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A nonlocal Cauchy problem for nonlinear fractional integro-differential equations with positive constant coefficient

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Abstract. In this paper, we study the existence, uniqueness and stability of solutions of a nonlocal Cauchy problem for nonlinear fractional integro-differential equations with positive constant coefficient. The results heavily depend on the Banach contraction principle, Schaefer's fixed point theorem and Pachpatte's integral inequality. In the last, results are illustrated with suitable example.

Keywords: Fractional integro–differential equation, Existence of solution, Fixed point, Pachpatte's integral inequality, Stability.
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1 Introduction

The idea of fractional differentiation was introduced by Riemann and Liouville in the nineteenth century. It is the generalization of ordinary differentiation and integration to arbitrary non-integer order, for details, see [1, 2, 6, 8, 9, 30, 32] and the references therein.

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The area of fractional differential equations is now considered to be very important due to its various applications in different fields of science and technology such as control theory, rheology, signal processing, modelling, fractals, chaotic dynamics, bioengineering and biomedical and so on, for example see [9, 24, 37] and the references therein. Recently, many researchers studied the fractional differential and integro-differential equations and obtained many interesting existence and uniqueness results, see [4, 11, 23, 38].

The stability problem of functional equations was introduced by Ulam [39, 40] and Hyers [18] which is known as Hyers-Ulam stability. Rassias [33] studied the Hyers-Ulam stability of linear and nonlinear mapping. Jung [19, 20] established Hyers-Ulam stability for more general mapping on restricted domain. Obloza [29] was the first who studied the Hyers-Ulam stability of linear differential equations. Later many researchers studied the Ulam type stability, for detail see [3, 5, 7, 15–17, 21, 22, 26, 33–36, 41–43].

In [14], Castro and Simões studied different kinds of Hyers-Ulam-Rassias stabilities for a class of nonlinear integro-differential equations. In [10], Benchohra and Bouriahi investigated existence and stability of solutions for a class of boundary value problem for implicit Caputo fractional differential equations of the type:

$$
\begin{aligned}
{}^cD^{\alpha}y(t) &= f(t, y(t), {}^cD^{\alpha}y(t)), \ t \in J := [0, T], \ T > 0, \\
& ay(0) + by(T) = c.
\end{aligned}
$$

Kucche and Shikhare [27] studied the Ulam-Hyers stabilities for Volterra integro-differential equations and Volterra delay integro-differential equations in Banach spaces on both finite and infinite intervals by using Pachpatte's inequality.

The above results motivates us and therefore, in this paper, we obtain the existence, uniqueness and various types of Ulam stability of the following nonlinear Caputo fractional integro–differential equations of order α $(0 < \alpha \leq 1)$ with constant coefficient $\lambda > 0$ of the type:

$$
{}^cD^{\alpha}y(t) = \lambda y(t) + f\left(t, y(t), \int_0^t h(t,s)y(s)ds\right), \ t \in J := [0, T], \ T > 0, \quad (1)
$$

$$
y(0) + g(y) = y_0, \quad (2)
$$

where $f : J \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is a continuous function, $g : C(J, \mathbf{R}) \to \mathbf{R}$ is a continuous function and y_0 is a real constant. This type of nonlocal Cauchy problem was introduced by Byszewski [12, 13]. The nonlocal condition can be more useful than the classical initial condition to describe some physical phenomena [12, 13] . We take an example of nonlocal conditions as follows:

$$g(y) = \sum_{i=1}^{p} c_i y(t_i), \tag{3}$$

where c_i, $i = 1, 2, \ldots, p$ are constants and $0 < t_1 < \ldots < t_p \leq T$.

The rest of the paper is organized as follows. In Section 2, some definitions, notations and basic results are given. Section 3 is devoted to study the existence, uniqueness and stability of the problem (1)-(2). An illustrative example is given in the last section.

2 Preliminaries

In this section, we introduce some definitions, notations and results which are useful for further discussion. For $T > 0$ and $J = [0, T]$, $C(J, \mathbf{R})$ denotes the Banach space of all continuous functions from J into $\mathbf{R}$ with the norm $||y||_\infty = \sup\{|y(t)| : t \in J\}$. Also $L^1(J)$ denotes the space of Lebesgue-integrable functions $y : J \to \mathbf{R}$ with the norm

$$||y||_{L^1} = \int_0^T |y(t)|\, dt.$$

Definition 1. *[32]* The Riemann-Liouville fractional integral of a function $h \in L^1([0, T], \mathbf{R}_+)$ of order $\alpha \in \mathbf{R}_+$ is defined by

$$I^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds,$$

where Γ is the Euler gamma function.

Definition 2. *[24]* The Caputo fractional derivative of order $\alpha > 0$ of a function $h \in L^1([0, T], \mathbf{R}_+)$ is defined as

$${}^cD^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} h^{(n)}(s)\, ds,$$

where n=[α]+1 and [α] denotes the integer part of the real number α.

Lemma 1. *[24] Let $\alpha > 0$ and n=[α]+1. Then*

$$I^\alpha({}^cD^\alpha f(t)) = f(t) - \sum_{k=0}^{n-1} \frac{f^k(0)}{k!} t^k,$$

where $f^k(t)$ is the usual derivative of $f(t)$ of order k.

Lemma 2. *[32] Let $\alpha > 0$. Then the fractional differential equation*

$$^cD^\alpha h(t) = 0,$$

has the solution $h(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1}$, where c_i, $i = 0, 1, 2, \ldots, n-1$ are constants and n=[α]+1.

The following Pachpatte's inequality plays an important role in obtaining our main results.

Theorem 1. *([31, page 39]) Let $u(t)$, $f(t)$ and $q(t)$ be nonnegative continuous functions defined on $\mathbf{R}_+$, and $n(t)$ be a positive and nondecreasing continuous function defined on $\mathbf{R}_+$ for which the inequality*

$$u(t) \leq n(t) + \int_0^t f(s)\Big[u(s) + \int_0^s q(\tau)u(\tau)\, d\tau\Big] ds,$$

holds for $t \in \mathbf{R}_+$. Then

$$u(t) \leq n(t)\Big[1 + \int_0^t f(s) \exp\Big(\int_0^s [f(\tau) + q(\tau)]\, d\tau\Big) ds\Big],$$

for $t \in \mathbf{R}_+$.

The following definitions are useful in the study of stability results.

Definition 3. *[10, 36]* The equation (1) is Ulam-Hyers stable if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C^1(J, \mathbf{R})$ of the inequality

$$\left|\left| {^cD^\alpha} z(t) - \lambda z(t) - f\Big(t, z(t), \int_0^t h(t,s)z(s)ds\Big)\right|\right| \leq \epsilon,\ t \in J,$$

there exists a solution $y \in C^1(J, \mathbf{R})$ of equation (1) with $||z(t) - y(t)|| \leq c_f \epsilon,\ t \in J$.

Definition 4. *[10, 36]* The equation (1) is generalized Ulam-Hyers stable if there exists $\psi_f \in C(\mathbf{R}_+, \mathbf{R}_+)$, $\psi_f(0) = 0$, such that for each solution $z \in C^1(J, \mathbf{R})$ of the inequality

$$\left|\left| {^cD^\alpha} z(t) - \lambda z(t) - f\Big(t, z(t), \int_0^t h(t,s)z(s)ds\Big)\right|\right| \leq \epsilon,\ t \in J,$$

there exists a solution $y \in C^1(J, \mathbf{R})$ of equation (1) with

$$||z(t) - y(t)|| \leq \psi_f(\epsilon),\ t \in J.$$

Definition 5. *[10,36]* The equation (1) is Ulam-Hyers-Rassias stable with respect to $\varphi \in C(J, \mathbf{R}_+)$ if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C^1(J, \mathbf{R})$ of the inequality

$$\left|\left|{}^cD^\alpha z(t) - \lambda z(t) - f\left(t, z(t), \int_0^t h(t,s)z(s)ds\right)\right|\right| \leq \epsilon\varphi(t),\ t \in J,$$

there exists a solution $y \in C^1(J, \mathbf{R})$ of equation (1) with

$$||z(t) - y(t)|| \leq c_f \epsilon \varphi(t),\ t \in J.$$

Definition 6. *[10,36]* The equation (1) is generalized Ulam-Hyers-Rassias stable with respect to $\varphi \in C(J, \mathbf{R}_+)$ if there exists a real number $c_{f,\varphi} > 0$ such that for each solution $z \in C^1(J, \mathbf{R})$ of the inequality

$$\left|\left|{}^cD^\alpha z(t) - \lambda z(t) - f\left(t, z(t), \int_0^t h(t,s)z(s)ds\right)\right|\right| \leq \varphi(t),\ t \in J,$$

there exists a solution $y \in C^1(J, \mathbf{R})$ of equation (1) with

$$||z(t) - y(t)|| \leq c_{f,\varphi}\varphi(t),\ t \in J.$$

Remark 1. *A function $z \in C^1(J, \mathbf{R})$ is a solution of the inequality*

$$\left|\left|{}^cD^\alpha z(t) - \lambda z(t) - f\left(t, z(t), \int_0^t h(t,s)z(s)ds\right)\right|\right| \leq \epsilon,\ t \in J,$$

if and only if there exists a function $g \in C(J, \mathbf{R})$ (which depends on solution z) such that

i) $||g(t)|| \leq \epsilon,\ \forall t \in J.$

ii) ${}^cD^\alpha z(t) = \lambda z(t) + f(t, z(t), \int_0^t h(t,s)z(s)ds) + g(t),\ t \in J.$

Remark 2. *Clearly,*

i): Definition 3. implies Definition 4.

ii): Definition 5. implies Definition 6.

Remark 3. *A solution of the fractional differential inequality*

$$\left|\left|{}^cD^\alpha z(t) - \lambda z(t) - f\left(t, z(t), \int_0^t h(t,s)z(s)ds\right)\right|\right| \leq \epsilon,\ t \in J,$$

is called an fractional $\epsilon-$solution of the nonlinear fractional integro-differential equation (1).

3 Existence and Ulam-Hyers stability of the nonlocal problem

In this section we obtain existence, uniqueness and stability results for the nonlocal problem (1)-(2). Now we introduce the following set of conditions:

(H1) There exists a constant $L > 0$ such that

$$||f(t,x,y) - f(t,\bar{x},\bar{y})|| \le L(||x-\bar{x}|| + ||y-\bar{y}||),$$

for each $t \in J$ and $x, y, \bar{x}, \bar{y} \in \mathbf{R}$.

(H2) The function $f : J \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is continuous.

(H3) There exists a constant $a_f > 0$ such that

$$||f(t,x,y)|| \le a_f(1 + ||x|| + ||y||),$$

for each $t \in J$ and $x, y \in \mathbf{R}$.

(H4) There exists a constant $G > 0$ such that $||g(y)|| \le G$, for each $y \in C(J,\mathbf{R})$.

(H5) There exists a constant $\bar{K} > 0$ such that $||g(y) - g(\bar{y})|| \le \bar{K}\,||y-\bar{y}||$, for each $y, \bar{y} \in C(J,\mathbf{R})$.

Lemma 3. *[10] Let $0 < \alpha \le 1$ and $h : [0,T] \to \mathbf{R}$ be a continuous function. Then the linear problem*

$$\begin{aligned} {}^cD^\alpha y(t) &= h(t),\ t \in [0,T],\ T > 0, \\ y(0) + g(y) &= y_0, \end{aligned}$$

has a unique solution which is given by

$$y(t) = y_0 - g(y) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds.$$

As a consequence of Lemma 3 and [23], we have the following result which is useful in our main results.

Lemma 4. *Let $f : J \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be a continuous function. Then the problem (1)-(2) is equivalent to the following integral equation*

$$\begin{aligned} y(t) &= y_0 - g(y) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, y(s), \int_0^s h(t,\tau) y(\tau) d\tau\right) ds,\ t \in J. \quad (4) \end{aligned}$$

Theorem 2. *Assume that (H1), (H2), (H5) hold. If*

$$\left[\bar{K} + \frac{(\lambda + L)T^{\alpha} + Lh_T T^{\alpha+1}}{\Gamma(\alpha+1)}\right] < 1, \tag{5}$$

where $h_T = \sup\{|h(t,s)|\,|0 \le s \le t \le T\}$*, then the nonlocal problem (1)-(2) has a unique solution on J.*

Proof. We transform problem (1)-(2) into a fixed point problem. For this, consider the operator $\bar{F} : C(J, \mathbf{R}) \to C(J, \mathbf{R})$ defined by

$$\begin{aligned}\bar{F}(y)(t) \;=\; & y_0 - g(y) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, y(s), \int_0^s h(t,\tau) y(\tau) d\tau\right) ds. \end{aligned} \tag{6}$$

Let $x, y \in C(J, \mathbf{R})$. Then for each $t \in J$, we have
$\left|\left|\bar{F}(x)(t) - \bar{F}(y)(t)\right|\right|$

$$\begin{aligned} &\le ||g(x) - g(y)|| + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \,||x(s) - y(s)||\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left|\left| f\left(s, x(s), \int_0^s h(t,\tau)x(\tau)d\tau\right)\right.\right. \\ &\quad \left.\left. - f\left(s, y(s), \int_0^s h(t,\tau)y(\tau)d\tau\right)\right|\right| ds \\ &\le \bar{K}\, ||x(t) - y(t)|| + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \,|x(s) - y(s)|\, ds \\ &\quad + \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(||x(s) - y(s)|| + \int_0^s |h(s,\tau)|\, ||x(\tau) - y(\tau)||\, d\tau \right) ds \\ &\le \bar{K}\, ||x(t) - y(t)|| + \frac{(\lambda + L)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \,||x(s) - y(s)||\, ds \\ &\quad + \frac{Lh_T T}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \,||x(\tau) - y(\tau)||\, ds \\ &\le \left[\bar{K} + \frac{(\lambda + L)T^{\alpha} + Lh_T T^{\alpha+1}}{\Gamma(\alpha+1)}\right] ||x - y||_{\infty}. \end{aligned}$$

Thus

$$\left|\left|\bar{F}(x) - \bar{F}(y)\right|\right|_{\infty} \le \left[\bar{K} + \frac{(\lambda + L)T^{\alpha} + Lh_T T^{\alpha+1}}{\Gamma(\alpha+1)}\right] ||x - y||_{\infty}.$$

This implies that $\bar{F}$ is a contraction due to the inequality (5). By Banach contraction principle, we deduce that $\bar{F}$ has a unique fixed point which is a solution of the problem (1)-(2) . □

The next result is based on Schaefer's fixed point theorem.

Theorem 3. *Assume that (H2), (H3), (H4) hold. Then the nonlocal problem (1)-(2) has at least one solution on J .*

Proof. We complete the proof in the following four steps.

Step 1: $\bar{F}$ is continuous.
Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C(J, \mathbf{R})$. Then for each $t \in J$, we have

$$\begin{aligned} &\left|\left|\bar{F}(y_n)(t) - \bar{F}(y)(t)\right|\right| \\ &\quad \le ||g(y_n) - g(y)|| + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ||y_n(s) - y(s)|| \, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sup_{t \in J} \left|\left| f\left(s, y_n(s), \int_0^s h(t,\tau) y_n(\tau) d\tau\right) \right.\right. \\ &\quad \left.\left. - f\left(s, y(s), \int_0^s h(t,\tau) y(\tau) d\tau\right)\right|\right| ds. \end{aligned}$$

Since f and g are continuous functions and $y_n \to y$, then we have

$$\left|\left|\bar{F}(y_n)(t) - \bar{F}(y)(t)\right|\right|_\infty \to 0,$$

as $n \to \infty$. Consequently, $\bar{F}$ is continuous.

Step 2: $\bar{F}$ maps bounded sets into bounded sets in $C(J, \mathbf{R})$.
We need to show that for any $\eta^* > 0$, there exists a positive constant l such that for each $y \in B_{\eta^*} = \{y \in C(J, \mathbf{R}) : ||y||_\infty \le \eta^*\}$, we have $\left|\left|\bar{F}(y)\right|\right|_\infty \le l$.
By (H3) and (H4), for each $t \in J$, we have

$$\begin{aligned} \left|\left|\bar{F}(y)(t)\right|\right| &\le ||y_0|| + ||g(y)|| + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ||y(s)|| \, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left|\left| f\left(s, y(s), \int_0^s h(t,\tau) y(\tau) d\tau\right)\right|\right| ds \\ &\le ||y_0|| + G + \frac{\lambda \eta^* T^\alpha}{\Gamma(\alpha+1)} \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} a_f (1 + ||y(s)|| + \int_0^s |h(t,\tau)| \, ||y(\tau)|| \, d\tau) \, ds \\ &\le ||y_0|| + G + \frac{\lambda \eta^* T^\alpha}{\Gamma(\alpha+1)} + \frac{a_f(1 + \eta^* + h_T \eta^* T) T^\alpha}{\Gamma(\alpha+1)}. \end{aligned}$$

Thus

$$\left|\left|\bar{F}(y)\right|\right|_\infty \le ||y_0|| + G + \frac{\lambda \eta^* T^\alpha}{\Gamma(\alpha+1)} + \frac{a_f(1 + \eta^* + h_T \eta^* T) T^\alpha}{\Gamma(\alpha+1)} := l.$$

Step 3: $\bar{F}$ maps bounded sets into equicontinuous sets of $C(J,\mathbf{R})$.

Let $t_1, t_2 \in (0,T]$, $t_1 < t_2$, B_{η^*} be a bounded set of $C(J,\mathbf{R})$ as in step 2, and let $y \in B_{\eta^*}$. Then

$$
\begin{aligned}
&\big|\big|\bar{F}(y)(t_1) - \bar{F}(y)(t_2)\big|\big| \\
&\quad \leq \frac{\lambda}{\Gamma(\alpha)}\int_0^{t_1}\{(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\}||y(s)||\,ds \\
&\qquad + \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\{(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\}||f\left(s, y(s), \int_0^s h(t,\tau)y(\tau)d\tau\right)||\,ds \\
&\qquad + \frac{\lambda}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}||y(s)||\,ds \\
&\qquad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}||f\left(s, y(s), \int_0^s h(t,\tau)y(\tau)d\tau\right)||\,ds \\
&\quad \leq \frac{\lambda\eta^*}{\Gamma(\alpha)}\int_0^{t_1}\{(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\}\,ds \\
&\qquad + \frac{a_f(1+\eta^*+h_T\eta^*T)}{\Gamma(\alpha)}\int_0^{t_1}\{(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\}\,ds \\
&\qquad + \frac{\lambda\eta^*}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\,ds + \frac{a_f(1+\eta^*+h_T\eta^*T)}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\,ds \\
&\quad \leq \frac{(\lambda\eta^*+a_f(1+\eta^*+h_T\eta^*T))}{\Gamma(\alpha+1)}\{2(t_2-t_1)^\alpha + (t_1^\alpha - t_2^\alpha)\}.
\end{aligned}
$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. As a consequence of steps 1 to 3 together with the Arzela-Ascoli theorem, we can conclude that $\bar{F} : C(J,\mathbf{R}) \to C(J,\mathbf{R})$ is continuous and completely continuous.

Step 4: A priori bounds.

Now it remains to show that the set

$$\mathcal{E} = \{y \in C(J,\mathbf{R}) : y = \beta\bar{F}(y), \text{for some } \beta \in (0,1)\},$$

is bounded. Let $y \in \mathcal{E}$, then $y = \beta\bar{F}(y)$, for some $\beta \in (0,1)$. Thus, for each $t \in J$ we have

$$
\begin{aligned}
y(t) \;=\; \beta\Bigg\{ & y_0 - g(y) + \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,ds \\
& + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s, y(s), \int_0^s h(t,\tau)y(\tau)d\tau\right)ds\Bigg\}.
\end{aligned}
$$

This implies by (H3) and (H4) that for each $t \in J$ we have

$$\left|\left|\bar{F}(y)(t)\right|\right| \le ||y_0|| + G + \frac{\lambda\eta^* T^\alpha}{\Gamma(\alpha+1)} + \frac{a_f(1+\eta^* + h_T\eta^* T)T^\alpha}{\Gamma(\alpha+1)}.$$

Thus for every $t \in J$, we have

$$\left|\left|\bar{F}(y)\right|\right|_\infty \le ||y_0|| + G + \frac{\lambda\eta^* T^\alpha}{\Gamma(\alpha+1)} + \frac{a_f(1+\eta^* + h_T\eta^* T)T^\alpha}{\Gamma(\alpha+1)} := R.$$

This shows that the set $\mathcal{E}$ is bounded. Now applying Schaefer's fixed point theorem, we deduce that $\bar{F}$ has a fixed point which is a solution of the problem (1)-(2). □

Theorem 4. *Assume that (H1), (H5) and the inequality* (5) *hold. Then the nonlocal problem (1)-(2) is Ulam-Hyers stable.*

Proof. Let $\epsilon > 0$ and let $z \in C^1(J, \mathbf{R})$ be a function which satisfies the inequality

$$\left|\left|{}^cD^\alpha z(t) - \lambda z(t) - f\left(s, z(s), \int_0^s h(t,\tau)z(\tau)d\tau\right)\right|\right| \le \epsilon, \tag{7}$$

for every $t \in J$ and let $y \in C(J, \mathbf{R})$ be the unique solution of the following Cauchy problem

$$\begin{aligned} {}^cD^\alpha y(t) &= \lambda y(t) + f\left(s, y(s), \int_0^s h(t,\tau)y(\tau)d\tau\right),\ t \in J,\ 0 < \alpha \le 1, \\ & z(0) + g(y) = y_0. \end{aligned}$$

By Lemma 4, we have

$$\begin{aligned} y(t) &= y_0 - g(y) + \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} y(s)\, ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s, y(s), \int_0^s h(t,\tau)y(\tau)d\tau\right) ds. \end{aligned}$$

By integrating (7), we obtain

$$\begin{aligned} &\Bigg|\Bigg| z(t) - y_0 + g(z) - \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} z(s)\, ds \\ &- \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s, z(s), \int_0^s h(t,\tau)z(\tau)d\tau\right) ds \Bigg|\Bigg| \le \frac{\epsilon t^\alpha}{\Gamma(\alpha+1)}. \end{aligned} \tag{8}$$

Using (H1), (H5) and the inequality (8), for every $t \in J$, we have

$$
\begin{aligned}
||z(t)-y(t)|| \leq & \left|\left| z(t) - y_0 + g(z) - \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s)\, ds \right.\right. \\
& \left.\left. - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, z(s), \int_0^s h(t,\tau) z(\tau) d\tau\right) ds \right|\right| \\
& + ||g(z)-g(y)|| + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ||z(s)-y(s)||\, ds \\
& + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left|\left| f\left(s, z(s), \int_0^s h(t,\tau) z(\tau) d\tau\right) \right.\right. \\
& \left.\left. - f\left(s, y(s), \int_0^s h(t,\tau) y(\tau) d\tau\right) \right|\right| ds, \\
\leq & \frac{\epsilon t^\alpha}{\Gamma(\alpha+1)} + \bar{K}\, ||z(t)-y(t)|| \\
& + \frac{(\lambda+L)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ||z(s)-y(s)||\, ds \\
& + \frac{L h_T}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\int_0^s ||z(\tau)-y(\tau)||\, d\tau \right) ds.
\end{aligned}
$$

Thus

$$
\begin{aligned}
||z(t)-y(t)|| \leq & \frac{\epsilon t^\alpha}{\Gamma(\alpha+1)(1-\bar{K})} \\
& + \frac{(\lambda+L)}{\Gamma(\alpha)(1-\bar{K})} \int_0^t (t-s)^{\alpha-1} ||z(s)-y(s)||\, ds \\
& + \frac{L h_T}{\Gamma(\alpha)(1-\bar{K})} \int_0^t (t-s)^{\alpha-1} \left(\int_0^s ||z(\tau)-y(\tau)||\, d\tau \right) ds, \\
\leq & \frac{\epsilon t^\alpha}{\Gamma(\alpha+1)(1-\bar{K})} \\
& + \int_0^t \frac{(\lambda+L)}{\Gamma(\alpha)(1-\bar{K})} (T-s)^{\alpha-1} \Big[||z(s)-y(s)|| \\
& + \int_0^s \frac{L h_T}{(\lambda+L)} ||z(\tau)-y(\tau)||\, d\tau \Big] ds.
\end{aligned} \tag{9}
$$

By applying Pachpatte's inequality given in Theorem 1 to the inequality (9) with

$$
u(t) = ||z(t)-y(t)||, \; n(t) = \frac{\epsilon t^\alpha}{\Gamma(\alpha+1)(1-\bar{K})},
$$

$$
f(s) = \frac{(\lambda+L)}{\Gamma(\alpha)(1-\bar{K})} (T-s)^{\alpha-1}, \; q(\tau) = \frac{L h_T}{(\lambda+L)},
$$

we obtain

$$\begin{aligned}||z(t)-y(t)|| \leq & \frac{\epsilon t^{\alpha}}{\Gamma(\alpha+1)(1-\bar{K})}\Bigg[1+\int_0^T \frac{(\lambda+L)}{\Gamma(\alpha)(1-\bar{K})}(T-s)^{\alpha-1} \\ & \times \exp\Bigg(\int_0^s \Bigg\{\frac{(\lambda+L)}{\Gamma(\alpha)(1-\bar{K})}(T-\tau)^{\alpha-1}+\frac{Lh_T}{(\lambda+L)}\Bigg\}d\tau\Bigg)ds\Bigg], \\ \leq & \frac{\epsilon T^{\alpha}}{\Gamma(\alpha+1)(1-\bar{K})}\Bigg[1+\int_0^T \frac{(\lambda+L)}{\Gamma(\alpha)(1-\bar{K})}(T-s)^{\alpha-1} \\ & \times \exp\Bigg(\int_0^s \Bigg\{\frac{(\lambda+L)}{\Gamma(\alpha)(1-\bar{K})}(T-\tau)^{\alpha-1}+\frac{Lh_T}{(\lambda+L)}\Bigg\}d\tau\Bigg)ds\Bigg].\end{aligned}$$

Putting

$$\begin{aligned}C = & \frac{T^{\alpha}}{\Gamma(\alpha+1)(1-\bar{K})}\Bigg[1+\int_0^T \frac{(\lambda+L)}{\Gamma(\alpha)(1-\bar{K})}(T-s)^{\alpha-1} \\ & \times \exp\Bigg(\int_0^s \Bigg\{\frac{(\lambda+L)}{\Gamma(\alpha)(1-\bar{K})}(T-\tau)^{\alpha-1}+\frac{Lh_T}{(\lambda+L)}\Bigg\}d\tau\Bigg)ds\Bigg],\end{aligned}$$

we obtain $||z(t)-y(t)|| \leq C\epsilon$, $\forall t \in J$. Thus the problem (1)-(2) is Ulam-Hyers stable. □

Corollary 1. *If f and g in the nonlocal problem (1)-(2) satisfy the conditions (H1), (H5) and the inequality (5) hold, then the nonlocal problem (1)-(2) is generalized Ulam-Hyers stable.*

Theorem 5. *Assume that (H1), (H5) and inequality (5) hold. Further suppose there exist an increasing function $\varphi \in C(J, \mathbf{R}_+)$ and $\kappa_\varphi > 0$ such that $I^{\alpha}\varphi(t) \leq \kappa_\varphi \varphi(t)$, for any $t \in J$. Then the nonlocal problem (1)-(2) is Ulam-Hyers-Rassias stable.*

Proof. Let $z \in C^1(J, \mathbf{R})$ be a solution of the following inequality

$$\left|\left|{}^cD^{\alpha}z(t)-\lambda z(t)-f\left(t, z(t), \int_0^t h(t,\tau)z(\tau)d\tau\right)\right|\right| \leq \epsilon\varphi(t), \tag{10}$$

for any $t \in J$, $\epsilon > 0$. Let $y \in C(J, \mathbf{R})$ be the unique solution of the following Cauchy problem

$$\begin{aligned}{}^cD^{\alpha}y(t) = & \lambda y(t)+f\left(t, y(t), \int_0^t h(t,\tau)y(\tau)d\tau\right),\ t \in J;\ 0<\alpha\leq 1, \\ & z(0)+g(y)=y_0.\end{aligned}$$

By Lemma 4, we have

$$\begin{aligned}y(t) = & y_0-g(y)+\frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,ds \\ & +\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f\left(s, y(s), \int_0^s h(t,\tau)y(\tau)d\tau\right)ds.\end{aligned}$$

By integrating (10), we obtain

$$\begin{aligned} \Bigg\| z(t) - y_0 + g(z) - \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s)\, ds \\ - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, z(s), \int_0^s h(t,\tau) z(\tau) d\tau\right) ds \Bigg\| \\ \leq \frac{\epsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(t) ds = \epsilon I^\alpha \varphi(t) \leq \epsilon \kappa_\varphi \varphi(t). \qquad (11) \end{aligned}$$

Further for any $t \in J$ we have

$$\begin{aligned} ||z(t) - y(t)|| \leq & \Bigg\| z(t) - y_0 + g(z) - \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s)\, ds \\ & - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, z(s), \int_0^s h(t,\tau) z(\tau) d\tau\right) ds \Bigg\| \\ & + ||g(z) - g(y)|| + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ||z(s) - y(s)||\, ds \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Bigg\| f\left(s, z(s), \int_0^s h(t,\tau) z(\tau) d\tau\right)) \\ & - f\left(s, y(s), \int_0^s h(t,\tau) y(\tau) d\tau\right) \Bigg\| ds. \end{aligned}$$

Using inequality (11), conditions (H1) and (H5), we obtain

$$\begin{aligned} ||z(t) - y(t)|| \leq\ & \epsilon \kappa_\varphi \varphi(t) + \bar{K} \, |z(t) - y(t)| \\ & + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ||z(s) - y(s)||\, ds \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} L\Bigg(||z(s) - y(s)|| \\ & + \int_0^s |h(t,\tau)|\, ||z(\tau) - y(\tau)||\, d\tau \Bigg) ds \\ \leq\ & \epsilon \kappa_\varphi \varphi(t) + \bar{K} \, |z(t) - y(t)| \\ & + \frac{(\lambda + L)}{\Gamma(\alpha)} \int_0^t (T-s)^{\alpha-1} ||z(s) - y(s)||\, ds \\ & + \frac{L h_T}{\Gamma(\alpha)} \int_0^t (T-s)^{\alpha-1} \left(\int_0^s ||z(\tau) - y(\tau)||\, d\tau \right) ds. \end{aligned}$$

Thus

$$\begin{aligned}||z(t)-y(t)|| &\le \frac{\epsilon\kappa_\varphi\varphi(t)}{(1-\bar{K})}+\frac{(\lambda+L)}{\Gamma(\alpha)(1-\bar{K})}\int_0^t (T-s)^{\alpha-1}\,||z(s)-y(s)|)|\,ds\\ &\quad+\frac{Lh_T}{\Gamma(\alpha)(1-\bar{K})}\int_0^t (T-s)^{\alpha-1}\left(\int_0^s ||z(\tau)-y(\tau)||\,d\tau\right)ds\\ &\le \frac{\epsilon\kappa_\varphi\varphi(t)}{(1-\bar{K})}+\int_0^t \frac{(\lambda+L)}{\Gamma(\alpha)(1-\bar{K})}(T-s)^{\alpha-1}\Bigg[||z(s)-y(s)||\\ &\quad+\int_0^s \frac{Lh_T}{(\lambda+L)}\,||z(\tau)-y(\tau)||\,d\tau\Bigg]\,ds.\end{aligned}$$

Now by applying Pachpatte's inequality given in the Theorem 1 with

$$\begin{aligned}u(t) &= ||z(t)-y(t)||\,,\quad n(t)=\frac{\epsilon\kappa_\varphi\varphi(t)}{(1-\bar{K})},\\ f(s) &= \frac{(\lambda+L)}{\Gamma(\alpha)(1-\bar{K})}(T-s)^{\alpha-1},\quad q(\tau)=\frac{Lh_T}{(\lambda+L)},\end{aligned}$$

we obtain

$$\begin{aligned}||z(t)-y(t)|| &\le \frac{\epsilon\kappa_\varphi\varphi(t)}{(1-\bar{K})}\Bigg[1+\int_0^t \frac{(\lambda+L)}{\Gamma(\alpha)(1-\bar{K})}(T-s)^{\alpha-1}\\ &\quad\times\exp\left(\int_0^s\left\{\frac{(\lambda+L)}{\Gamma(\alpha)(1-\bar{K})}(T-\tau)^{\alpha-1}+\frac{Lh_T}{(\lambda+L)}\right\}d\tau\right)ds\Bigg]\\ &\le \frac{\epsilon\kappa_\varphi\varphi(t)}{(1-\bar{K})}\Bigg[1+\int_0^T \frac{(\lambda+L)}{\Gamma(\alpha)(1-\bar{K})}(T-s)^{\alpha-1}\\ &\quad\times\exp\left(\int_0^s\left\{\frac{(\lambda+L)}{\Gamma(\alpha)(1-\bar{K})}(T-\tau)^{\alpha-1}+\frac{Lh_T}{(\lambda+L)}\right\}d\tau\right)ds\Bigg].\end{aligned}$$

Thus we have $||z(t)-y(t)||\le C\epsilon\varphi(t)$, $\forall t\in J$, where

$$\begin{aligned}C &= \frac{\kappa_\varphi}{(1-\bar{K})}\Bigg[1+\int_0^T \frac{(\lambda+L)}{\Gamma(\alpha)(1-\bar{K})}(T-s)^{\alpha-1}\\ &\quad\times\exp\left(\int_0^s\left\{\frac{(\lambda+L)}{\Gamma(\alpha)(1-\bar{K})}(T-\tau)^{\alpha-1}+\frac{Lh_T}{(\lambda+L)}\right\}d\tau\right)ds\Bigg].\end{aligned}$$

□

Corollary 2. *Under the assumptions of Theorem 5, the nonlocal problem (1)-(2) is generalized Ulam-Hyers-Rassias stable.*

4 Example

In this section, we illustrate our main results with the help of following example.

Consider the nonlocal problem:

$$
\begin{aligned}
{}^{c}D^{1/2}y(t) &= \frac{1}{10}y(t) + \frac{e^{-t}}{(9+e^{t})}\left[\frac{|y(t)|}{1+|y(t)|}\right] + \frac{1}{10}\int_0^t \frac{e^{-t}}{(3+t)^2}y(s)ds,\ t\in[0,1], && (12)\\
& y(0) + \sum_{i=1}^{n} c_i y(t_i) = 1, && (13)
\end{aligned}
$$

where $0 < t_1 < \ldots < t_n < 1$ and c_i, $i = 1, 2, \ldots, n$ are positive constants with

$$\sum_{i=1}^{n} c_i \leq \frac{1}{5}. \tag{14}$$

Problem (12)-(13) is of the form (1)-(2) with $\alpha = \frac{1}{2},\ \lambda = \frac{1}{10}$,

$$f(t, y(t), Hy(t)) = \frac{e^{-t}}{(9+e^{t})}\left[\frac{|y(t)|}{1+|y(t)|}\right] + \frac{1}{10}Hy(t),\ t\in[0,1],\ y\in[0,\infty),$$

where

$$Hy(t) = \int_0^t \frac{e^{-t}}{(3+t)^2}y(s)ds.$$

Clearly, the function f is continuous. For each $y, \bar{y} \in \mathbf{R}$ and $t \in [0,1]$

$$||f(t, y, Hy(t)) - f(t, \bar{y}, H\bar{y}(t))|| \leq \frac{1}{10}\Big[\ ||y-\bar{y}|| + ||Hy - H\bar{y}||\Big].$$

Also, we have

$$||g(y) - g(\bar{y})|| \leq \left|\left|\sum_{i=1}^{n} c_i y - \sum_{i=1}^{n} c_i \bar{y}\right|\right| \leq \sum_{i=1}^{n} c_i\, ||y-\bar{y}|| \leq \frac{1}{5}\, ||y-\bar{y}||\,.$$

Hence conditions (H1) and (H5) are satisfied with $L = \frac{1}{10}$, $\bar{K} = \frac{1}{5}$, $h_T = \frac{1}{9}$ and $\lambda = \frac{1}{10}$. We have

$$\left[\bar{K} + \frac{(\lambda+L)T^{\alpha} + Lh_T T^{\alpha+1}}{\Gamma(\alpha+1)}\right] = \left[\frac{1}{5} + \frac{(\frac{1}{10}+\frac{1}{10}) + \frac{1}{90}}{\Gamma(\frac{3}{2})}\right] = \frac{1}{5} + \frac{19}{45\sqrt{\pi}} < 1.$$

It follows from Theorem 2 that the problem (12)-(13) has a unique solution on [0,1] and by Theorem 4, the problem (12)-(13) is Ulam-Hyers stable.

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