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Solving two-dimensional nonlinear mixed Volterra Fredholm integral equations by using rationalized Haar functions in the complex plane

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Abstract. We present a method for calculating the numerical approximation of the two-dimensional mixed Volterra Fredholm integral equations, using the properties of the rationalized Haar (RH) wavelets and the matrix operator. Attaining this purpose, first, an operator and then an orthogonal projection should be defined. Regarding the characteristics of Haar wavelet, we solve the integral equation without using common mathematical methods. An upper bound and the convergence of the mentioned method have been proved, by using the Banach fixed point. Moreover, the rate of the convergence method is $O(n(2q)^n)$. Finally, several examples of different kinds of functions are presented and solved by this method.

Keywords: Nonlinear, mixed, Volterra, Fredholm, integral equation, Haar Wavelet, error estimation.

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1 Introduction

The complex numbers are used in many fields of science, such as physics, quantum mechanics, electrical engineering, biology, chemistry, economics,

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control theory, mathematics, AC's voltage and calculating AC circuits, signal analysis, fluid dynamics, improper integrals, current and resistance alternate in both amplitude and direction, transmitting radio waves, cell phones, fractals, and statistics. They are also very important in investigating the currents, which operate electrically such as the movement of the shock absorbers in the vehicles, the design of dynamo sand electric motors, liquid flow related to the obstacles, water around a pipe, analysis of stress on beams, the study of resonance of structures, and wave length; see [30].

Recently, several articles have been published in the complex plane to approximate integral equations, such as using the RH wavelet method [6, 7, 14, 16], with collocation method based on the Bernoulli operational [29], or the Cubic Bspline finite element method [17], with high order linear complex differential equations in [24], and by using a system of Cauchy type for singular integral equations [27].

We start the present study by introducing some definitions.

Definition 1. [28] If $z = x + iy$ and P are complex numbers and $r > 0$ is radius, then we denote the open and close disc with center P, respectively, by $D(P, r) = \{z \in \mathbb{C} : |z - P| < r\}$ and $\bar{D}(P, r) = \{z \in \mathbb{C} : |z - P| \leq r\}$.

In addition, we say that a set $E \subset \mathbb{C}$ is connected if there are no non-empty disjoint open sets U and V such as $E = (U \cap E) \cup (V \cap E)$.

Definition 2. [28] A continuously differentiable function $f : U \to \mathbb{C}$ defined in an open subset U of $\mathbb{C}$ is said to be holomorphic at every point of U, if $\partial f / \partial \bar{z} = 0$.

Definition 3. [28] A real-valued function $f : U \to \mathbb{C}$ is called a C^k function, if all partial derivatives of f up to and including order k exist and they are continuous on U, and a function $f = u + iv$ is a C^k function if $u, v \in C^k$.

Hence a C^k function is k times continuously differentiable and C^0 function is just a continuous function.

Definition 4. [28] If $U \subseteq \mathbb{C}$ is an open set and $f : U \to \mathbb{C}$ is a C^1 function, then

$$f(z) = f(x + iy) \equiv \widetilde{f}(x, y) = u(x, y) + iv(x, y), \tag{1}$$

where u and v are real-valued functions.

This article is organized as follows. Section 2 contains the notations and some properties of RH wavelet and complex analysis. Also, we formulate a problem. Moreover we review the RH wavelet method and approximate the solution for the integral equation in Section 3. We analyze the error of the suggested approach. In fact, we investigate the convergence analysis. In addition, we compute the order of method in Section 4. Finally, we illustrate the proposed methodology in numerical examples. We conclude our work in Section 5.

2 Preliminaries

Many types of problems are in engineering science such as seepage diffusion equation, acoustic, and engineering science, for examples, consolidation equation [8], Darboux problem [11], electrical engineering, and electrodynamic [12]. Also, in physics science, for examples, heat and mass transfer, electromagnetic, molecular physics [5] are another type of problems. These problems can be solved if they turn into the two-dimensional integral equation.

Several methods have been used for approximating the solution of two-dimensional integral equations as follows: the meshless local discrete Galerkin (MLDG) scheme [1, 2], the Bernstein collocation method [21–23, 25, 26] and the Haar wavelet in [4, 15]. The aim of this study is to present a numerical method for approximating the solution of second-kind two-dimensional nonlinear mixed Volterra Fredholm integral equations in the complex plane as follows:

$$z(x,t) = k(x,t) + \int_0^x \int_0^1 Z(x,t,s,y,z(s,y))dsdy, \tag{2}$$

where $(x,t) \in [0,1] \times [0,1]$ and $z(x,t) = v(x,t) + \mathrm{i}w(x,t)$ is an unknown function to be determined, where

$$z \in C([0,1]^2, \mathbb{C}), \quad v, w : C([0,1]^2) \longrightarrow \mathbb{R}^2,$$

and

$$k(x,t) = k_1(x,t) + \mathrm{i}k_2(x,t), \qquad k_j(x,t) : [0,1]^2 \to \mathbb{R}^2, \quad j = 1,2.$$

Also by using (1), we have the following properties

$$Z(x,t,s,y,z(s,y)) = Z_1(x,t,s,y,f(s,y)) + \mathrm{i}Z_2(x,t,s,y,g(s,y)),$$

$$Z_j : C([0,1]^4 \times C([0,1]^2), \mathbb{R}^2), \quad j = 1,2,$$

$$f(s,y), g(s,y) : C([0,1]^2) \longrightarrow \mathbb{R}^2,$$

and they are assumed to be known as continuous functions, satisfying the Lipschitz condition, that is, there are $M_1, M_2 \geqslant 0$ such that:

$$|Z_1(x,t,s,y,f_1(s,y)) - Z_1(x,t,s,y,f_2(s,y))| \leqslant M_1|f_1(s,y) - f_2(s,y)|, \quad (3)$$

$$|Z_2(x,t,s,y,g_1(s,y)) - Z_2(x,t,s,y,g_2(s,y))| \leqslant M_2|g_1(s,y) - g_2(s,y)|, \quad (4)$$

where

$$f_1(s,y), f_2(s,y) : C([0,1]^2) \longrightarrow \mathbb{R}^2,$$
$$g_1(s,y), g_2(s,y) : C([0,1]^2) \longrightarrow \mathbb{R}^2.$$

Therefore, in equation (2), we have

$$v(x,t) = k_1(x,t) + \int_0^x \int_0^1 Z_1(x,t,s,y,f(s,y))dsdy, \quad (5)$$

$$w(x,t) = k_2(x,t) + \int_0^x \int_0^1 Z_2(x,t,s,y,g(s,y))dsdy. \quad (6)$$

Thus

$$z(x,t) = k(x,t) + \int_0^x \int_0^1 Z_1(x,t,s,y,f(s,y))dsdy$$
$$+ \mathrm{i} \int_0^x \int_0^1 Z_2(x,t,s,y,g(s,y))dsdy. \quad (7)$$

The numerical results presented in this paper demonstrate a fast convergence method with low CPU runtime, when it is applied to integral equations. To achieve this purpose it is necessary to define the following integral operator:

$$T : (X, \|.\|_\infty) \to (X, \|.\|_\infty).$$

We are going to describe the idea of our proposed numerical method. The first point lies in the operator formulation of the two-dimensional nonlinear mixed Fredholm Volterra integral equations, by using an initial function $z_0 \in C[0,1]$. Since calculating Tz_0, is not generally possible, we approximate this function in Eq. (2) as follows:

$$(Tz)(x,t) = k(x,t) + \int_0^x \int_0^1 Z(x,t,s,y,z(s,y))dsdy. \quad (8)$$

The Banach fixed point theorem guarantees that under certain assumptions in reference [3], T has a unique fixed point, which means the two-dimensional mixed Fredholm Volterra integral equation in complex plane has exactly one solution. Thus for Eq. (7), we define for $n \geq 1$, recursively,

$$\begin{aligned} z_n(x,t) = k(x,t) &+ \int_0^x \int_0^1 Z_1(x,t,s,y,f_{n-1}(s,y))dsdy \\ &+ \mathrm{i} \int_0^x \int_0^1 Z_2(x,t,s,y,g_{n-1}(s,y))dsdy. \end{aligned} \tag{9}$$

We assume

$$\begin{aligned} \psi_{n-1}(x,t,s,y) &:= Z_1(x,t,s,y,f_{n-1}(s,y)), \\ \phi_{n-1}(x,t,s,y) &= Z_2(x,t,s,y,g_{n-1}(s,y)). \end{aligned} \tag{10}$$

Thus from Eqs. (8) and (9) and using Eq. (10), we have

$$\begin{aligned} (Tz_n)(x,t) = k(x,t) &+ \int_0^x \int_0^1 \psi_{n-1}(x,t,s,y)dsdy \\ &+ \mathrm{i} \int_0^x \int_0^1 \phi_{n-1}(x,t,s,y)dsdy. \end{aligned} \tag{11}$$

In this work, we use the 2D rational Haar (RH) wavelet method for approximating the solution of integral equations. There are some researches on approximating the solution of integral equations by using the Haar wavelet. For example, the solution of the Fredholm, Volterra, and mixed integral equations in [9,13,18,20], and also the integro-differential equation in [10, 19] were approximated. The Haar wavelets are one of the simplest wavelets among various types of wavelets, and the RH functions are family on $[0,1]$ of only three values $+1, -1$, and 0.

Definition 5. *The RH functions $h_n(t)$ for $n = 2^j + k, \quad j = 0,1,\ldots,$ and $k = 0,1,\ldots,2^j - 1$ are defined by*

$$h_n(t) = H(2^j t - k)|_{[0,1]},$$

where

$$H(t) = \begin{cases} 1, & 0 < t \leq \frac{1}{2}, \\ -1, & \frac{1}{2} < t < 1, \\ 0, & \text{otherwise}, \end{cases} \tag{12}$$

and for all $t \in [0,1)$, we have $h_0(t) = 1$.

By expanding ψ_{n-1} and ϕ_{n-1} for $n = 1, 2, 3, \ldots$, in terms of RH functions, we have

$$\psi_{n-1}(x,t,s,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\sum_{q=0}^{\infty} k_{ijrq}^{(1)} h_{ij}(x,t) h_{rq}(s,y),$$
$$\phi_{n-1}(x,t,s,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}\sum_{q=0}^{\infty} k_{ijrq}^{(2)} h_{ij}(x,t) h_{rq}(s,y).$$

Let $m = 2^{n+1}$ and Q_m be the orthogonal projection with the following interpolation property

$$Q_m(\psi_{n-1}(x,t,s,y)) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1}\sum_{r=0}^{m-1}\sum_{q=0}^{m-1} k_{ijrq}^{(1)} h_{ij}(x,t) h_{rq}(s,y),$$
$$Q_m(\phi_{n-1}(x,t,s,y)) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1}\sum_{r=0}^{m-1}\sum_{q=0}^{m-1} k_{ijrq}^{(2)} h_{ij}(x,t) h_{rq}(s,y), \tag{13}$$

or

$$Q_m(\psi_{n-1}(x,t,s,y)) = h^t(x,t) K_1 h(s,y),$$
$$Q_m(\phi_{n-1}(x,t,s,y)) = h^t(x,t) K_2 h(s,y). \tag{14}$$

where K_1 and K_2 are block matrices of the form,

$$K_1 = [K_1^{(i,j)}]_{i,j=1}^{m-1}, \qquad K_2 = [K_2^{(i,j)}]_{i,j=1}^{m-1} \tag{15}$$

in which

$$K_1^{(i,j)} = [k_{ijrq}^{(1)}]_{i,j,r,q=1}^{m-1}, \qquad K_2^{(i,j)} = [k_{ijrq}^{(2)}]_{i,j,r,q=1}^{m-1} \tag{16}$$

and the coefficients k_{ijrq}, are given by

$$k_{ijrq}^{(1)} = \frac{\langle Z_1(x,t,s,y,f_{n-1}(s,y)), h_{rq}(s,y)\rangle, h_{ij}(x,t)\rangle}{\langle h_{ij}(x,t), h_{ij}(x,t)\rangle \langle h_{rq}(s,y), h_{rq}(s,y)}, \tag{17}$$

$$k_{ijrq}^{(2)} = \frac{\langle Z_2(x,t,s,y,g_{n-1}(s,y)), h_{rq}(s,y)\rangle, h_{ij}(x,t)\rangle}{\langle h_{ij}(x,t), h_{ij}(x,t)\rangle \langle h_{rq}(s,y), h_{rq}(s,y)}. \tag{18}$$

Also, any function $f(x,t)$ of two variables in X can be similarly approximated in terms of RH functions as

$$f(x,t) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} l_{ij} h_{ij}(x,t) = \mathbf{L}^T \mathbf{h}(x,t), \tag{19}$$

where

$$\begin{aligned}\mathbf{L} &= [l_{00}, l_{01}, \ldots, l_{m-1,m-1}]^T_{(m-1)\times(m-1)},\\ \mathbf{h}(x,t) &= [h_{00}, h_{01}, \ldots, h_{m-1,m-1}]^T_{(m-1)\times(m-1)}(x,t),\end{aligned}$$

and for two-dimensional RH functions, we have

$$h_{ij}(x,t) = h_i(x)h_j(t),$$

where the RH function coefficients l_{ij} are given by

$$l_{ij} = \frac{\langle f(x,t), h_{ij}(x,t)\rangle}{\|h_{ij}(x,t)\|_2^2}.$$

Thus by using Eqs. (13), (14) and (9) for the nonlinear two-dimensional Volterra integral equations (2) in the complex plane, we have

$$\begin{aligned}\mathbf{z}_n(x,t) = k(x,t) &+ \int_0^x \int_0^1 Q_m(\psi_{n-1}(x,t,s,y))dsdy\\ &+ \mathrm{i}\int_0^x \int_0^1 Q_m(\phi_{n-1}(x,t,s,y))dsdy. \qquad (20)\end{aligned}$$

3 Error analysis

In this section, an upper bound for the error of our method will be calculated, by using the Banach fixed point theorem. Moreover, the order of convergence of our method is $O(n(2q)^n)$. Suppose that, $g(x,t) : [0,1]^2 \to \mathbb{R}^2$ is an arbitrary continuous function; then we define

$$||g||_\infty = \sup\{|g(x,t)|; (x,t) \in [0,1] \times [0,1]\}. \qquad (21)$$

Lemma 1. *Let $Z_1, Z_2 : [0,1]^4 \times \mathbb{R}^2 \to \mathbb{R}^2$ be continuous and Lipschitzian functions with respect to their fifth variables, with Lipschitz constants M_1 and M_2. Then T has an unique fixed point and for all $z_0(s,t) \in C([0,1]^2, \mathbb{C})$ and*

$$q = \max\{M_1, M_2\} < \frac{1}{2}, \qquad (22)$$

we have

$$\|z - T^n z_0\|_\infty \leq \|Tz_0 - \mathbf{z}_0\|_\infty \times \sum_{j=n}^{\infty} q^j. \qquad (23)$$

Proof. By Eq. (20) and using the triangle inequality and Eqs. (3) and (4), we have

$$
\begin{aligned}
&|(Tz_1)(x,t) - (Tz_2)(x,t)| \qquad (24)\\
&= |\int_0^x \int_0^1 (Z_1(x,t,s,y,f_1(s,y)) - Z_1(x,t,s,y,f_2(s,y)))dsdy\\
&\quad + \mathrm{i}\int_0^x \int_0^1 (Z_2(x,t,s,y,g_1(s,y)) - Z_2(x,t,s,y,g_2(s,y)))dsdy|\\
&\le \int_0^x \int_0^1 |Z_1(x,t,s,y,f_1(s,y)) - Z_1(x,t,s,y,f_2(s,y))|dsdy\\
&\quad + \mathrm{i}\int_0^x \int_0^1 |Z_2(x,t,s,y,g_1(s,y)) - Z_2(x,t,s,y,g_2(s,y))|dsdy\\
&\le \int_0^x \int_0^1 |(Z_1(x,t,s,y,f_1(s,y)) - Z_1(x,t,s,y,f_2(s,y)))|dsdy\\
&\quad + \int_0^x \int_0^1 |(Z_2(x,t,s,y,g_1(s,y)) - Z_2(x,t,s,y,g_2(s,y)))|dsdy\\
&\le M_1 \int_0^x \int_0^1 |f_1(s,y) - f_2(s,y)|dsdy + M_2 \int_0^x \int_0^1 |g_1(s,y) - g_2(s,y)|dsdy\\
&\le M_1||f_1(x,t) - f_2(x,t)||_\infty + M_2|||g_1(x,t) - g_2(x,t)||_\infty.
\end{aligned}
$$

We define $z_1^* = f_1 + \mathrm{i}g_1$ and $z_2^* = f_2 + \mathrm{i}g_2$. Also by using Eq. (24) and the triangle inequality, we have

$$
\begin{aligned}
\Big|(Tz_1)(x,t) - (Tz_2)(x,t)\Big| &\le M_1\|f_1(x,t) - f_2(x,t)\|_\infty + M_2\|g_1(x,t) - g_2(x,t)\|_\infty\\
&\le 2q\|z_1^*(x,t) - z_2^*(x,t)\|_\infty.
\end{aligned}
$$

By induction on $n \in \mathbb{N}$, we obtain

$$\|T^n z_1 - T^n z_2\|_\infty \le (2q)^n \|z_1 - z_2\|_\infty. \qquad (25)$$

Since $q < \frac{1}{2}$, so T is a contraction mapping. Therefore,

$$\sum_{n=1}^{\infty} \|T^n z_1 - T^n z_2\|_\infty < \infty.$$

Then T has a unique fixed point, which means that Eq. (8) has a unique solution and Eq. (23) satisfies the Banach fixed-point theorem. □

Theorem 1. *If $\varphi_{n-1}, \psi_{n-1} \in C([0,1]^4)$, and $Z_1, Z_2 : [0,1]^4 \times \mathbb{R}^2 \to \mathbb{R}^2$, are continuous and Lipschitzian functions and $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_i > 0$ for $i \ge 1$, then*

$$\|z - \mathbf{z}_n\|_\infty \le \|Tz_0 - \mathbf{z}_0\|_\infty \sum_{j=n}^{\infty} q^j + \sum_{j=1}^{n} q^{n-j}\varepsilon_j. \qquad (26)$$

Moreover, the rate of convergence method is $O(n(2q)^n)$.

Proof. If

$$L_{n-1} = \max\{\|\frac{\partial\psi_{n-1}}{\partial t}\|_\infty, \|\frac{\partial\psi_{n-1}}{\partial s}\|_\infty \|\frac{\partial\psi_{n-1}}{\partial x}\|_\infty, \|\frac{\partial\psi_{n-1}}{\partial y}\|_\infty\}, \qquad (27)$$

for $n = 1, 2, \ldots$ and $m = 2^{n+1}$, then, using the triangle inequality, we have

$$\begin{aligned}
&\|T(z_{n-1}) - \mathbf{z}_n\|_\infty \\
&\leq \ \|\int_0^x \int_0^1 \psi_{n-1}(x,t,s,y) - Q_m(\psi_{n-1}(x,t,s,y))dsdy \\
&\quad + \mathrm{i}\int_0^x \int_0^1 \phi_{n-1}(x,t,s,y) - Q_m(\phi_{n-1}(x,t,s,y))dsdy\|_\infty \\
&\leq \ \|\int_0^x \int_0^1 \psi_{n-1}(x,t,s,y) - Q_m(\psi_{n-1}(x,t,s,y))dsdy\|_\infty \\
&\quad + \ \|\int_0^x \int_0^1 \phi_{n-1}(x,t,s,y) - Q_m(\phi_{n-1}(x,t,s,y))dsdy\|_\infty \\
&\leq \|\psi_{n-1} - Q_m(\psi_{n-1})\|_\infty + \|\phi_{n-1} - Q_m(\phi_{n-1})\|_\infty.
\end{aligned}$$

If we define

$$g(t,s,x,y) := \psi_{n-1} - Q_m(\psi_{n-1}),$$

then by using the interpolating property and the mean-value theorem for four variables at $t_0, s_0, x_0, y_0 = 0$ with

$$\begin{aligned}
t_i &= \frac{1}{2^{n_1+1}} + \frac{v_1}{2^{n_1}}, \quad \text{for} \quad i = 2^{n_1} + v_1, \\
s_j &= \frac{1}{2^{n_2+1}} + \frac{v_2}{2^{n_2}}, \quad \text{for} \quad j = 2^{n_2} + v_2, \\
x_k &= \frac{1}{2^{n_3+1}} + \frac{v_3}{2^{n_3}}, \quad \text{for} \quad k = 2^{n_3} + v_3, \\
y_l &= \frac{1}{2^{n_4+1}} + \frac{v_4}{2^{n_4}}, \quad \text{for} \quad l = 2^{n_4} + v_4,
\end{aligned}$$

where $n_1, n_2, n_3, n_4 \geq 1$, and $i, j, k, l \leq m - 1$, we have

$$\begin{aligned}
&\|\psi_{n-1} - Q_m(\psi_{n-1})\|_\infty \\
&= \|g(t_i, s_j, x_k, y_l) + \frac{\partial g}{\partial t}(\xi,\gamma,\tau,\upsilon)(\xi - t_i) + \frac{\partial g}{\partial s}(\xi,\gamma,\tau,\upsilon)(\gamma - s_j) \\
&\quad + \frac{\partial g}{\partial x}(\xi,\gamma,\tau,\upsilon)(\tau - x_k) + \frac{\partial g}{\partial y}(\xi,\gamma,\tau,\upsilon)(\upsilon - y_l)\|_\infty
\end{aligned}$$

$$= \|(I-Q_m)\frac{\partial\psi_{n-1}}{\partial t}(\xi,\gamma,\tau,\upsilon)+(I-Q_m)\frac{\partial\psi_{n-1}}{\partial s}(\xi,\gamma,\tau,\upsilon)$$
$$+(I-Q_m)\frac{\partial\psi_{n-1}}{\partial x}(\xi,\gamma,\tau,\upsilon)+(I-Q_m)\frac{\partial\psi_{n-1}}{\partial y}(\xi,\gamma,\tau,\upsilon)\|_\infty$$
$$\max\{\|\xi-t_i\|_\infty,\|\gamma-s_j\|_\infty,\|\tau-x_k\|_\infty,\|\upsilon-y_l\|_\infty\}$$
$$\leq \frac{2}{2^n}\|(I-Q_m)\|_\infty\|\|\frac{\partial\psi_{n-1}}{\partial t}(\xi,\gamma,\tau,\upsilon)+\frac{\partial\psi_{n-1}}{\partial s}(\xi,\gamma,\tau,\upsilon)$$
$$+\frac{\partial\psi_{n-1}}{\partial x}(\xi,\gamma,\tau,\upsilon)+\frac{\partial\psi_{n-1}}{\partial y}(\xi,\gamma,\tau,\upsilon)\|_\infty \leq \frac{8L_{n-1}}{2^n}.$$

The same proof holds for ϕ_{n-1}, so we have

$$\|T(z_{n-1})-\mathbf{z}_n\|_\infty \leq \frac{16L_{n-1}}{2^n}. \tag{28}$$

Choose fixed constants $\varepsilon_1,\varepsilon_2,\dots,\varepsilon_i>0$, $i\geq 1$, with

$$\frac{16L_{k-1}}{2^k}<\varepsilon_k \quad \text{for} \quad k=1,2,\dots,n, \tag{29}$$

Then we have

$$\|T(z_{n-1})-\mathbf{z}_n\|_\infty<\varepsilon_n, \quad n\geq 1. \tag{30}$$

By applying the triangle inequality, we achieve

$$\|z-\mathbf{z}_n\|_\infty \leq \|z-T^n(\mathbf{z}_0)\|_\infty+\sum_{j=1}^{n} q^j\|T(z_{j-1})-\mathbf{z}_j\|_\infty. \tag{31}$$

From Eqs. (23) and (30), we conclude

$$\|z-\mathbf{z}_n\|_\infty \leq \|T(z_0)-\mathbf{z}_0\|_\infty\sum_{j=n}^{\infty} q^j+\sum_{j=1}^{n} q^{n-j}\varepsilon_j. \tag{32}$$

Also, since $q<1$, the geometric series $\sum_{j=n}^{\infty} q^j=\frac{q^n}{1-q}$ is convergent. By using Eqs. (28), (29), and (22), if we set

$$q=\frac{1}{2}-\frac{1}{2^{l+1}}<\frac{1}{2}, \qquad l\in\mathbb{N}, \tag{33}$$

then applying Eq. (32) implies

$$\|z-\mathbf{z}_n\|_\infty \leq \|Tz_0-\mathbf{z}_0\|_\infty\frac{q^n}{1-q}+\sum_{j=1}^{n}(\frac{1}{2}-\frac{1}{2^{l+1}})^{n-j}\frac{16L_{j-1}}{2^j}$$
$$=\|Tz_0-\mathbf{z}_0\|_\infty\frac{q^n}{1-q}+\sum_{j=1}^{n}(\frac{1}{2}-\frac{1}{2^{l+1}})^{n}(\frac{1}{2}-\frac{1}{2^{l+1}})^{-j}\frac{16L_{j-1}}{2^j}$$
$$=\|Tz_0-\mathbf{z}_0\|_\infty\frac{q^n}{1-q}+16(\frac{1}{2}-\frac{1}{2^{l+1}})^{n}\sum_{j=1}^{n}(\frac{1}{2}-\frac{1}{2^{l+1}})^{-j}\frac{L_{j-1}}{2^j}, \tag{34}$$

in which, from Eq. (27), the sequence L_{j-1} for any $j \in \mathbb{N}$ is uniformly bounded. Therefore, for every $1 \leq j \leq n$, there exists $N < \infty$ such that $|L_{j-1}| \leq N$. Thus from Eq. (34), we have

$$\begin{aligned}\|z - \mathbf{z}_n\|_\infty &\leq \|Tz_0 - \mathbf{z}_0\|_\infty \frac{q^n}{1-q} + 16(q)^n \sum_{j=1}^{n} (\frac{1}{2} - \frac{1}{2^{l+1}})^{-j} \frac{N}{2^j} \\ &= \|Tz_0 - \mathbf{z}_0\|_\infty \frac{q^n}{1-q} + 16N(q)^n \sum_{j=1}^{n} (1 + \frac{1}{2^l - 1})^j \frac{2^j}{2^j}. \quad (35)\end{aligned}$$

Since $(1 + \frac{1}{2^l-1}) \leq 2$ for any $l \in \mathbb{N}$ and by using (35), we have

$$\begin{aligned}\|z - \mathbf{z}_n\|_\infty &\leq \|Tz_0 - \mathbf{z}_0\|_\infty \frac{q^n}{1-q} + 16N(q)^n \sum_{j=1}^{n} 2^j \\ &\leq \|Tz_0 - \mathbf{z}_0\|_\infty \frac{q^n}{1-q} + 16N(q)^n n 2^n. \quad (36)\end{aligned}$$

If $n \to \infty$ and $q < \frac{1}{2}$, then the last summand of the right-hand side of (36) vanishes, and so we have $\|z - \mathbf{z}_n\|_\infty \leq 16Nn(2q)^n$. Thus the rate of convergence of our method is $O(n(2q)^n)$. □

4 Numerical examples

We illustrate the behavior of the proposed numerical method using three examples. In this work, using the equations of Section 2, for approximating z we obtain a sequence of functions $\{z_n\}_{n\in\mathbb{N}}$. Also, we set the exact solution $z = v + \mathrm{i}w$ with an initial function $z_0(s,t) \in C([0,1]^2, \mathbb{C})$ that $z_0(s,t) = g(s,t)$ and $m = 2^{n+1}$ for all $n = 0, 1, \ldots$. When we approximate the exact solution z, iteratively, we exhibit the absolute errors which have been committed in points $(x_i, t_i) \in [0,1]^2$. In this section, points are proposed as $(x_i, t_i) = (i/10, i/10)$ for $i = 1, 2, \ldots, 9$.

As we know so far, no study has yet been attempted in order to solve the nonlinear two-dimensional mixed Volterra Fredholm integral equation in the complex plane. So this method cannot be compared with other numerical methods and ones for that matter. We have solved and computed all of the examples with software Maple 18 on a machine with Intel core i7 Duo processor 2.4 GHz and 8 GB RAM. Moreover, we introduce an algorithm, based on the method presented in Section 2 that is used to solve all examples. We know that time-complexity is a formal construct that we use for any kind of algorithm that depends on the asymptotic number of steps (when considering worst-case, average-case, or best-case analysis). By using this algorithm, we do not need to solve any linear system for evaluating the wavelet coefficients, the CPU time is very low and the solving of equations with this method is an economical.

Algorithm 1.

1. *Define* j
2. *For* $n = 1$ *to* 4
3. *Set* $m = 2^{n+1}$
4. *Calculate* $Q_m(\varphi_{n-1}(x,t,s,y)), Q_m(\psi_{n-1}(x,t,s,y))$ *from Eqs.* (13) *or* (14).
5. $k := 1$
6. *While* $k \leq 9$
7. $(x_k, t_k) := (k/10, k/10)$
8. *Approximate* $\mathbf{z}_n(x,t)$ *from equations of* (20).
9. *Calculate of the absolute errors from equations of* $$|error| = \sqrt{(v - Re\ \mathbf{z}_n)^2 + (w - Im\ \mathbf{z}_n)^2}.$$
10. $k := k + 1$,
11. *Go to step 2.*

Example 1. In the first example, we consider the nonlinear two-dimensional mixed Volterra Fredholm integral equation of the second kind as

$$z(x,t) = k(x,t) + \frac{1}{25}\int_0^x \int_0^1 \frac{s^2 y^3 z^3(s,y)}{5 + x^2 t} dsdy$$

where the exact solution is $z(x,t) = \sin(\frac{x}{2} + t) + \mathrm{i}\cos(x - \frac{t}{2})$.

In Table 1, for the grid points $(x_i, t_i) = (i/10, i/10)$, for $i = 1, 2, \ldots, 9$, we have to obtain the absolute error. Moreover, CPU runtime for $n = 2$ or 2^3 Haar wavelet basis is 0.047 seconds, and for $n = 4$ or 2^5 Haar wavelet basis is 0.640 seconds.

Low CPU runtime is an efficiency of this method. Absolute errors of Example 1 and comparison between the numerical and exact solution have been demonstrated in Figures 1 and 2, respectively. Also, in Table 1, the numerical results for Example 1 have been presented.

Example 2. We consider the nonlinear two-dimensional mixed Volterra Fredholm integral equation of the second kind as

$$z(x,t) = k(x,t) + \frac{1}{20}x^2 t \int_0^x \int_0^1 sy^2 \frac{\sin^2(z(s,y))}{\cos(1 - z(s,y))} dsdy$$

with exact solution $z(x,t) = t - x^2 + \mathrm{i}(t^3 - x)$.

Table 1: Absolute errors of Example 1.

x_i	Our method for $n = 2$	Our method for $n = 4$
0.1	$2.14E{-11}$	$2.36E{-12}$
0.2	$1.37E{-10}$	$2.08E{-10}$
0.3	$4.16E{-9}$	$1.85E{-9}$
0.4	$1.16E{-8}$	$1.23E{-8}$
0.5	$3.28E{-8}$	$3.27E{-8}$
0.6	$1.46E{-7}$	$1.33E{-7}$
0.7	$2.86E{-7}$	$3.36E{-7}$
0.8	$7.56E{-7}$	$6.41E{-7}$
0.9	$1.13E{-6}$	$1.19E{-6}$

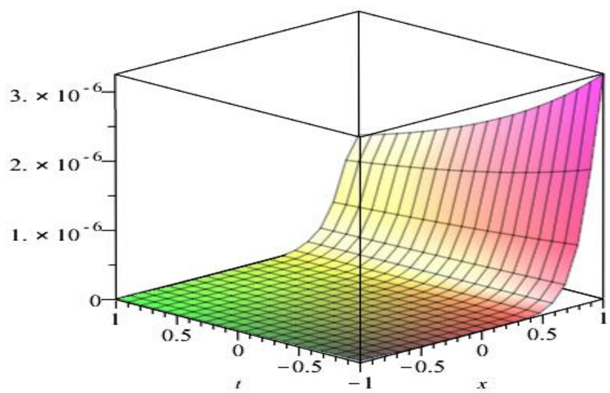

Figure 1: Absolute errors for $n = 4$ in Example 1.

In Table 2, for the grid points of $(x_i, t_i) = (i/10, i/10)$, $i = 1, 2, \ldots, 9$, we obtain the absolute error. Moreover, CPU runtime for $n = 3$ or 2^4 Haar wavelet basis is 43.531 seconds, and for $n = 4$ or 2^5 Haar wavelet basis is 134.172 seconds. The CPU runtime proves efficient veracity of the method. In Figure 3 the absolute errors of Example 2 and in Figure 4 comparison between the numerical and exact solution have been demonstrated.

Also, in Table 2, the numerical results of Example 2 have been presented.

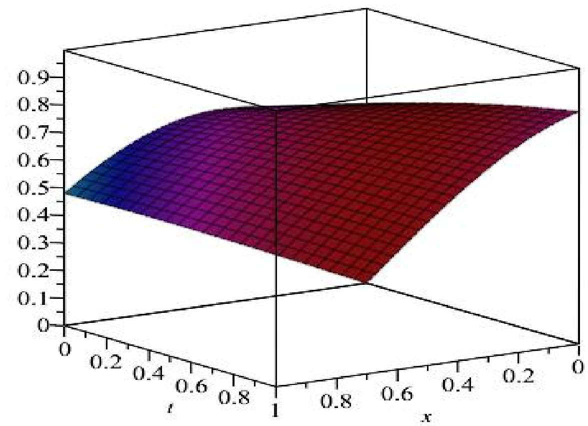

Figure 2: The exact solution for $n = 4$ in Example 1.

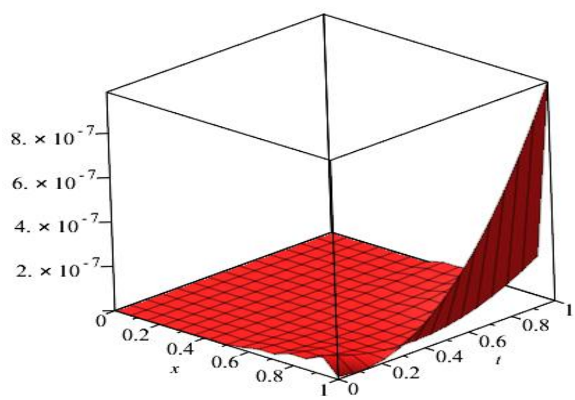

Figure 3: Absolute errors for $n = 3$ in Example 2

5 Conclusions

It is a fact that the solutions of the nonlinear 2D mixed Volterra Fredholm integral equations are complicated in general. So, the mathematicians often attempt to achieve the approximate solutions. One of the most efficient and suitable methods of solving the Eq. (2) is the Haar wavelets' basis. The context of this study is the Haar wavelet basis in the complex plane. We proved the convergence of the method by getting an upper bound and using the Banach fixed point theorem for the mentioned equations in Error

Table 2: Absolute errors of Example 2.

x_i	Our method for $n = 3$	Our method for $n = 4$
0.1	$2.17E{-}10$	$3.40E{-}10$
0.2	$5.33E{-}9$	$8.38E{-}9$
0.3	$3.48E{-}8$	$2.13E{-}8$
0.4	$1.23E{-}7$	$8.73E{-}8$
0.5	$1.61E{-}8$	$1.53E{-}8$
0.6	$2.76E{-}7$	$3.45E{-}7$
0.7	$5.81E{-}7$	$7.03E{-}7$
0.8	$1.00E{-}6$	$8.22E{-}7$
0.9	$1.35E{-}6$	$1.11E{-}6$

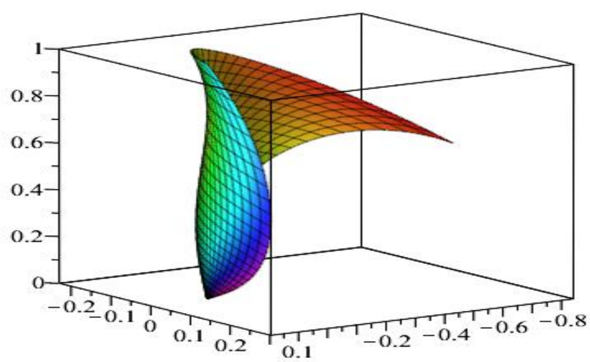

Figure 4: The exact solution for $n = 4$ in Example 2.

analysis section. Also, in Numerical examples section, it has been observed that the approximate solutions, which have been obtained by using the Haar wavelet basis in the nodes specified $x_i = i/10$, $i = 1, \ldots, 9$, are of great accuracy in two numerical examples of 2D mixed Volterra Fredholm integral equations. In addition, CPU runtimes in seconds have been presented. It is worth noting that the problems have been solved by using Algorithm 1.

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References

[1] P. Assari,*On the numerical solution of two-dimensional integral equations using a meshless local discrete Galerkin scheme with error analysis*, Eng. Comput. **35**(3) (2019) 893–916.

[2] P. Assari and M. Dehghan, *A meshless local discrete Galerkin (MLDG) scheme for numerically solving two-dimensional nonlinear Volterra integral equations*, Appl. Math. Comput. **350** (2019) 249–265.

[3] K.E. Atkinson, *The Numerical solution of integral equations of the second kind*, Cambridge University Press, 1997.

[4] A. Babaaghaie and K. Maleknejad, *Numerical solutions of nonlinear two-dimensional partial Volterra integro-differential equations by Haar wavelet*, J. Comput. Appl. Math. **317** (2017) 643–651.

[5] W.C. Chew, M.S. Tong and B. Hu, *Integral equation methods for electromagnetic and elastic waves*, Synthesis Lectures on Computational Electromagnetics, 2008.

[6] M. Erfanian, *The approximate solution of nonlinear mixed Volterra-Fredholm Hammerstein integral equations with RH wavelet bases in a complex plane*, Math. Method Appl. Sci. **41**(18) (2018) 8942–8952.

[7] M. Erfanian, *The approximate solution of nonlinear integral equations with the RH wavelet bases in a complex plane*, Int. J. Appl. Comput. Math. **4** (2018) 31.

[8] M. Rastegar, A. Bazrafshan Moghaddam, M. Erfanian and B. Bazrafshan Moghaddam, *Using of matrix-based rationalized Haar wavelet method for solving consolidation equation*, Asian-Eur. J. Math. **12**(1) (2019) 1950086.

[9] M. Erfanian, M. Gachpazan and H. Beiglo, *Rationalized Haar wavelet bases to approximate solution of nonlinear Fredholm integral equations with error analysis.* Appl. Math. Comput. **265** (2015) 304–312.

[10] M. Erfanian, M. Gachpazan and M. Beiglo, *A new sequential approach for solving the integro-differential equation via Haar wavelet bases*, Comput. Math. Math. Phys. **57(2)**(2017) 297–305.

[11] M. Erfanian, M. Gachpazan and S. Kosari, *A new method for solving of Darboux problem with Haar wavelet*, SeMA Journal **74(4)** (2017) 475–487.

[12] R.T. Lynch and J.J. Reis, *Haar transform image coding*, Proceedings of the National Telecommunications Conference, Dallas, TX, (1976) 441–443.

[13] M. Erfanian and A. Mansoori, *Rationalized Haar wavelet bases to approximate the solution of Painleve equations (PI)*, J. Math. Model. **7**(1) (2019) 107–116.

[14] M. Erfanian and A. Mansoori, *Solving the nonlinear integro-differential equation in complex plane with rationalized Haar wavelet*, Math. Comput. Simul. **165** (2019) 223–237.

[15] M. Erfanian, M. Parsamanesh and A. Akrami, *Solving two-dimensional nonlinear Fredholm integral equations using rationalized Haar functions in the complex plane*, Int. J. Appl. Comput. Math. **5** (2019) 47.

[16] M. Erfanian and H. Zeidabadi, *Solving of nonlinear Fredholm integro-differential equation in a complex plane with rationalized Haar wavelet bases*, Asian-Eur. J. Math. **12** (4) (2019) 1950055.

[17] M. Erfanian and H. Zeidabadi, *Approximate solution of linear Volterra integro-differential equation by using Cubic Bspline finite element method in the complex*, Adv. Difference Equ. **62** (2019) 10.1186/s13662-019-2012-9.

[18] I. Singh and S. Kumar, *Haar wavelet method for some nonlinear Volterra integral equations of the first kind*, J. Comput. Appl. Math. **292** (2016) 541–552.

[19] K. Maleknejad and F. Mirzaee, *Numerical solution of integro-differential equations by using rationalized Haar functions method*, Kybernetes **35**(10) (2006) 1735–1744.

[20] K. Maleknejad and F. Mirzaee, *Numerical solution of linear Fredholm integral equations system by rationalized Haar functions method*, Int. J. Comput. Math. **80**(11) (2010) 1397–1405.

[21] F. Mirzaee and E. Hadadiyan, *Using operational Matrix for solving nonlinear class of mixed Volterra Fredholm integral equations*, Math. Method Appl. Sci. **40**(10) (2017) 3433–3444.

[22] F. Mirzaee and E. Hadadiyan, *A new numerical method for solving two-dimensional Volterra Fredholm integral equations*, J. Appl. Math. Comput. **52**(1-2) (2015) 489–513.

[23] F. Mirzaee and E. Hadadiyan, *Numerical solution of linear Fredholm integral equations via two-dimensional modification of hat functions*, Appl. Math. Comput. **250** (2015) 805–816.

[24] F. Mirzaee, N. Samadyar and S.R. Alipour, *Numerical solution of high order linear complex differential equations via complex operational matrix method*, SeMA Journal **76** (1) (2019) 1–13.

[25] F. Mirzaee and N. Samadyar, *Numerical solution based on two-dimensional orthonormal Bernstein polynomials for solving some classes of two-dimensional nonlinear integral equations of fractional order*, Appl. Math. Comput. **344** (2019) 191–203.

[26] F. Mirzaee and N. Samadyar, *Convergence of 2D-orthonormal Bernstein collocation method for solving 2D-mixed Volterra Fredholm integral equations*, Trans. A. Razmadze Math. Inst. **172** (3) (2018) 631–641.

[27] V. Sharma, A. Setia and R.P. Agarwal, *Numerical solution for system of Cauchy type singular integral equations with its error analysis in complex plane*, Appl. Math. Comput. **328** (2018) 338–352.

[28] S.G. Krantz , *A guide to complex variables*, ISBN: 978-0-88385-338-2, 2007.

[29] F. Toutounian, E. Tohidi and S. Shateyi, *A collocation method based on the Bernoulli operational matrix for solving high-order linear complex differential equations in a rectangular domain*, Abstr. Appl. Anal. (2013) Article ID 823098.

[30] Y.K. Kwok, *Applied complex variables for scientists and engineers*, Second Edition, Cambridge University Press, 2010.