∗Corresponding author .

$v_0, v_1, \ldots, v_k$ of vertices which consecutive terms are adjacent. A walk is *closed* if its end points coincide. A walk is called *positive* if it contains an even number of negative edges, otherwise it is called *negative*. Let $w^+_{i,j}(k)$ be the number of positive walks of length k from v_i to v_j and let $w^-_{i,j}(k)$ be the number of negative walks. A cycle of length k is called a *k-cycle* and denoted by C_k. Let t^+_r (resp., t^-_r) be the number of positive (resp., negative) r-cycles in a signed graph Γ. A signed graph is called *balanced* if all of its cycles are positive; otherwise, it is *unbalanced*. The number of negative cycles in a signed complete graph have been studied extensively by many authors, see [9, 11]. The *adjacency matrix* of a signed graph $\Gamma = (G, \sigma)$ is a square matrix $A(\Gamma) = A(G, \sigma) = (a^\sigma_{ij})$, where $a^\sigma_{ij} = \sigma(v_i v_j)a_{ij}$ and $A(G) = (a_{ij})$ is the adjacency matrix of G. The nullity of a graph is the nullity of its adjacency matrix and we use $\mathrm{null}(G)$ for the nullity of graph G. If Γ is a signed graph, then $\varphi(\Gamma, \lambda)$ is the characteristic polynomial of $A(\Gamma)$ which referred to as the characteristic polynomial of Γ. The eigenvalues of the adjacency matrix of a graph are often referred to as the eigenvalues of the graph. The spectrum of a signed graph Γ is the set of all eigenvalues of Γ along with their multiplicities. The spectrum of graphs, in particular, signed graphs has been studied by many authors, for instance see [1, 3, 4]. If the distinct eigenvalues of Γ are $\lambda_1 > \cdots > \lambda_s$, and their multiplicities are $m(\lambda_1), \ldots, m(\lambda_s)$, then we write

$$\mathrm{Spec}\,(\Gamma) = \begin{pmatrix} \lambda_1 & \ldots & \lambda_s \\ m(\lambda_1) & \ldots & m(\lambda_s) \end{pmatrix}.$$

Let $\mathrm{tr}(A^k(\Gamma)) = \sum_{i=1}^n \lambda_i^k$ $(k = 0, 1, 2, \ldots)$ be the k-th spectral moment of a signed graph Γ with n vertices. In this paper, we study the k-th spectral moment of (K_n, σ), for a signature σ. Also, the number of negative cycles and the k-th spectral moment of $(K_n, K^-_{n_1,n_2} \cup K^-_{n_3,n_4})$ are investigated.

2. Negative Cycles in $(K_n, K^-_{n_1,n_2} \cup K^-_{n_3,n_4})$

In this section, we investigate the k-th spectral moment of (K_n, σ), for a signature σ. Also, we study some properties of $(K_n, K^-_{n_1,n_2} \cup K^-_{n_3,n_4})$. First of all we recall certain necessary results:

Lemma 2.1. [10, Theorem II.1] *Let Γ be a signed graph. Then the (i, j)-entry of the matrix $A^k(\Gamma)$ is $w^+_{i,j}(k) - w^-_{i.j}(k)$.*

Lemma 2.2. [4, Corollary 3.3] *Let Γ be a signed graph with n vertices and m edges. Then $\mathrm{tr}(A^2(\Gamma)) = 2m$ and $\mathrm{tr}(A^3(\Gamma)) = 6(t^+_3 - t^-_3)$.*

Lemma 2.3. [2, Lemma 1] *Let* $\Gamma = (K_n, \sigma)$ *be a signed complete graph. Then*

$$\mathrm{tr}(A^4(\Gamma)) = 8(t_4^+ - t_4^-) + n(n-1)(2n-3).$$

Lemma 2.4. [3, Theorem 1] *Let* (K_n, H^-) *be a signed complete graph and* $|V(H)| = k < n$. *Then* $m(-1) = n - k - 1 + \mathrm{null}(H)$.

Our first result on the 5-th spectral moment of (K_n, σ) is:

Theorem 2.5. *Let* $\Gamma = (K_n, \sigma)$ *be a signed complete graph. Then*

$$\mathrm{tr}(A^5(\Gamma)) = 10(t_5^+ - t_5^-) + 30(n-2)(t_3^+ - t_3^-).$$

Proof. By Lemma 2.1, we get

$$\mathrm{tr}(A^5(\Gamma)) = \sum_{i=1}^{i=n} \big(w_{i,i}^+(5) - w_{i.i}^-(5)\big).$$

There are three different types of closed walks of length 5 in Γ as follows:

Type 1. The 5-cycles: the walks of length 5 from the vertex v_i to the vertex v_i in a 5-cycle are as follows: $v_iv_jv_kv_lv_mv_i$ or $v_iv_mv_lv_kv_jv_i$, see Fig. 1. Thus the number of positive (resp., negative) closed walks of length 5 of this kind is $10t_5^+$ (resp., $10t_5^-$).

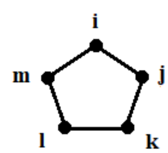

FIGURE 1. The 5-cycle.

Type 2. The triangles: the walks of length 5 from the vertex v_i to the vertex v_i in a triangle are as follows: $v_iv_jv_kv_iv_kv_i$; $v_iv_kv_jv_iv_jv_i$; $v_iv_jv_iv_jv_kv_i$; $v_iv_kv_iv_kv_jv_i$; $v_iv_jv_kv_jv_kv_i$; $v_iv_kv_jv_kv_jv_i$; $v_iv_jv_kv_iv_jv_i$; $v_iv_kv_j v_iv_kv_i$; $v_iv_jv_iv_kv_jv_i$ or $v_iv_kv_iv_jv_kv_i$, see Fig. 2. Hence the number of positive (resp., negative) closed walks of length 5 of this kind is $30t_3^+$ (resp., $30t_3^-$).

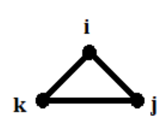

FIGURE 2. The triangle.

Type 3. The paw graphs: the closed walks of length 5 in a paw graph are as follows: $v_iv_jv_kv_lv_jv_i$; $v_iv_jv_lv_kv_jv_i$; $v_jv_kv_lv_jv_iv_j$; $v_jv_lv_kv_jv_iv_j$; $v_jv_iv_jv_lv_kv_j$; $v_jv_iv_jv_kv_lv_j$; $v_lv_jv_iv_jv_kv_l$; $v_lv_kv_jv_iv_jv_l$; $v_kv_lv_jv_iv_jv_k$ or $v_kv_jv_iv_jv_lv_k$, see Fig. 3. Since the number of balanced (resp., unbalanced) paw graphs in Γ is $3(n-3)t_3^+$ (resp., $3(n-3)t_3^-$), one can see that the number of positive (resp., negative) closed walks of length 5 of this kind is $30(n-3)t_3^+$ (resp., $30(n-3)t_3^-$).

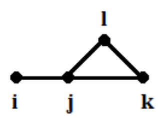

FIGURE 3. The paw graph.

Therefore, the following holds:

$$\sum_{i=1}^{i=n}\left(w_{i,i}^+(5)-w_{i.i}^-(5)\right)=10(t_5^+-t_5^-)+30(n-3)(t_3^+-t_3^-)+30(t_3^+-t_3^-),$$

as desired. □

Let (K_n,H^+) be a signed complete graph whose positive edges induce a subgraph H. We have the following example.

Example 2.6. Let (K_5,σ) be a signed complete graph. Then, by considering the results of [8] we have seven switching classes for (K_5,σ) with the following spectrums:

(1) $\mathrm{Spec}\,(K_5,C_5^-)=\begin{pmatrix}\sqrt{5} & 0 & -\sqrt{5}\\ 2 & 1 & 2\end{pmatrix}$,

(2) $\mathrm{Spec}\,(K_5,K_{1,2}^-)=\begin{pmatrix}3 & \frac{\sqrt{17}-1}{2} & -1 & -\frac{\sqrt{17}+1}{2}\\ 1 & 1 & 2 & 1\end{pmatrix}$,

(3) $\mathrm{Spec}\,(K_5,K_{1,2}^+)=\begin{pmatrix}\frac{\sqrt{17}+1}{2} & 1 & \frac{1-\sqrt{17}}{2} & -3\\ 1 & 2 & 1 & 1\end{pmatrix}$,

(4) $\mathrm{Spec}\,(K_5,K_{1,1}^-)=\begin{pmatrix}\frac{\sqrt{33}+1}{2} & 1 & -1 & \frac{1-\sqrt{33}}{2}\\ 1 & 1 & 2 & 1\end{pmatrix}$,

(5) $\mathrm{Spec}\,(K_5,K_{1,1}^+)=\begin{pmatrix}\frac{\sqrt{33}-1}{2} & 1 & -1 & -\frac{\sqrt{33}+1}{2}\\ 1 & 2 & 1 & 1\end{pmatrix}$,

and for two cases $(K_5,+)$ and $(K_5,-)$, the spectrums are well-known. We would like to obtain t_5^- in (K_5,σ). By Theorem 2.5 and Lemma 2.2 we conclude that $t_5^-=12,8,6,6,6,4,0$ in $(K_5,-)$, $(K_5,K_{1,2}^-)$, (K_5,C_5^-), $(K_5,K_{1,1}^-)$, $(K_5,K_{1,1}^+)$, $(K_5,K_{1,2}^+)$ and $(K_5,+)$, respectively.

Here, we need to introduce an additional notation. An $(n_1,\dots,n_k)$-choice of $X_1,\dots,X_k$ is the subset $S_i \subseteq X_i$, $i=1,\dots,k$ such that $|S_i| = n_i$, $i=1,\dots,k$. Now, we would like to obtain t_3^-, t_4^- and t_5^- in $(K_n, K^-_{n_1,n_2} \cup K^-_{n_3,n_4})$. Our second main result is:

Theorem 2.7. *Let $(K_n, K^-_{n_1,n_2} \cup K^-_{n_3,n_4})$ be a signed complete graph. Then the following statements hold:*

(1) $t_3^- = A + mB$,

(2) $t_4^- = (n-3)(A+mB) - 4C$,

(3) $t_5^- = (n^2 - 7n + 12)(A+mB) - 4(n+m-4)C$,

where $A = \sum_{1\le i<j<l\le 4} n_i n_j n_l$, $B = n_1n_2 + n_3n_4$, $C = \prod_{i=1}^4 n_i$ *and* $m = n - \sum_{i=1}^4 n_i$.

Proof. Let $\Gamma = (K_n, K^-_{n_1,n_2} \cup K^-_{n_3,n_4})$. Suppose that the set of vertices of K_{n_1,n_2} and K_{n_3,n_4} are partitioned into the parts X, Y and Z, W, respectively. Let $m = n - \sum_{i=1}^4 n_i$, and

$$U = V(K_n) \setminus (X \cup Y \cup Z \cup W).$$

Now, we determine the number of negative 3-cycles in Γ. It is not hard to see that Γ has no negative 3-cycle with exactly 3 negative edges. In the sequel we determine the number of negative 3-cycles in Γ with exactly one negative edge. The negative edges are between X, Y or between Z, W. Therefore,

$$t_3^- = n_1n_2(m + n_3 + n_4) + n_3n_4(m + n_1 + n_2) = A + mB,$$

where $A = \sum_{1\le i<j<l\le 4} n_i n_j n_l$, $B = n_1n_2 + n_3n_4$.

For the determination of t_4^-, one can choose four vertices of $V(K_n)$ and consider the signed graph K_4 induced by them and then find its number of negative 4-cycles. The partitions of 4 are $1+1+1+1$, $2+1+1$, $2+2$, $3+1$ and 4. So we have the following cases:

(1) $1+1+1+1$: We choose one vertex from U and consider a $(1,1)-$choice of X, Y or Z, W and one remaining vertex from the other parts. Thus the number of $(K_4, K^-_{1,1})$ induced by them is $mn_1n_2(n_3+n_4) + mn_3n_4(n_1+n_2)$.

The other choices contain no negative 4-cycle.

(2) $2+1+1$: Consider a $(1,1)-$choice of X, Y or Z, W and two remaining vertices from one of the other parts. So the number of $(K_4, K_{1,1}^-)$ induced by them is $n_1n_2\Big(\binom{n_3}{2}+\binom{n_4}{2}+\binom{m}{2}\Big)+n_3n_4\Big(\binom{n_1}{2}+\binom{n_2}{2}+\binom{m}{2}\Big)$.

Also, consider a $(2,1)-$choice of X, Y; Y, X; Z, W or W, Z, and one remaining vertex from the other parts. Hence the number of $(K_4, K_{1,2}^-)$ induced by them is $(n_3+n_4+m)\Big(\binom{n_1}{2}n_2+\binom{n_2}{2}n_1\Big)+(n_1+n_2+m)\Big(\binom{n_3}{2}n_4+\binom{n_4}{2}n_3\Big)$.

Note that the cases $2+2$, $3+1$ and 4 contain no negative 4-cycle. The number of negative 4-cycles in $(K_4, K_{1,1}^-)$ and $(K_4, K_{1,2}^-)$ are 2. Therefore, one can conclude that

$$t_4^- = (n-3)(A+mB) - 4C,$$

where $C = \prod_{i=1}^{4} n_i$.

Similarly, for the determination of t_5^-, we can choose five vertices of $V(K_n)$ and consider the signed graph K_5 induced by them and then find its number of negative 5-cycles. The partitions of 5 are $1+1+1+1+1$, $2+1+1+1$, $2+2+1$, $3+1+1$, $3+2$, $4+1$, 5. Thus we have the following cases:

(1) $1+1+1+1+1$: Consider all signed graphs K_5 obtained by choosing one vertex from each parts X, Y, Z, W and U. Thus we have mC signed complete graphs with two non-adjacent negative edges which they are switching equivalent of $(K_5, K_{1,2}^+)$.

(2) $2+1+1+1$: We have 4 cases:

Case 1. Consider a $(2,1,1,1)-$choice of X, Y, Z, W; Y, X, Z, W; Z, X, Y, W or W, X, Y, Z. Hence we have $\sum_{1\le i<j<l\le 4}\binom{n_s}{2}n_in_jn_l$, $(s\neq i,j,l; 1\le s\le 4)$, signed complete graphs which they are switching equivalent of $(K_5, K_{1,1}^+)$.

Case 2. Choose two vertices from U and consider a $(1,1,1)-$choice of X, Y, Z; X, Y, W; X, Z, W or Y, Z, W. Then the number of $(K_5, K_{1,1}^-)$ induced by them is $\binom{m}{2}\sum_{1\le i<j<l\le 4} n_in_jn_l$.

Case 3. Choose one vertex from U and consider a $(2,1)-$choice of X, Y; Y, X; Z, W or W, Z and the last vertex from the other parts. So the number of $(K_5, K_{1,2}^-)$ induced by them is

$m(n_3+n_4)\Big(\binom{n_1}{2}n_2+\binom{n_2}{2}n_1\Big)+m(n_1+n_2)\Big(\binom{n_3}{2}n_4+\binom{n_4}{2}n_3\Big)$.

Case 4. Choose one vertex from U and consider a $(1,1)-$choice of X, Y or Z, W and two remaining vertices from one of the other parts. Thus the number of $(K_5, K_{1,1}^-)$ induced by them is $mn_3n_4\Big(\binom{n_1}{2}+\binom{n_2}{2}\Big)+mn_1n_2\Big(\binom{n_3}{2}+\binom{n_4}{2}\Big)$.

(3) $2+2+1$: We have 3 cases:

Case 1. Choose two vertices from X or Y and two vertices from Z or W and the last vertex from the other parts expect U. Hence the number of $(K_5, K_{1,2}^-)$ induced by them is $\binom{n_3}{2}\Big(\binom{n_1}{2}(n_2+n_4)+\binom{n_2}{2}(n_1+n_4)\Big)+\binom{n_4}{2}\Big(\binom{n_1}{2}(n_2+n_3)+\binom{n_2}{2}(n_1+n_3)\Big)$.

Case 2. Choose two vertices from U and consider a $(2,1)-$choice of X, Y; Y, X; Z, W or W, Z. Then the number of $(K_5, K_{1,2}^-)$ induced by them is $\binom{m}{2}\Big(\binom{n_1}{2}n_2+\binom{n_2}{2}n_1+\binom{n_3}{2}n_4+\binom{n_4}{2}n_3\Big)$.

Case 3. Consider a $(2,2)-$choice of X, Y or Z, W and the last vertex from the other parts. So we have $\binom{n_1}{2}\binom{n_2}{2}(n_3+n_4+m)+\binom{n_3}{2}\binom{n_4}{2}(n_1+n_2+m)$, signed complete graphs which they are switching equivalent of $(K_5, K_{1,2}^-)$.

The other choices contain no negative 5-cycle.

(4) $3+1+1$: Consider a $(3,1)-$choice of X, Y; Y, X; Z, W or W, Z and the last vertex from the other parts. Hence we have $(n_3+n_4+m)\Big(\binom{n_1}{3}n_2+\binom{n_2}{3}n_1\Big)+(n_1+n_2+m)\Big(\binom{n_3}{3}n_4+\binom{n_4}{3}n_3\Big)$, signed complete graphs which they are switching equivalent of $(K_5, K_{1,1}^-)$.

Also we can consider a $(1,1)-$choice of X, Y or Z, W and three vertices from one of the other parts. Thus the number of $(K_5, K_{1,1}^-)$ induced by them is $n_1n_2\Big(\binom{n_3}{3}+\binom{n_4}{3}+\binom{m}{3}\Big)+n_3n_4\Big(\binom{n_1}{3}+\binom{n_2}{3}+\binom{m}{3}\Big)$.

Obviously, in this case the other choices contain no negative 5-cycle.

The cases $3+2$, $4+1$ and 5 contain no negative 5-cycle. By Example 2.6, we get the number of negative 5-cycles of (K_5, σ). Therefore, one may deduce that

$$t_5^- = (n^2 - 7n + 12)(A + mB) - 4(n + m - 4)C.$$

This completes the proof. □

Corollary 2.8. *Let* $\Gamma = (K_n, K^-_{n_1,n_2} \cup K^-_{n_3,n_4})$ *be a signed complete graph. Then the* k*-th spectral moment of* Γ *for* $k = 3, 4, 5$ *are as follows:*

(1) $\mathrm{tr}(A^3(\Gamma)) = (n-1)^3 - 12(A + mB) - (n-1)$,

(2) $\mathrm{tr}(A^4(\Gamma)) = (n-1)^4 - 16(n-3)(A + mB) + 64C + (n-1)$,

(3) $\mathrm{tr}(A^5(\Gamma)) = (n-1)^5 - 20(n^2 - 4n + 6)(A + mB) + 80(n + m - 4)C - (n-1)$,

where $A = \sum_{1\le i<j<l\le 4} n_i n_j n_l$, $B = n_1n_2 + n_3n_4$, $C = \prod_{i=1}^4 n_i$ *and* $m = n - \sum_{i=1}^4 n_i$.

Now, we have the following immediate result.

Corollary 2.9. *Let* $\Gamma = (K_n, K^-_{n_1,n_2} \cup K^-_{n_3,n_4})$ *be a signed complete graph. Then*

$$\varphi(\Gamma,\lambda) = (\lambda+1)^{n-5}\Big(\lambda^5 + (5-n)\lambda^4 + (10-4n)\lambda^3 + (4A + 4mB + 10 - 6n)\lambda^2 + (8A + 8mB - 16C + 5 - 4n)\lambda + 4(A + mB) - 16(m+1)C + 1 - n\Big),$$

where $A = \sum_{1\le i<j<l\le 4} n_i n_j n_l$, $B = n_1n_2 + n_3n_4$, $C = \prod_{i=1}^4 n_i$ *and* $m = n - \sum_{i=1}^4 n_i$.

Proof. Let $\Gamma = (K_n, K^-_{n_1,n_2} \cup K^-_{n_3,n_4})$. By Lemma 2.4 if $n > \sum_{i=1}^4 n_i$, then we have $m(-1) = n - 5$. If $n = \sum_{i=1}^4 n_i$, then two signed graphs Γ and $(K_n, K^-_{n_1,n_2,n_3})$ are switching equivalent. Since $\mathrm{null}(K_{n_1,n_2,n_3}) = n_1 + n_2 + n_3 - 3$, (see [5]) then by the Lemma 2.4 we get $m(-1) = n - 4$. Hence $m(-1) \ge n - 5$. Now, we obtain five remaining eigenvalues of Γ, say λ_1, λ_2, λ_3, λ_4 and λ_5. Hence Corollary 2.8, implies that

$$\sum_{i=1}^5 \lambda_i^2 = (n-1)^2 + 4,$$

$$\sum_{i=1}^5 \lambda_i^3 = (n-1)^3 - 12(A + mB) - 4,$$

$$\sum_{i=1}^{5}\lambda_i^4=(n-1)^4-16(n-3)(A+mB)+64C+4,$$

$$\sum_{i=1}^{5}\lambda_i^5=(n-1)^5-20(n^2-4n+6)(A+mB)+80(n+m-4)C-4.$$

Since $\sum_{i=1}^{5}\lambda_i=n-5$, it is not hard to see that $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ and λ_5 are the roots of the polynomial

$$\lambda^5+(5-n)\lambda^4+(10-4n)\lambda^3+(4A+4mB+10-6n)\lambda^2+$$

$$(8A+8mB-16C+5-4n)\lambda+4(A+mB)-16(m+1)C+1-n.$$

This completes the proof. □

Acknowledgments

The authors wish to thank Professor Saieed Akbari for his fruitful comments. The authors would also like to thank the referee for careful reading.

References

1. S. Akbari, F. Belardo, F. Heydari, M. Maghasedi and M. Souri, *On the largest eigenvalue of signed unicyclic graphs*, Linear Algebra and its Applications, **581** (2019), 145-162.
2. S. Akbari, S. Dalvandi, F. Heydari and M. Maghasedi, *On the eigenvalues of signed complete graphs*, Linear and Multilinear Algebra, **67** (2019), 433-441.
3. S. Akbari, S. Dalvandi, F. Heydari and M. Maghasedi, *On the multiplicity of* -1 *and* 1 *in signed complete graphs*, Utilitas Mathematica, accepted.
4. F. Belardo and P. Petecki, *Spectral characterizations of signed lollipop graphs*, Linear Algebra and its Applications, **480** (2015), 144-167.
5. D. Cvetkovic, P. Rowlinson and S. Simic, *An Introduction to the Theory of Graph Spectra*, Cambridge University Press, Cambridge, 2010.
6. F. Harary, *On the notion of balance in a signed graph*, Michigan Mathematical Journal, **2** (1953), 143-146.
7. F. Heider, *Attitude and cognitive organization*, Psychologia, **21** (1946), 107-112.
8. C.L. Mallows and N.J.A. Sloane, *Two-graphs, switching classes and euler graphs are equal in number*, Siam Journal Applications Mathematics, **28** (1975), 876-880.
9. D.R. Popescu and I. Tomescu, *Negative cycles in complete signed graphs*, Discrete Applied Mathematics, **68** (1996), 145-152.
10. T. Zaslavsky, *Matrices in the theory of signed simple graphs*, Advances in Discrete Mathematics and Applications, Mysore, 2008, Ramanujan Math. Soc. Lect. Notes Ser., vol. 13, Ramanujan Math. Soc., 2010, pp. 207-229. arXiv: 1303.30837 (2013).

11. T. Zaslavsky, *Negative (and positive) circles in signed graphs: A problem collection*, AKCE International Journal of Graphs and Combinatorics, **15** (2018), 31-48.

Soudabeh Dalvandi
Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran.
Email: s.dalvandi@kiau.ac.ir

Farideh Heydari
Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran.
Email: f-heydari@kiau.ac.ir

Mohammad Maghasedi
Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran.
Email: maghasedi@kiau.ac.ir