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ON THE CLASS OF SUBSETS OF RESIDUATED LATTICE WHICH INDUCES A CONGRUENCE RELATION

H. HARIZAVI

ABSTRACT. In this manuscript, we study the class of special subsets connected with a subset in a residuated lattice and investigate some related properties. We describe the union of elements of this class. Using the intersection of all special subsets connected with a subset, we give a necessary and sufficient condition for a subset to be a filter. Finally, by defining some operations, we endow this class with a residuated lattice structure and prove that it is isomorphic to the set of all congruence classes with respect to a filter.

1. INTRODUCTION

The concept of residuated lattice was firstly introduced by M. Ward and R. P. Dilworth [14] as generalization of ideals of rings. The properties of a residuated lattice were presented in [9]. Recently, these structures have been studied in [5] and [8]. The quotient residuated lattice with respect to a filter was defined and studied in [12]. In 2009, a class of special subset connected with an order filter of a MV-algebra was defined and studied by Colin G. Bailey (see ([3]). In this paper, following [3], we consider a class of special subsets connected with a subset of a residuated lattice and investigate some related properties. We describe the union of two of these subsets. We consider the intersection of all special subsets in a residuated lattice and investigate some

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related properties. Also, we give a characterization of this intersection. Finally, we consider a filter with an additional condition, namely, the complement-closed and prove that for any complement-closed filter F there is a close connection between the class of special subsets connected with F and the set of all congruence classes induced by F in a residuated lattice.

2. Preliminaries

We first recall some basic definitions and theorems which required in the sequel. For more details we refer the reader to [2, 8, 12].

Definition 2.1. [2] A residuated lattice is an algebra $(L; \wedge, \vee, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ satisfying the following:
(i) $(L; \wedge, \vee, 0, 1)$ is a bounded lattice;
(ii) $(A; \odot, 1)$ is a commutative ordered monoid;
(iii) $\odot$ and $\wedge$ form an adjoint pair, i.e. $a \leq b \rightarrow c$ if and only if $a \odot b \leq c$ for all $a, b, c \in L$.

In the sequel, a residuated lattice $(L; \wedge, \vee, \odot, \rightarrow, 0, 1)$ is represented by its support set L unless otherwise stated.

Theorem 2.2. [8, 12] *Let $x, y.z \in L$. Then we have the following rules of calculus:*
(r_1) $1 \rightarrow x = x,\ x \rightarrow x = 1,\ x \rightarrow 1 = 1, 0 \rightarrow x = 1$;
(r_2) $x \leq y$ *if and only if* $x \rightarrow y = 1$;
(r_3) $x \odot y \leq x, y$, *hence* $x \odot y \leq x \wedge y$ *and* $x \odot 0 = 0$;
(r_4) $x \rightarrow y = 1$ *and* $y \rightarrow x = 1$ *imply* $x = y$;
(r_5) *if* $x \leq y$ *then* $z \rightarrow x \leq z \rightarrow y$ *and* $y \rightarrow z \leq x \rightarrow z$;
(r_6) $x \leq y \rightarrow x$;
(r_7) $x \odot (x \rightarrow y) \leq y$;
(r_8) $x \leq (x \rightarrow y) \rightarrow y$;
(r_9) $((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y$;
(r_{10}) $x \odot (y \rightarrow z) \leq y \rightarrow (x \odot z) \leq (x \odot y) \rightarrow (x \odot z)$;
(r_{11}) $x \rightarrow y \leq (x \odot z) \rightarrow (y \odot z$;
(r_{12}) $x \leq y$ *implies* $x \odot z \leq y \odot z$;
(r_{13}) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z) = (x \odot y) \rightarrow z$;
(r_{14}) $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)$;
(r_{15}) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$;
(r_{16}) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$;
(r_{17}) $x \odot (y \wedge z) \leq (x \odot y) \wedge (x \odot z)$;
(r_{18}) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$;
(r_{19}) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$.

Definition 2.3. [13] A non-empty subset F of L is called a *filter* if
$(F1)$ $1 \in F$;
$(F2)$ if $a \in F$ and $a \leq b$, then $b \in F$;
$(F3)$ if $a, b \in F$, then $a \odot b \in F$.

Theorem 2.4. [13] *A non-empty subset F of L is a filter of L if and only if it satisfies the following conditions:*
$(F1)$ $1 \in F$;
$(F4)$ $x \in F$ *and* $x \to y \in F$ *imply* $y \in F$.

Definition 2.5. [13] A filter F of L is called a *prime* filter if $x \vee y \in F$ implies $x \in F$ or $y \in F$ for all $x, y \in L$.

Theorem 2.6. [13] *Let F be a filter of L. Define the relation $\equiv_F$ on L by*

$$x \equiv_F y \text{ if and only } \; x \to y \in F \textit{ and } y \to x \in F.$$

Then $\equiv_F$ is a congruence relation on L.

For every congruence relation $\equiv_F$ and $x \in L$, we denote the equivalence class of x by x/F and the set of all classes by L/F.

Theorem 2.7. [13] *Let F be a filter of L. Then L/F, endowed with the natural operations induced from those L, become a residuated lattice which is called the quotient residuated lattice with respect to F.*

3. Main results

In this section, we define the special subset of L and investigate some related properties. In the sequel, we denote the complement of a subset E by E^c.

Definition 3.1. For any non-empty subset E of L and for any $a \in L$, we denote

$$E_a := \begin{cases} E & \text{if } \; a \in E \\ \{x \in L \mid \; x \to a \in E^c\} & \text{if } \; a \in E^c. \end{cases}$$

Proposition 3.2. *If E is a non-empty subset of L, then we have*
(i) $1 \in E_a$ *for all* $a \in E^c$;
(ii) $1 \in E$ *if and only if* $a \notin E_a$ *for all* $a \in E^c$.

Proof. (i) Let $a \in E^c$. By Theorem 2.2(r_1), $1 \to a = a$, and so by Definition 3.1, we get $1 \in E_a$.
(ii) Using the rule $a \to a = 1$, the result is obvious. □

Proposition 3.3. *If F is a filter of L, then*

$$(\forall a, b \in L) \quad a \leq b \Rightarrow F_b \subseteq F_a.$$

Proof. Let F be a filter of L. We investigate the following cases:

Case 1: $a \in F$.

In this case, from $a \leq b$ we get $b \in F$ and so $F_a = F = F_b$.

Case 2: $a \in F^c$ and $b \in F$.

In this case, we have $F_b = F$. Now, let $x \in F_b$. If $x \to a \in F$, then since F is a filter and $x \in F$, we get $a \in F$, which is a contradiction. Hence $x \to a \in F^c$ and so $x \in F_a$. Therefore $F_b \subseteq F_a$.

Case 3: $a, b \in F^c$.

Assume that $x \in F_b$. Then $x \to b \in F^c$. Applying Theorem 2.2(r_5) to $a \leq b$, we obtain $x \to a \leq x \to b$. If $x \to a \in F$, then, since F is a filter, we get $x \to b \in F$, which is a contradiction. Hence $x \to a \in F^c$ and so $x \in F_a$. Therefore $F_b \subseteq F_a$. □

Corollary 3.4. *If F is a filter of L, then for all $a \in L$, $F \subseteq F_a$.*

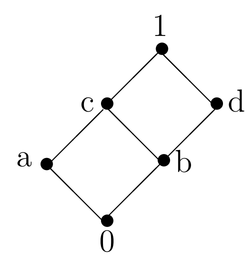
Proof. From $a \leq 1$ and $F_1 = F$ the result holds by Proposition 3.3(ii). □

The following example shows that the condition "F being a filter" in Proposition 3.3 is necessary.

Example 3.5. [7] Let $L = \{0, a, b, c, d, 1\}$ be the residuated lattice defined by the following tables:

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1



$E := \{c, d, 1\}$ is not a filter of L because $d \to b = c \in E$ but $b \notin E$. By a simple calculation, we obtain $E_b = \{1, a\}$ and $E_c = E$. Hence, $b \leq c$ does not imply $E_c \subseteq E_b$.

Definition 3.6. A subset E of L is said to be $\wedge$-*closed* if $a, b \in E$ implies $a \wedge b \in E$.

Proposition 3.7. *Let F be a filter of L. Then*

$$F_a \cup F_b \subseteq F_{a \wedge b} \quad \text{for all } a, b \in L.$$

In addition, if F is a $\wedge$-closed, then

$$F_a \cup F_b = F_{a \wedge b} \quad \text{for all } a, b \in L.$$

Proof. Using Proposition 3.3, it follows from $a \wedge b \leq a$ that $F_a \subseteq F_{a \wedge b}$. Similarly, we have $F_b \subseteq F_{a \wedge b}$ and so $F_a \cup F_b \subseteq F_{a \wedge b}$. To show the second part of proposition it suffices to prove the inverse inclusion. Assume that $x \in F_{a\wedge b}$ and consider the following cases:

Case (1) $a \in F^c$ or $b \in F^c$.

In this case, since $a \wedge b \leq a, b$ we have $a \wedge b \in F^c$ and so from $x \in F_{a\wedge b}$, we get $x \to (a \wedge b) \in F^c$. Hence, by Theorem 2.2($r_{19}$), we have $(x \to a) \wedge (x \to b) \in F^c$. Thus it follows from F is a $\wedge$-closed that $x \to a \in F^c$ or $x \to b \in F^c$. Thus $x \in F_a$ or $x \in F_b$ and so $x \in F_a \cup F_b$. Therefore $F_{a\wedge b} \subseteq F_a \cup F_b$.

Case (2) $a, b \in F$.

In this case, since F is a $\wedge$-closed, we have $a \wedge b \in F$ and so by Definition 3.1, we get $F_a \cup F_b = F = F_{a\wedge b}$. □

Definition 3.8. For any non-empty subset E of L, we denote

$$\Gamma(E) := \{x \in L | x \to a \in E^c, \forall a \in E^c\}.$$

Proposition 3.9. *Let E be a non-empty sunset of L. Then the following statements hold:*

(i) $\Gamma(E) = \bigcap_{a\in E^c} E_a$.

(ii) $1 \in E$ *if and only if* $\Gamma(E) \subseteq E$.

Proof. (i) By Definitions 3.8 and 3.1, the result is obvious.

(ii) Let $1 \in E$. Assume to the contrary that $\Gamma(E) \nsubseteq E$. Then there exists $x \in \Gamma(E)$ such that $x \in E^c$. Hence it follows from $x \in \Gamma(E)$ that $x \in E_x$, that is, $x \to x \in E^c$. Thus $1 \in E^c$, which is a contradiction. Therefore $\Gamma(E) \subseteq E$.

Conversely, by Proposition 3.2(i), the proof is straightforward. □

The following theorem introduces the relationship between $\Gamma(E)$ and filter E.

Theorem 3.10. *Let E be a subset of L. Then the following are equivalent:*

(i) E is a filter of L;

(ii) $1 \in E$ and $E \subseteq \Gamma(E)$;

(iii) $1 \in E$ and $\Gamma(E) = E$.

Proof. $(i) \Rightarrow (ii)$ Let E be a filter of L. Clearly, $1 \in E$. Assume that $x \in E$ such that $x \notin E_a$ for some $a \in E^c$. It follows from $x \notin E_a$ that $x \to a \in E$. Thus, since E is a filter and $x \in E$, we get $a \in E$, which is a contradiction. Therefore $x \in E_a$ and so $E \subseteq E_a$ for any $a \in E^c$. Therefore $E \subseteq \Gamma(E)$.

$(ii) \Rightarrow (iii)$ By Proposition 3.9(ii), since $1 \in E$, we get $\Gamma(E) \subseteq E$ and so by hypothesis $\Gamma(E) = E$.

$(iii) \Rightarrow (i)$ Assume to the contrary that E is not a filter of L. Then there exist $x, y \in L$ such that $x \to y \in E$ and $x \in E$ but $y \notin E$. Since $x \in E \subseteq \Gamma(E)$, we get $x \in E_y$. This implies $x \to y \in E^c$, which is a contradiction. Therefore E is a filter of L. □

The next theorem gives a characterization of $\Gamma(F)$.

Theorem 3.11. *Let F be a filter of L. Then*

$$\Gamma(F) = \{x \in L | x \odot e \in F, \forall e \in F\}.$$

Proof. We have

$$\begin{aligned}
x \in \Gamma(F) &\Leftrightarrow x \in \bigcap_{a \in E^c} F_a && \text{by Proposition 3.9(i)}\\
&\Leftrightarrow (\forall a \in F^c)\ x \in F_a\\
&\Leftrightarrow (\forall a \in F^c)\ x \to a \in F^c && \text{by Definition 3.1}\\
&\Leftrightarrow (\forall a \in F^c)\ (\forall e \in F)\ e \nleq x \to a && \text{F is a filter}\\
&\Leftrightarrow (\forall a \in F^c)\ (\forall e \in F)\ e \odot x \nleq a && \text{by Definition 2.1(iii)}\\
&\Leftrightarrow (\forall e \in F)\ x \odot e \in F && \text{by } x \odot e \leq e\\
&\Leftrightarrow x \in \{x \in L | x \odot e \in F, \forall e \in F\}.
\end{aligned}$$

Therefore $\Gamma(F) = \{x \in L | x \odot e \in F, \forall e \in F\}$.

□

To describe the connection between the special subsets and the congruence classes, we define:

Definition 3.12. A non-empty subset E of L is called complement-closed if

$$(\forall a \in E^c)\ (\exists x \in E^c)\ x \to a \in E^c.$$

Lemma 3.13. *For any filter F of L, the following are equivalent:*
(i) F is complement-closed;
(ii) For any $a \in F^c$, $F_a \neq F$.

Proof. $(i) \Rightarrow (ii)$ Let $a \in F^c$. Then by Definition 3.12, there exists $x \in F^c$ such that $x \to a \in F^c$. This implies that $x \in F_a$. But $x \notin F$, hence $F_a \neq F$.

$(ii) \Rightarrow (i)$ Let $a \in F^c$. Then by (ii) and Corollary 3.4, we have $F \subsetneq F_a$. Hence there exists $x \in F_a$ such that $x \in F^c$. Thus from $x \in F_a$, we conclude $x \to a \in F^c$. Therefore F is a complement-closed. □

Corollary 3.14. *Let F be a complement-closed filter of L. Then*

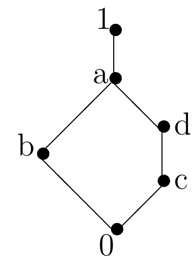
$$(\forall a, b \in L) \quad F_a = F_b \Rightarrow a, b \in F \text{ or } a, b \in F^c.$$

Proof. Using Lemma 3.13(ii) and Definition 3.1, the proof is straightforward. □

Example 3.15. Let $L = \{0, a, b, c, d, 1\}$ be the residuated lattice defined by the following tables (see [8]):

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	b	c	c	1
b	c	a	1	c	c	1
c	b	a	b	1	a	1
d	b	a	b	a	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	b	d	d	a
b	0	b	b	0	0	b
c	0	d	0	d	d	c
d	0	d	0	d	d	d
1	0	a	b	c	d	1



Consider the filters $E := \{a, 1\}$ and $F := \{a, b, 1\}$. It is easily to check that:
(i) $E_b = \{a, c, d, 1\}$, $E_c = \{a, b, 1\}$ and $E_d = \{a, b, 1\}$. Hence E is a complement-closed filter;
(ii) $F_c = F$ and so F is not a complement-closed filter.

Theorem 3.16. *If F is a filter of L, then we have*

$$(\forall a, b \in F^c) \quad F_a \subseteq F_b \;\Leftrightarrow b \rightarrow a \in F.$$

Proof. ($\Rightarrow$) Let $F_a \subseteq F_b$ for some $a, b \in F^c$. By Proposition 3.2(ii), since $1 \in F$, we have $b \notin F_b$. From this follows that $b \notin F_a$. Therefore $b \rightarrow a \in F$.

($\Leftarrow$) Let $b \rightarrow a \in F$ for some $a, b \in F^c$. Suppose that $x \in F_a$. Then $x \rightarrow a \in F^c$. If $x \notin F_b$, then $x \rightarrow b \in F$. By Theorem 2.2(r_{14}), we have $b \rightarrow a \leq (x \rightarrow b) \rightarrow (x \rightarrow a)$. Since F is a filter, it follows from $b \rightarrow a \in F$ that $(x \rightarrow b) \rightarrow (x \rightarrow a) \in F$. Then from $x \rightarrow b \in F$, we get $x \rightarrow a \in F$, which is a contradiction. Hence $x \in F_b$ and so $F_a \subseteq F_b$. □

As a consequence from Theorem 3.16, we have:

Corollary 3.17. *For any filter F of L, we have*

$$(\forall a, b \in F^c) \quad F_a = F_b \;\Leftrightarrow b \rightarrow a \in F \text{ and } a \rightarrow b \in F.$$

Notation 3.18. For any non-empty subset E of L, we denote

$$L(E) := \{E_a : a \in L\}.$$

It is clear that $E \in L(E)$.

To introduce some operations on $L(E)$, we state and prove some rules of calculus in residuated lattice as follows:

Lemma 3.19. *For any $a, b, c \in L$, we have*

(i) $a \odot (a \to b) \leq a \wedge b$;
(ii) $a \to b \leq a \wedge c \to b \wedge c$.

Proof. (i) Using Theorem 2.2(r_3, r_7), the proof is straightforward.

(ii) By Definition 2.1(iii), it suffices to show that $(a \to b) \odot (a \wedge c) \leq b \wedge c$. For this purpose, we have

$$\begin{aligned}(a \to b) \odot (a \wedge c) &\leq ((a \to b) \odot a) \wedge ((a \to b) \odot c), && \text{by Theorem 2.2}(r_{17}) \\ &\leq (a \wedge b) \wedge c, && \text{by (i) and Theorem 2.2}(r_3) \\ &\leq b \wedge c.\end{aligned}$$

□

Lemma 3.20. *Let F be a complement-closed filter of L. Then we have*

$$(\forall a, b \in L) \quad F_a = F_b \Rightarrow a \to b \in F \text{ and } b \to a \in F.$$

Proof. Let $F_a = F_b$. Then by Lemma 3.14, we get $a, b \in F$ or $a, b \in F^c$. If $a, b \in F$, then, since F is a filter, it follows from $b \leq a \to b$ that $a \to b \in F$. Similarly, from $a \leq b \to a$ we get $b \to a \in F$. If $a, b \in F^c$, then by Corollary 3.17, we also conclude $b \to a \in F$ and $a \to b \in F$. □

In order to endow $L(F)$ with a residuated lattice structure, we define operations "$\sqcap, \sqcup, \hookrightarrow, \otimes$" on $L(E)$ as follows:

Proposition 3.21. *Let F be a complement-closed filter of L. Then the operations "$\sqcap, \sqcup, \hookrightarrow, \otimes$" on $L(E)$ defined by, $(\forall F_a, F_b \in L(F))$,*
(i) $F_a \sqcap F_b = F_{a \wedge b}$,
(ii) $F_a \sqcup F_b = F_{a \vee b}$,
(iii) $F_a \otimes F_b = F_{a \odot b}$,
(iv) $F_a \hookrightarrow F_b = F_{a \to b}$
are well-defined.

Proof. Let $F_a = F_c$ and $F_b = F_d$ for some $a, b, c, d \in L$. Then by Lemma 3.20, we have

$$(a \to c \in F \text{ and } c \to a \in F) \quad ; \quad (b \to d \in F \text{ and } d \to b \in F).$$

(i) By Lemma 3.19(ii), we have

$$a \to c \leq a \wedge b \to c \wedge b;$$
$$b \to d \leq b \wedge c \to d \wedge c.$$

Then, since F is a filter, it follows from $a \to c \in F$ and $b \to d \in F$ that

$$a \wedge b \to c \wedge b \in F; \tag{3.1}$$
$$b \wedge c \to d \wedge c \in F. \tag{3.2}$$

By Theorem 2.2(r_{15}), we have $a \wedge b \to c \wedge b \leq (c \wedge b \to d \wedge c) \to (a \wedge b \to d \wedge c)$. Then it follows from (3.1) that $(b \wedge c \to d \wedge c) \to (a \wedge b \to d \wedge c) \in F$. Thus from (3.2), we get

$$a \wedge b \to d \wedge c \in F. \tag{3.3}$$

By a similar argument as above, we obtain

$$d \wedge c \to a \wedge b \in F. \tag{3.4}$$

Since F is a filter, it follows from (3.3) and (3.4) that

$$a \wedge b \in F \Leftrightarrow c \wedge d \in F.$$

Thus by Corollary 3.17, we conclude $F_{a \wedge b} = F_{c \wedge d}$ and so $F_a \sqcap F_b = E_c \sqcap F_d$, e.i. the operation $\sqcap$ is well-defined.

(ii) Applying Theorem 2.2(r_{14}) on $c \leq b \vee c$, we get $a \to c \leq a \to (b \vee c)$. Then this follows from $a \to c \in F$ that $a \to (b \vee c) \in F$. By Theorem 2.2(r_{18}), we have

$$\begin{aligned} (a \vee b) \to (b \vee c) &= (a \to (b \vee c)) \wedge (b \to (b \vee c)) \\ &= (a \to (b \vee c)) \wedge 1 \\ &= a \to (b \vee c). \end{aligned}$$

Then from $a \to (b \vee c) \in F$, we get

$$(a \vee b) \to (b \vee c) \in F.$$

By a similar argument as above, using $b \to d \in F$, we obtain

$$(b \vee c) \to (c \vee d) \in F.$$

Using Theorem 2.2(r_{14}), similar to the proof of (i), we get

$$(a \vee b) \to (c \vee d) \in F \text{ and } (c \vee d) \to (a \vee b) \in F. \tag{3.5}$$

Since F is a filter, it follows from (3.5) that

$$a \vee b \in F \Leftrightarrow c \vee d \in F.$$

Thus by Corollary 3.17, we conclude $F_{a \vee b} = F_{c \vee d}$ and so $F_a \sqcup F_b = E_c \sqcup F_d$, e.i. the operation $\sqcup$ is well-defined.

(iii) Applying Theorem 2.2(r_{11}) on $b \to d \in F$ and $c \to a \in F$, we get

$$b \to d \leq a \odot b \to a \odot d;$$

$$a \to c \leq a \odot d \to c \odot d.$$

Hence, since F is a filter, we get

$$a \odot b \to a \odot d \in F.$$

$$a \odot d \to c \odot d \in F.$$

Similar to the proof of (i), we have

$$a \odot b \in F \Leftrightarrow c \odot d \in F.$$

Thus by Corollary 3.17, we conclude $F_{a\odot b} = F_{c\odot d}$ and so $F_a \otimes F_b = F_c \otimes F_d$, e.i. the operation $\otimes$ is well-defined.

(iv) We have

$$\begin{aligned} b \to d &\leq (a \to b) \to (a \to d) \text{ by Theorem 2.2}(r_{14}) \\ &\leq (a \to b) \to ((c \to a) \to (c \to d)) \text{ by Theorem 2.2}(r_{14}, r_5) \\ &\leq (c \to a) \to ((a \to b) \to (c \to d)) \text{ by Theorem 2.2}(\mathrm{r}_{13}). \end{aligned}$$

Then it follows from $b \to d \in F$ that $(c \to a) \to ((a \to b) \to (c \to d)) \in F$ and so from $c \to a \in F$, we conclude

$$(a \to b) \to (c \to d) \in F. \tag{3.6}$$

Similarly, we obtain

$$(c \to d) \to (a \to b) \in F. \tag{3.7}$$

Applying Lemma 3.20, from (3.6) and (3.7), we obtain $F_{a\to b} = F_{c\to d}$ and so $F_a \hookrightarrow F_b = F_c \hookrightarrow F_d$, e.i. the operation $\hookrightarrow$ is well-defined. □

Theorem 3.22. *Let F be a complement-closed filter of L. Then $(L(F); \sqcap, \sqcup, \otimes, \hookrightarrow, F_0, F)$ is a residuated lattice, where the operations "$\sqcap, \sqcup, \otimes, \hookrightarrow$" are defined as Proposition 3.21.*

Moreover, $L(F) \simeq L/F$, where L/F is the quotient residuated lattice with respect to F.

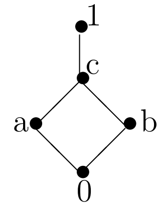
Proof. Define the mapping $\varphi : L(F) \to L/F$ by $\varphi(F_a) = a/F$. Assume that $F_a = F_b$ for some $a, b \in L$. Then by Lemma 3.20, we have $a \to b \in F$ and $b \to a \in F$. This implies that $a/F = b/F$ and so $\varphi(F_a) = \varphi(F_b)$. Hence φ is well-defined. Now, let $\varphi(F_a) = \varphi(F_b)$. Then $a/F = b/F$ and so by the property of congruence classes, we obtain $a \to b \in F$ and $b \to a \in F$. Hence $a \in F$ if and only if $b \in F$. Then by Corollary 3.17, we get $F_a = F_b$. Therefore φ is injective. Obviously, φ is onto. We note that the operations defined on $L(F)$ and L/F are the natural operations induced from L. Therefore φ is a bijective function preserving the operations of $L(F)$. Then $L(F) \simeq L/F$ and so, since L/F is a residuated lattice, it follows that $(L(F); \sqcap, \sqcup, \otimes, \hookrightarrow, F_0, F)$ is a residuated lattice too. □

We now give an example to illustrate the previous theorem.

Example 3.23. [7] Let $L = \{0, a, b, c, 1\}$ be the residuated lattice defined by the following tables:

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1



It is not difficult to check that $F := \{c, 1\}$ is a complement-closed filter of L. By a simple calculation, we obtain:

$$L(F) = \{F_1 = F_c = F,\ F_a = \{b, c, 1\},\ F_b = \{a, c, 1\},\ F_0 = \{a, b, c, 1\}\};$$
$$L/F = \{1/F = c/F = F,\ a/F = \{a\},\ b/F = \{b\},\ 0/F = \{0\}\};$$
$$L(F) \cong L/F \text{ in which } F_x \longmapsto x/F\ (\forall x \in L).$$

The following example shows that the condition complement-closed in Theorem 3.22 is necessary.

Example 3.24. Let $L = \{0, a, b, c, d, 1\}$ be the residulated lattice as in Example 3.15. Then $F := \{a, b, 1\}$ is a filter of L, but is not a complement-closed because $x \to c = 1 \in F$ for any $x \in F^c$. By a simple calculation, we obtain

$$\mid L(F) \mid = \mid \{F_0 = F_a = F_b = F_c = F_d = F_1 = F\} \mid = 1,$$
$$\mid L/F \mid = \mid \{a/F = b/F = 1/F = \{a, b, 1\}, 0/F = \{0, c, d\}\} \mid = 2.$$

From this follows that $L(F) \ncong L/F$.

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References

1. G. Birkhoff, *Lattice Theory,* American Mathematical Society Colloquium Publication, third edition, 1967.
2. T. S. Blyth and M. F. Janovitz, *Residuation Theory,* Pregamon Press, 1972.
3. C. G. Bailey, *Prime filters in MV-algebras,* arXiv:0907.3328v1 [math.RA] , 2009.
4. E. Eslami, H. Shojaei and J. J. Buckley, *Filters in Residuated Algebras*, Southeast Asian Bull. Math., **27** (2003), 429-438.
5. N. Galatos, P. Jipsen, T. Kowalski and H. Ono, *Residuated lattices: an algebraic glimpse at substructural logics*, Elsevier, 2007.
6. P. Hájek, *Matemathematics of Fuzzy Logic,* Kluwer Academic Publishers, Dordrecht, 1998.
7. A. Iorgulescu, *Classes of BCK algebras-Part III,* Preprint Series of the Institute of Mathematics of the Romanian Academy, preprint, **3** (2004), 1-37.
8. T. Kowalski and H. Ono, *Residuated Lattices:* An algebraic glimpse at logics without contraction, 2001.

9. H. Ono, *Substructural logics and residuated lattices- an introduction, 50 Years of Studia Logica*, Trends in Logic, Kluwer Academic Publischer, 21 (2003), 193-228.
10. D. Piciu, *Algebras of Fuzzy Logic,* Editura Universitaria Craiova, 2007.
11. S. Tanaka, *A new class of algebras,* Math. Seminar Notes, **3** (1975), 37-43.
12. E. Turunen, *Mathematics behind fuzzy logic*, Physica-Verlag, Heidelberg, 1999.
13. E. Turunen, *Boolean deductive systems of BL-algebras,* Archive for Mathematical Logic, **40** (2001), 467-473.
14. M. Ward and R. P. Dilworth, *lattices,* rans. Amer. Math. Soc., **45** (1939), 335-354.

Habib Harizavi
Department of Mathematics, Shahid Chamran University of Ahvaz, Ahvaz, Iran
Email: harizavi@scu.ac.ir