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On the complete convergence of channel hardening and favorable propagation properties in massive-MIMO communications systems

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Abstract. Massive MIMO is known as a core technology for future 5G networks. The major advantage of massive MIMO over the conventional MIMO systems is that different mobile users are allowed to communicate in the same time-frequency resources while the resultant severe interferences can be eliminated using linear signal processing schemes. This is a consequence of the favorable propagation condition and channel hardening which are known as two basic limiting results in mathematics. In this paper we propose new stochastic convergence proofs for these limiting results in terms of the complete convergence in a massive MIMO system with uncorrelated Rayleigh fading.

Keywords: Massive MIMO systems, favorable propagation condition, channel hardening, stochastic convergence, Rayleigh fading.
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1 Introduction

By the increase in popularity of portable communication devices such as smart phones and tablets, demand for high quality wireless services at any time is exponentially increasing. However, the available frequency and

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time network resources remain limited. Therefore, any approach that theoretically increases data rate and reliability could potentially be a candidate to the development of the next generation of wireless networks. Massive MIMO which refers to implementing large antenna arrays at the base stations (BS), is known as a core technology for the development of next generation wireless networks and could potentially increase data rates of mobile users' equipment (UEs) by simultaneously serving multiple UEs communicating in the same time-frequency resources [8].

In the literature, several authors (e.g., [8], [10] and [11]) have discussed on the theoretical aspects of the massive MIMO technology and its appropriateness for using in 5G cellular communications systems. In particular, as illustrated in [11], when the number of antennas at the BS is very large the wireless channels between the BS and UEs show interesting properties such as the channel hardening and the favorable propagation condition for some specific fading scenarios. The channel hardening helps to mitigate the severe effects of small-scale fading while the favorable propagation condition makes it possible to eliminate multi-user interference at the BS using a simple linear processing scheme. These important properties are exploited recently by researchers to develop new communications protocols for massive MIMO wireless systems with Rayleigh fading channels (see, e.g., [5], [7] and [12]). From a mathematical perspective, the channel hardening and favorable propagation properties are known as limiting results on random channel gains when the number of antennas at the BS goes to infinity. Despite the importance of these basic properties, detailed convergence results have yet not been presented in the literature. Although it is always possible to check the correctness of these limiting properties by applying Monte-Carlo simulation at finite large numbers of BS antennas, for a rigorous investigation of these properties in the case of infinite number of BS antennas, we need to use the stochastic convergence theory. In the literature, some authors resort to the strong law of large numbers as a reasoning tool to get an almost sure (*a.s.*) convergence (see, e.g., [2] and [11]).

In this paper, the channel hardening and favorable propagation properties are studied in the sense of complete convergence (*c.c.* in brief), for independent Rayleigh fading channels. In the state-of-the-art in engineering systems, such as communication networks, the *c.c.* concept is not yet taken into account. This concept originally introduced by [6], is a stronger convergence result than the almost sure convergence. Hence this paper presents tighter convergence analysis for the channel hardening and favorable propagation properties in the sense of the complete convergence.

The proposed convergence analysis requires determining the higher order moments of radio channel gains which are directly calculated and utilized.

The rest of paper is organized as follows. In Section 2, some preliminaries including a description of the data transmission and channel model, propagation conditions in massive MIMO systems, and a literature review of the existing convergence results are provided. New convergence results in terms of the complete convergence are proposed in Section 3, and the concluding remarks are given in Section 4.

2 System model

We consider a wireless cellular communication system consisting of L separate communication cells, in which a cell refers to a geometrical area with one multiple antenna BS at the center and numerous single antenna UEs distributed in the area. The number of antennas per BS and the number of UEs per cell are denoted with M and K, respectively.

2.1 Channel and data transmission model

There exist M wireless channels between each UE and M antennas of each BS whose fading gains are assumed to be independent and identically distributed (i.i.d.) random variables (r.v.s). Packing these M channels into a single vector with M elements gives what is called the channel vector between each UE and each BS. For instance, the vector $h_{jlk} \in C^{M\times 1}$ represents the channel gain vector between UE k from cell l and the BS j as follows:

$$h_{jlk} \triangleq \begin{bmatrix} h_{jlk}^{(1)} & h_{jlk}^{(2)} & \cdots & h_{jlk}^{(M)} \end{bmatrix}^T. \tag{1}$$

In isotropic scattering environments with none-line-of-sight path, the r.v.s $h_{jlk}^{(1)}, h_{jlk}^{(2)}, \ldots, h_{jlk}^{(M)}$ are modeled as complex circularly symmetric Gaussian r.v.s with zero-mean and the same variances ([3], [11]) denoted by C_{jlk}, i.e.,

$$h_{jlk}^{(r)} \sim CN(0, C_{jlk}), \quad r = 1, 2, \ldots, M\ . \tag{2}$$

Since the magnitude variable $|h_{jlk}^{(r)}|$ has Rayleigh distribution, the described channels in (2) are also known as Rayleigh fading channels. Because of i.i.d. assumption on the channel gains, the covariance matrix of the random vector h_{jlk} is diagonal, and its distribution is given by:

$$h_{jlk} \sim CN(0, C_{jlk}.I_M)\ , \tag{3}$$

where I_M denotes the $M \times M$ identity matrix.

We consider uplink data transmission in a cellular network where at one time slot each active UE transmits one data symbol. Since each antenna of the BS j receives symbols from all UEs in all cells, hence the received symbol vector $Y_j \in C^{M\times 1}$ at BSj at one time slot is represented by:

$$Y_j = \sum_{l=1}^{L}\sum_{k=1}^{K}(\sqrt{p_{lk}}.h_{jlk}.x_{lk}) + N_j \ , \tag{4}$$

where $x_{lk} \in C^1$ is the data symbol transmitted by the UE k from cell l with power p_{lk}, and $N_j \in C^{M\times 1}$ is the additive white noise vector. Eq.(4) can be rearranged as follows:

$$Y_j = \sum_{k=1}^{K}\left(\sqrt{p_{jk}}.h_{jjk}.x_{jk}\right) + \sum_{\substack{l=1\\ l\neq j}}^{L}\sum_{k=1}^{K}(\sqrt{p_{lk}}.h_{jlk}.x_{lk}) + N_j \ . \tag{5}$$

The first desired term in (5) is the combination of symbols received from UEs inside the cell j. The second term is the combination of symbols received at BS j from UEs of the other cells considered as the inter-cell interference.

2.2 Propagation conditions in massive MIMO systems

In the literature, a scenario in which the UE channel gain vectors are assumed orthogonal is known as the favorable propagation condition ([10] and [11]). In this case, to detect symbols of the UE m in cell j, the BS j can eliminate the interference terms in equation (5) by implementing a linear signal processing scheme like:

$$\tilde{Y}_{jm} \triangleq \frac{h_{jjm}^{H}}{\|h_{jjm}\|^2}.Y_j = \sqrt{p_{jm}}.x_{jm} + \frac{h_{jjm}^{H}}{\|h_{jjm}\|^2}.N_j \ , \tag{6}$$

where the superscript H indicates the Hermitian transpose operation and $\|.\|$ is the Euclidean norm. Of course, using (6) one may use a maximum likelihood (ML) detection rule to detect data symbols of the UE m. Unfortunately, in real wireless communication scenarios channel orthogonality will never be exactly satisfied. In such scenarios, asymptotically favorable propagation condition states that with a huge number of antennas at the BS (massive MIMO case), users' channel vectors are almost orthogonal and therefore a linear signal processing scheme is still able to eliminate the interferences of undesired UEs effectively [11]. Asymptotically favorable propagation condition and the channel hardening property for radio channels with independent Rayleigh fading have been already studied in the literature [2] [11]. Here, we briefly introduce these properties in terms of two lemmas (Lemma 1 and Lemma 2) as follows:

Lemma 1. (Asymptotically favorable propagation condition) *Let* $h_{jl_1k_1} = [h^{(1)}_{jl_1k_1}, \ldots, h^{(M)}_{jl_1k_1}]^T$ *and* $h_{jl_2k_2} = [h^{(1)}_{jl_2k_2}, \ldots, h^{(M)}_{jl_2k_2}]^T$ *represent random channel gain vectors of two different users in the massive MIMO system, i.e,* $(l_1, k_1) \neq (l_2, k_2)$. *These vectors are statistically independent and distributed as* $h_{jl_1k_1} \sim CN(0, C_{jl_1k_1}.I_M)$ *and* $h_{jl_2k_2} \sim CN(0, C_{jl_2k_2}.I_M)$. *The following a.s. convergence holds for* $M \to \infty$*:*

$$\frac{1}{M} h^H_{jl_1k_1}.h_{jl_2k_2} = \frac{1}{M} \sum_{r=1}^{M} \left({h^{(r)}_{jl_1k_1}}^* . h^{(r)}_{jl_2k_2} \right) \xrightarrow{a.s.} 0 \; . \tag{7}$$

Proof. For the assumed channel gains the entries $\left({h^{(r)}_{jl_1k_1}}^* . h^{(r)}_{jl_2k_2} \right), \forall r = 1, 2, \ldots, M$ appeared in the summation term in (7) construct a sequence of independent random variables with the same distribution function and zero first-order moment ($E\left[{h^{(r)}_{jl_1k_1}}^* . h^{(r)}_{jl_2k_2} \right] = E\left[{h^{(r)}_{jl_1k_1}}^* \right] . E\left[h^{(r)}_{jl_2k_2} \right] = 0$). Therefore using the strong law of large numbers (see, [9, theorem 4.3.3]), the *a.s.* convergence in (7) is expected. □

Lemma 2. (Channel hardening property) *Let* $h_{jlk} = \left[h^{(1)}_{jlk}, \ldots, h^{(M)}_{jlk} \right]^T$ *represent the channel gain vector of a typical user in the massive MIMO system with the probability distribution function given by (3). The following a.s. convergence holds for* $M \to \infty$*:*

$$\frac{1}{M} \|h_{jlk}\|^2 = \frac{1}{M} \sum_{r=1}^{M} \left| h^{(r)}_{jlk} \right|^2 \xrightarrow{a.s.} C_{jlk} \; . \tag{8}$$

Proof. Since the entries $|h^{(r)}_{jlk}|^2, \forall r = 1, 2, \ldots, M$ appeared in the summation term in (8) build a sequence of independent random variables with the same distribution function and finite first-order moment (because from (3) we know that the expectation $E[|h^{(r)}_{jlk}|^2] = C_{jlk} < \infty$ is bounded), therefore the *a.s.* convergence in (8) is expected using the strong law of large numbers (see, [9, theorem 4.3.3]). □

Using the channel hardening property in (8) the receiver side (BSs) can perform some signal processing tasks to get a more stable signal quality for the UE m in the sense that the received signal-to-interference-plus-noise ratio (SINR) of this user does not fluctuate with small-scale fading in the channel. To more clarify on the effects of the channel hardening and favorable propagation properties on the massive MIMO transmission scheme, in the sequel it is briefly shown that by utilizing (7) and (8) a linear signal processing scheme is still capable of eliminating the undesired interferences in the BS receiver. Let us start with the following linear operation on the received vector Y_j in (5) to detect symbols of the UE m:

$$\frac{h_{jjm}^{H}}{\|h_{jjm}\|^2}.Y_j = \sqrt{p_{jm}}\, x_{jm} + \sum_{\substack{k=1\\k\neq m}}^{K}\left(\sqrt{p_{jk}}\,.\frac{h_{jjm}^{H}.\,h_{jjk}}{M}.\frac{M}{\|h_{jjm}\|^2}.x_{jk}\right)$$
$$+\sum_{\substack{l=1\\l\neq j}}^{L}\sum_{k=1}^{K}\left(\sqrt{p_{lk}}.\frac{h_{jjm}^{H}.\,h_{jlk}}{M}.\frac{M}{\|h_{jjm}\|^2}.x_{lk}\right)+\frac{h_{jjm}^{H}}{\|h_{jjm}\|^2}.N_j\ . \quad (9)$$

The second and the third terms in RHS of (9) are the interferences created by the undesired UEs on the intended UE m. Using the convergence results in (7) and (8), for $M \to \infty$ we get $\frac{h_{jjm}^{H}.\,h_{jjk}}{M} \xrightarrow{a.s.} 0, \forall m \neq k$, $\frac{h_{jjm}^{H}.\,h_{jlk}}{M} \xrightarrow{a.s.} 0, \forall j \neq l$, $\frac{\|h_{jjm}\|^2}{M} \xrightarrow{a.s.} C_{jjm}$ and therefore in the asymptotic case $M \to \infty$, we have:

$$\frac{h_{jjm}^{H}}{\|h_{jjm}\|^2}.Y_j \xrightarrow{a.s.} \sqrt{p_{jm}}.x_{jm} + \frac{h_{jjm}^{H}}{\|h_{jjm}\|^2}.N_j\ . \quad (10)$$

As a result of (10), in a massive MIMO system with very large M (e.g., $M > 100$), one can use the following equalized received symbol which is now ready for the ML detection of UE m symbols:

$$\tilde{Y}_{jm} \triangleq \frac{h_{jjm}^{H}}{\|h_{jjm}\|^2}.Y_j \approx \sqrt{p_{jm}}.x_{jm} + \frac{h_{jjm}^{H}}{\|h_{jjm}\|^2}.N_j\ . \quad (11)$$

Eq. (11) states that in a massive MIMO system the UE m can communicate with the BS in the same time-frequency resources that the other UEs are communicating with the BS, and the BS can still remove the resultant interferences by implementing a linear signal processing scheme, This leads to an excellent symbol detection performance similar to that of a UE which attends in the network alone.

3 New convergence results

In this section, we provide new convergence results for the Lemmas 1 and 2 in the previous section. In particular, we show that tighter convergence results can be obtained for the channel hardening and favorable propagation properties in terms of the complete convergence.

3.1 Basic convergence concepts

We start with a short review of the following two basic convergence concepts.

Definition 1. [9, p. 50] A sequence $\{U_n, n \geq 1\}$ of r.v.s is said to be completely convergent to constant θ if:

$$\lim_{N\to\infty} \sum_{n=N}^{\infty} \Pr\{|U_n - \theta| > \epsilon\} = 0, \quad \forall\epsilon > 0\,, \tag{12}$$

where $\Pr\{\cdot\}$ denotes probability. Equivalently, if the sequence $\{U_n, n \geq 1\}$ converges completely to θ, then the series $\sum_{n=1}^{\infty} \Pr\{|U_n - \theta| > \epsilon\}$ converges for every $\epsilon > 0$, i.e., [9, p. 52]:

$$\sum_{n=1}^{\infty} \Pr\{|U_n - \theta| > \epsilon\} < \infty, \quad \forall\epsilon > 0\,. \tag{13}$$

Definition 2. [9, p. 30] A sequence $\{U_n, n \geq 1\}$ of r.v.s is said to be almost surely (certainly) convergent to constant θ if:

$$\lim_{N\to\infty} \Pr\left\{ \bigcup_{n=N}^{\infty} (|U_n - \theta| > \epsilon) \right\} = 0, \quad \forall\epsilon > 0\,. \tag{14}$$

Remark 1. *Since the following inequality holds*

$$\Pr\left\{ \bigcup_{n=N}^{\infty} (|U_n - \theta| > \epsilon) \right\} \leq \sum_{n=N}^{\infty} \Pr\{|U_n - \theta| > \epsilon\}, \quad \forall\epsilon > 0\,, \tag{15}$$

hence we can always get (14) from (12), but the vice versa is not true. To more clarify, here we show a simple example. Let us consider a uniform probability space $\zeta \in [0, 1]$ and define the random sequence $\{U_n\}_{n=1}^{\infty}$ with

$$U_n = \begin{cases} 1, & 0 < \zeta < \frac{1}{n}\,, \\ 0, & \textit{otherwise}. \end{cases} \tag{16}$$

Since U_n is a sequence of dependent r.v.s, one can easily show that

$$\lim_{N\to\infty} \Pr\left\{\bigcup_{n=N}^{\infty} (|U_n| > \epsilon)\right\} = \lim_{N\to\infty} \frac{1}{N} = 0\,, \tag{17}$$

which means that the a.s. convergence holds. However, $\sum_{n=N}^{\infty} \Pr\{|U_n| > \epsilon\} = \sum_{n=N}^{\infty} \left(\frac{1}{n}\right)$ *diverges, and therefore the sequence does not converge completely. This example shows that for random sequences with dependent elements, (12) cannot be resulted from (14), and therefore the c.c. is stronger than the a.s. convergence. In particular, when the sequence* $\{U_n, n \geq 1\}$ *consists of i.i.d. random variables, both (12) and (14) are equivalent and the c.c. and a.s. convergence imply the same meaning.*

Remark 2. *The difference between the c.c. and a.s. convergence concepts may be intuitively explained as follows. From (12) we can infer that for all large* $n\,(n \geq N)$ *the event* $(|U_n - \theta| > \epsilon)$ *happens with zero probability, which means that all random variables* U_n *are close to* θ *enough. However, when (14) holds such implication is not necessarily true, i.e., from (14) we cannot say that for all large* $n\,(n \geq N)$ *the event* $(|U_n - \theta| > \epsilon)$ *happens with zero probability (there may be a large* n *for which the variable* U_n *is not close to* θ *enough).*

3.2 On the complete convergence of favorable propagation condition

Theorem 1. *Let*

$$h_{jl_1k_1} = [h_{jl_1k_1}^{(1)}, \ldots, h_{jl_1k_1}^{(M)}]^T \quad \textit{and} \quad h_{jl_2k_2} = [h_{jl_2k_2}^{(1)}, \ldots, h_{jl_2k_2}^{(M)}]^T,$$

represent channel gain vectors of two different users in the massive MIMO system, i.e., $(l_1, k_1) \neq (l_2, k_2)$*. The elements of the vectors* $h_{jl_1k_1}$ *and* $h_{jl_2k_2}$ *are independent zero-mean complex normal r.v.s with bounded second-order moments. For these vectors, the following complete convergence holds for* $M\to\infty$*:*

$$\frac{1}{M} h_{jl_1k_1}^H . h_{jl_2k_2} = \frac{1}{M} \sum_{r=1}^{M} \left({h_{jl_1k_1}^{(r)}}^{*} . h_{jl_2k_2}^{(r)} \right) \xrightarrow{c.c.} 0\,. \tag{18}$$

Proof. For simplicity, we use the following new brief notation:

$$X = [x_1 \ \cdots \ x_M]^T \triangleq h_{jl_1k_1} = \left[h_{jl_1k_1}^{(1)} \ \cdots \ h_{jl_1k_1}^{(M)} \right]^T\,, \tag{19}$$

$$Y = [y_1 \ \cdots \ y_M]^T \triangleq h_{jl_2k_2} = \left[h_{jl_2k_2}^{(1)} \ \ldots \ h_{jl_2k_2}^{(M)} \right]^T\,, \tag{20}$$

$$V_M \triangleq \frac{1}{M}\sum_{r=1}^{M}\left(h_{jl_1k_1}^{(r)}{}^{*}.h_{jl_2k_2}^{(r)}\right) = \frac{1}{M}\sum_{r=1}^{M}\left(x_r{}^{*}.y_r\right)\ . \tag{21}$$

In the considered model, X and Y are independent random vectors whose elements $x_1, x_2, \ldots, x_M$ and $y_1, y_2, \ldots, y_M$ are zero-mean independent Gaussian r.v.s. Of course, the elements of the sequence $\{V_M\}, \forall M = 1, 2, \ldots$ are dependent r.v.s, and therefore for this sequence the *c.c.* concept is different from the *a.s.* convergence as explained in Subsection 3.1.

Using the notation (19-21), the claim in (18) may be re-expressed as the complete convergence $V_M \xrightarrow{c.c.} 0$ for $M \to \infty$. To prove this claim, we need to show that the series $\sum_{M=1}^{\infty} \Pr\{|V_M| > \epsilon\}$ converges for every $\epsilon > 0$ (see, Eq. (13)). The proof is started with the evaluation of the fourth-order moment of the r.v. V_M as follows:

$$\begin{aligned}
E\Big[|V_M|^4\Big] &= E[V_M.V_M{}^{*}.V_M.V_M{}^{*}] \\
&= E\Bigg[\left(\frac{1}{M}\sum_{r_1=1}^{M}\left(x_{r_1}^{*}.y_{r_1}\right)\right).\left(\frac{1}{M}\sum_{r_2=1}^{M}\left(x_{r_2}^{*}.y_{r_2}\right)\right)^{*} \\
&\quad .\left(\frac{1}{M}\sum_{r_3=1}^{M}\left(x_{r_3}^{*}.y_{r_3}\right)\right).\left(\frac{1}{M}\sum_{r_4=1}^{M}\left(x_{r_4}^{*}.y_{r_4}\right)\right)^{*}\Bigg] \\
&= \frac{1}{M^4}\sum_{r_1=1}^{M}\sum_{r_2=1}^{M}\sum_{r_3=1}^{M}\sum_{r_4=1}^{M} E\big[x_{r_1}^{*}.x_{r_2}.x_{r_3}^{*}.x_{r_4}.y_{r_1}.y_{r_2}^{*}.y_{r_3}.y_{r_4}^{*}\big],
\end{aligned} \tag{22}$$

where x^* denotes the complex conjugate of x, and $E[.]$ is the expectation operator. The nested sums in (22) have in general M^4 components, however in our setup most of components are zero because the r.v.s $\{x_r\}$ and $\{y_r\}$ are independent and zero-mean. Still, the following components are non-zero:

$$\begin{aligned}
&\sum_{r_1=1}^{M}\sum_{r_2=1}^{M}\sum_{r_3=1}^{M}\sum_{r_4=1}^{M} E\left[x_{r_1}^{*}.x_{r_2}.x_{r_3}^{*}.x_{r_4}.y_{r_1}.y_{r_2}^{*}.y_{r_3}.y_{r_4}^{*}\right] \\
&= \underbrace{\sum_{r_1=1}^{M} E\left[x_{r_1}^{*}.x_{r_1}.x_{r_1}^{*}.x_{r_1}.y_{r_1}.y_{r_1}^{*}.y_{r_1}.y_{r_1}^{*}\right]}_{r_4=r_3=r_2=r_1} \\
&\quad + \underbrace{\sum_{r_1=1}^{M}\sum_{\substack{r_3=1\\ r_3\neq r_1}}^{M} E\left[x_{r_1}^{*}.x_{r_1}.x_{r_3}^{*}.x_{r_3}.y_{r_1}.y_{r_1}^{*}.y_{r_3}.y_{r_3}^{*}\right]}_{r_4=r_3\neq r_1=r_2}
\end{aligned}$$

$$+\underbrace{\sum_{r_1=1}^{M}\sum_{\substack{r_2=1\\r_2\neq r_1}}^{M} E\left[x_{r_1}^*.x_{r_2}.x_{r_1}^*.x_{r_2}.y_{r_1}.y_{r_2}^*.y_{r_1}.y_{r_2}^*\right]}_{r_4=r_2\neq r_1=r_3}$$

$$+\underbrace{\sum_{r_1=1}^{M}\sum_{\substack{r_2=1\\r_2\neq r_1}}^{M} E\left[x_{r_1}^*.x_{r_2}.x_{r_2}^*.x_{r_1}.y_{r_1}.y_{r_2}^*.y_{r_2}.y_{r_1}^*\right]}_{r_3=r_2\neq r_1=r_4}$$

$$=\sum_{r_1=1}^{M} E\left[|x_{r_1}|^4\right].E\left[|y_{r_1}|^4\right]$$

$$+\sum_{r_1=1}^{M}\sum_{\substack{r_3=1\\r_3\neq r_1}}^{M}\left(E\left[|x_{r_1}|^2\right].E\left[|x_{r_3}|^2\right].E\left[|y_{r_1}|^2\right].E\left[|y_{r_3}|^2\right]\right)$$

$$+\sum_{r_1=1}^{M}\sum_{\substack{r_2=1\\r_2\neq r_1}}^{M}\left(E\left[\left(x_{r_1}^*\right)^2\right].E\left[x_{r_2}^2\right].E\left[y_{r_1}^2\right].E\left[\left(y_{r_2}^*\right)^2\right]\right)$$

$$+\sum_{r_1=1}^{M}\sum_{\substack{r_2=1\\r_2\neq r_1}}^{M}\left(E\left[|x_{r_1}|^2\right].E\left[|x_{r_2}|^2\right].E\left[|y_{r_1}|^2\right].E\left[|y_{r_2}|^2\right]\right). \qquad (23)$$

To calculate (23) one needs to compute the moments appearing in nested sums. Since all elements of the vector X are zero-mean complex normal r.v.s with the same variances, for the r^{th} element of this vector we consider the distribution $x_r \sim CN\left(0, 2\sigma_1^2\right)$, where $2\sigma_1^2\,(=C_{jl_1k_1})$ denotes the variance of distribution. In a similar manner, the r^{th} element of the vector Y has distribution $y_r \sim CN\left(0, 2\sigma_2^2\right)$ with the variance $2\sigma_2^2\,(=C_{jl_2k_2})$. Using the notations $x_r = \Re\{x_r\} + j.\Im\{x_r\}$ and $y_r = \Re\{y_r\} + j.\Im\{y_r\}$ and considering the fact that the real and imaginary parts of x_r and y_r are independent real zero-mean normal r.v.s with variances σ_1^2 and σ_2^2 respectively, we can get:

$$E\left[\left(x_{r_1}^*\right)^2\right] = E\left[x_{r_2}^2\right] = E\left[y_{r_1}^2\right] = E\left[\left(y_{r_2}^*\right)^2\right] = 0\,. \qquad (24)$$

Moreover, the positive r.v.s $|x_r|$ and $|y_r|$ has Rayleigh distributions with parameters $\sqrt{\sigma_1^2}$ and $\sqrt{\sigma_2^2}$, respectively (see, [13, Eq. (6-70)]). In this case, the n^{th} order moment of $|x_r|$, i.e., $E\left[|x_r|^n\right]$ is given by [13, Eq. (5-76)]:

$$E[|x_r|^n] = \begin{cases}(1\times 3\times\cdots\times n)\,\sigma_1^2\sqrt{\frac{\pi}{2}}, & n = 2k+1\,,\\ 2^k k!\sigma_1^{2k}, & n = 2k\,.\end{cases} \qquad (25)$$

A similar equation like (25) can be written for the n^{th} order moment of $|y_r|$, i.e., $E[|y_r|^n]$ with the parameter $\sqrt{\sigma_2^2}$. Using (25), the absolute moments appearing in (23) are given by:

$$E[|x_{r_1}|^4] = 8\sigma_1^4 \, , \tag{26}$$

$$E[|y_{r_1}|^4] = 8\sigma_2^4 \, , \tag{27}$$

$$E[|x_{r_1}|^2] = E[|x_{r_2}|^2] = E[|x_{r_3}|^2] = 2\sigma_1^2 \, , \tag{28}$$

$$E[|y_{r_1}|^2] = E[|y_{r_2}|^2] = E[|y_{r_3}|^2] = 2\sigma_2^2 \, . \tag{29}$$

By substituting (24) and (26-29) into (23), the forth-order moment in (22) can be obtained as:

$$\begin{aligned} E\left[|V_M|^4\right] &= \frac{64}{M^4}\sum_{r_1=1}^{M}\sigma_1^4\sigma_2^4 + \frac{16}{M^4}\sum_{r_1=1}^{M}\sum_{\substack{r_3=1\\ r_3\neq r_1}}^{M}\sigma_1^4\sigma_2^4 + \frac{16}{M^4}\sum_{r_1=1}^{M}\sum_{\substack{r_2=1\\ r_2\neq r_1}}^{M}\sigma_1^4\sigma_2^4 \\ &= \frac{32(M+1)}{M^3}\sigma_1^4\sigma_2^4 \, . \end{aligned} \tag{30}$$

From Bienaymé inequality [14] we get:

$$\Pr\left\{|V_M| > \epsilon\right\} \le \left(\frac{1}{\epsilon^4}\right) E\left[|V_M|^4\right] = \frac{32\sigma_1^4\sigma_2^4}{\epsilon^4}\left(\frac{1}{M^2} + \frac{1}{M^3}\right) \, . \tag{31}$$

From (31), we get the following limited bound:

$$\begin{aligned} \sum_{M=1}^{\infty}\Pr\left\{|V_M| > \epsilon\right\} &\le \frac{32\sigma_1^4\sigma_2^4}{\epsilon^4}\left(\sum_{M=1}^{\infty}\frac{1}{M^2} + \sum_{M=1}^{\infty}\frac{1}{M^3}\right) \\ &= \frac{32\sigma_1^4\sigma_2^4}{\epsilon^4}\left(\frac{\pi^2}{6} + \zeta(3)\right) < \infty \, , \end{aligned} \tag{32}$$

where in (32), $\zeta(3) \cong 1.202$ is the Riemann zeta function, and it is assumed that the second-order moments of channel gains are bounded, i.e., $\sigma_1^2, \sigma_2^2 < \infty$. As a result of (32), the series $\sum_{M=1}^{\infty}\Pr\left\{|V_M| > \epsilon\right\}$ always converges and therefore the proof is completed. □

3.3 On the complete convergence of channel hardening property

Theorem 2. *Let $h_{jlk} = \left[h_{jlk}^{(1)}, \ldots, h_{jlk}^{(M)}\right]^T$ represent the channel gain vector of a typical user in the massive MIMO system with the probability distribution function indicated by (3). The following complete convergence holds for $M \to \infty$:*

$$\frac{1}{M}\|h_{jlk}\|^2 = \frac{1}{M}\sum_{r=1}^{M}\left|h_{jlk}^{(r)}\right|^2 \xrightarrow{c.c.} C_{jlk} \, . \tag{33}$$

Proof. For simplicity, using the brief notation (19), we define the sequence of r.v.s $\{W_M\}, \forall M = 1, 2, \ldots$ as below:

$$W_M \triangleq \frac{1}{M}\sum_{r=1}^{M}\left|h_{jlk}^{(r)}\right|^2 = \frac{1}{M}\sum_{r=1}^{M}|x_r|^2 , \tag{34}$$

hence, the aim is to prove the claim $W_M \xrightarrow{c.c.} \left(C_{jlk} = 2\sigma^2\right)$ for $M \to \infty$. It is worth noting that $\{W_M\}$ is a sequence of dependent r.v.s, and therefore for this sequence the *c.c.* concept is different from the *a.s.* convergence as explained in Subsection 3.1. To prove the complete convergence $W_M \xrightarrow{c.c.} 2\sigma^2$ for $M \to \infty$, we equivalently show that the series $\sum_{M=1}^{\infty} \Pr\left\{\left|W_M - 2\sigma^2\right| > \epsilon\right\}$ converges for every $\epsilon > 0$ (see, Eq. (13)). The proof is started by calculating the forth-order moment $E\left[\left|W_M - 2\sigma^2\right|^4\right]$. Since the r.v. W_M takes positive real values, the quantity $W_M - 2\sigma^2$ is also real, and therefore its fourth-order moment is given by:

$$\begin{aligned} E\left[\left|W_M - 2\sigma^2\right|^4\right] &= E\left[\left(W_M - 2\sigma^2\right)^4\right] \\ &= E\left[W_M^4\right] - 8\sigma^2.E\left[W_M^3\right] + 24\sigma^4.E\left[W_M^2\right] - 32\sigma^6.E[W_M] \\ &\quad + 16\sigma^8 . \end{aligned} \tag{35}$$

To compute the moments $E[W_M^n], n = 1, 2, 3, 4$ appearing in (35), we first obtain the probability density function (PDF) of the r.v. W_M in (34). Since $x_r \sim CN\left(0, 2\sigma^2\right)$ we know that the r.v. $|x_r|^2$ has exponential distribution with parameter $\beta = 1/2\sigma^2$ (see, [13, p.162]). In this case, since all r.v.s $|x_r|^2, \forall r = 1, 2, \ldots, M$ have the same distribution, from [1, Theorem 3.1] we can deduce that the r.v. $Z \triangleq \sum_{r=1}^{M}|x_r|^2 \sim Erlang(M, \beta)$ has the Erlang distribution with the following PDF:

$$f_Z(z) = \frac{\beta^M.z^{M-1}.e^{-\beta.z}}{(M-1)!} , \tag{36}$$

therefore, the r.v. $W_M = \frac{Z}{M}$ has the PDF:

$$f_{W_M}(w) = M.f_Z(M.w) = \frac{M^M.\beta^M.w^{M-1}.e^{-M\beta w}}{(M-1)!} . \tag{37}$$

Using (37), we can now compute the n^{th} order moment $E[W_M^n]$ as follows:

$$E\left[W_M^n\right] = \int_0^\infty w^n.f_{W_M}(w).dw = \frac{M^M.\beta^M}{(M-1)!}\int_0^\infty w^{n+M-1}.e^{-M\beta w}.dw . \tag{38}$$

Using [4, Eq.(2.321)], the integral in the right hand side of (38) is given by:

$$\begin{aligned}\int_0^\infty w^{n+M-1}e^{-M\beta w}dw &= \left[e^{-M\beta w} \sum_{k=0}^{n+M-1} \frac{(-1)^k k! \binom{n+M-1}{k}}{(-M\beta)^{k+1}} w^{n+M-1-k} \right]_{w=0}^{\infty} \\ &= \frac{(n+M-1)!}{(M\beta)^{n+M}} \,, \end{aligned} \tag{39}$$

hence, we get:

$$E\left[W_M^n\right] = \frac{M^M.\beta^M}{(M-1)!} \left(\frac{(n+M-1)!}{(M\beta)^{n+M}} \right) . \tag{40}$$

From (40), and for the special cases $n = 1, 2, 3, 4$ we obtain:

$$E[W_M] = \frac{1}{\beta} \,, \tag{41}$$

$$E[W_M^2] = \frac{M+1}{M\beta^2} \,, \tag{42}$$

$$E[W_M^3] = \frac{M(M+1)(M+2)}{(M\beta)^3} \,, \tag{43}$$

$$E[W_M^4] = \frac{M(M+1)(M+2)(M+3)}{(M\beta)^4} \,. \tag{44}$$

By substituting (41-44) in to (35), we get:

$$E\left[\left|W_M - 2\sigma^2\right|^4\right] = \frac{3}{M^2.\beta^4} + \frac{6}{M^3.\beta^4} \,. \tag{45}$$

From Bienaymé inequality ([14]) and using (45), $\Pr\left\{\left|W_M - 2\sigma^2\right| > \epsilon\right\}$ can be bounded as:

$$\begin{aligned}\Pr\left\{\left|W_M - 2\sigma^2\right| > \epsilon\right\} &\le \left(\frac{1}{\epsilon^4}\right).E\left[\left|W_M - 2\sigma^2\right|^4\right] \\ &= \left(\frac{3}{\epsilon^4.\beta^4}\right).\left(\frac{1}{M^2}\right) + \left(\frac{6}{\epsilon^4.\beta^4}\right).\left(\frac{1}{M^3}\right), \end{aligned} \tag{46}$$

hence, we obtain the following limited bound:

$$\begin{aligned}\sum_{M=1}^{\infty} \Pr\left\{\left|W_M - 2\sigma^2\right| > \epsilon\right\} &\le \frac{3}{\epsilon^4.\beta^4} \sum_{M=1}^{\infty} \frac{1}{M^2} + \frac{6}{\epsilon^4.\beta^4} \sum_{M=1}^{\infty} \frac{1}{M^3} \\ &= \frac{3}{\epsilon^4.\beta^4}.\frac{\pi^2}{6} + \frac{6}{\epsilon^4.\beta^4}.\zeta(3) < \infty \,. \end{aligned}$$

(47)

As a result of (47), the series $\sum_{M=1}^{\infty} \Pr\left\{\left|W_M - 2\sigma^2\right| > \epsilon\right\}$ always converges and therefore the proof is completed. □

3.4 Final remark

As the final note here we comment on the practical importance of the new proposed convergence results. At first, we should note that the whole massive MIMO success almost relies on the channel hardening and favorable propagation properties which are known as two limiting results. In the literature it is accepted that almost sure convergence is tight enough in practice for massive MIMO and nobody has even challenged this accepted rule-of-thumb. Considering that massive MIMO with very large antenna array might be implemented in very sensitive applications, e.g., in wireless surgeries, one needs to be fully confident that the favorable propagation condition and channel hardening always hold. Almost sure convergence does not guarantee that these two properties never break down. According to the definition of almost sure convergence (Eq. (14)) and Remark 2, it is probable to encounter finite number of situations that the almost sure convergence does not hold. From an obsessive point of view, much tighter convergence is required to prove the eligibility of massive MIMO wireless communication system for those sensitive applications. We think the complete convergence is a stronger convergence result which may be useful in this context.

4 Conclusions

The channel hardening and favorable propagation properties in massive MIMO cellular systems are two basic important limiting results which are utilized in the development of many communications systems protocols. In this paper, we first studied *a.s.* convergence results for these limiting properties when wireless channels are subject to uncorrelated Rayleigh fading (Lemmas 1 and 2). We then proposed new convergence results in terms of the complete convergence (Theorems 1 and 2). The proofs are based on the exact calculation of higher order moments of radio channel gains up to order eight. The proposed proofs provide a stronger convergence result than the almost sure convergence, and hence it is expected to be utilized in future performance analysis of massive MIMO communications systems.

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