Journal of Algebra and Related Topics
Vol. 6, No 2, (2018), pp 97-106

PRIME EXTENSION DIMENSION OF A MODULE

T. DURAIVEL, S. MANGAYARCARASSY, AND K. PREMKUMAR*

Abstract. We have that for a finitely generated module M over a Noetherian ring A any two RPE filtrations of M have same length. We call this length as prime extension dimension of M and denote it as $\text{pe.d}_A(M)$. This dimension measures how far a module is from torsion freeness. We show for every submodule N of M, $\text{pe.d}_A(N) \leq \text{pe.d}_A(M)$ and $\text{pe.d}_A(N)+\text{pe.d}_A(M/N) \geq \text{pe.d}_A(M)$. We compute the prime extension dimension of a module using the prime extension dimensions of its primary submodules which occurs in a minimal primary decomposition of 0 in M.

1. Introduction

Throughout this paper all rings are commutative, Noetherian with identity and all modules are finitely generated and unitary. For standard reference and notations see [5]. Let A be a ring and M be an A-module. Then we can have a filtration

$$\mathcal{F} : 0 = M_0 \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n = M$$

of submodules of M. If $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$ for $i = 1, \ldots, n$, then $\mathcal{F}$ is called a prime filtration of the module M. Various kinds of filtrations are studied in [1], [3] and [4]. In [2] Prime

This work was done while the third author was at Pondicherry University and was funded by CSIR-SRF, India. Most results of this paper are contained in the third author thesis [6] under the supervisor of the first author that was defended in 2017 at Pondicherry University, India.
MSC(2010): Primary 13C15; Secondary 13E05.
Keywords: prime Submodules, primary Decomposition, prime filtration and regular prime extension filtration.
Received: 5 September 2018, Accepted: 17 December 2018.
*Corresponding author .

extension filtration for a finitely generated module over a Noetherian ring is defined and studied. We say a submodule K of M is $\mathfrak{p}$-prime extension of a proper submodule N of M, if N is prime submodule of K with $(N : K) = \mathfrak{p}$, that is K/N is a $A/\mathfrak{p}$-torsion free module.

In [2], it is shown that, $\mathfrak{p}$-prime extension of a proper submodule N of M exists if and only if $\mathfrak{p} \in \mathrm{Ass}(M/N)$. A filtration of submodules $N = M_0 \subset M_1 \subset \cdots \subset M_n = M$ of M is called a prime extension filtration of M over N, if each M_i is a prime extension of M_{i-1} in M. Clearly, a prime filtration of a module M is a prime extension filtration of M over 0. If $N = 0$, a prime extension filtration of M is called as weak prime decomposition of M by Dress [1]. Further, if each M_i is a maximal $\mathfrak{p}_i$-prime extension of M_{i-1} in M, then the filtration is called a maximal prime extension (MPE) filtration of M over N and it is proved that $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} = \mathrm{Ass}(M)$ (see [2, Theorem 14]).

A submodule K is a regular prime extension of N, if K is maximal $\mathfrak{p}$-prime extension of N in M and $\mathfrak{p}$ is a maximal element in $\mathrm{Ass}(M/N)$. A prime extension filtration $0 \overset{\mathfrak{p}_1}{\subset} M_1 \subset \cdots \subset M_{n-1} \overset{\mathfrak{p}_n}{\subset} M_n = M$ is called a regular prime extension (RPE) filtration, if for each i, $1 \le i \le n$, M_i is regular $\mathfrak{p}_i$-prime extension of M_{i-1} in M. So we have that $\mathfrak{p}_i \not\subset \mathfrak{p}_j$, for $1 \le i < j \le n$.

In [2, Theorem 22], it is proved that every module M has a RPE filtration and any two RPE filtration have same length. So this length is a numerical invariant for the module M. In this article we call it as prime extension dimension of M and we denote it as $\mathrm{pe.d}_A(M)$. This dimension measures, how far a module is from torsion freeness. We have that M is torsion free $A/\mathrm{ann}(M)$-module if and only if $\mathrm{pe.d}_A(M) = 1$ and therefore A is an integral domain if and only if $\mathrm{pe.d}_A(A) = 1$. For a submodule L of M we prove $\mathrm{pe.d}_A(L) \le \mathrm{pe.d}_A(M)$ and $\mathrm{pe.d}_A(M) \le \mathrm{pe.d}_A(L) + \mathrm{pe.d}_A(M/L)$. We show that the prime extension dimension of a module is infimum of the lengths of prime extension filtrations of the module and we deduce that if M is a finite length module then $\mathrm{pe.d}_A(M) \le l(M)$, the length of the module. We also compute the prime extension dimension of a module using the prime extension dimensions of its primary submodules which occur in a minimal primary decomposition of 0 in M.

2. Prime Extension Dimension of a Submodule

We begin with a summary of the following result that will be useful in the following sections.

Proposition 2.1. [2, Theorem 22]
Let N be a proper submodule of M. For any prime ideal $\mathfrak{p}$ of A the

number of times $\mathfrak{p}$ occurs in any two RPE filtration of M over N are equal, and hence any two RPE filtration of M over N have same length.

From the above proposition we have the following definitions.

Definition 2.2. Let M be a nonzero A-module. The prime extension dimension (PE dimension) of M is the length of a RPE filtration of M, and we denote it as $\text{pe.d}_A(M)$. We define the prime extension dimension of the zero module is 0.

Definition 2.3. Let M be an A-module. For a prime ideal $\mathfrak{p}$ of A, the $\mathfrak{p}$-prime extension dimension of M is the number of times $\mathfrak{p}$-occurs in a RPE filtration of M and we denote it as $\mathfrak{p}\text{-pe.d}_A(M)$.

Proposition 2.4. *Let M be an A-module. Then*

$$\text{pe.d}_A(M) = \sum_{\mathfrak{p}\in \text{Spec}(A)} \mathfrak{p}\text{-pe.d}_A(M) = \sum_{\mathfrak{p}\in \text{Ass}(M)} \mathfrak{p}\text{-pe.d}_A(M)$$

Proof. The first equality follows from the definition of RPE filtration and Proposition 2.1. By [2, Corollary 15], the prime ideal $\mathfrak{p}$ occurs in an RPE filtration of M if and only if $\mathfrak{p} \in \text{Ass}(M)$. That is $\mathfrak{p}\text{-pe.d}_A(M) \neq 0$ if and only if $\mathfrak{p} \in \text{Ass}(M)$. This proves the second equality. □

Remark 2.5. If $0 \overset{\mathfrak{p}_1}{\subset} M_1 \overset{\mathfrak{p}_2}{\subset} M_2 \subset \cdots \subset M_{n-1} \overset{\mathfrak{p}_n}{\subset} M_n = M$ is a RPE filtration of M and $\mathfrak{p}_{i_1}, \dots, \mathfrak{p}_{i_n}$ is a permutation of $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ satisfying the condition that $\mathfrak{p}_{i_j} \not\subset \mathfrak{p}_{i_k}$, where $1 \leq j < k \leq n$, then we can have a RPE filtration $0 \overset{\mathfrak{p}_{i_1}}{\subset} M'_1 \subset \cdots \subset M'_{n-1} \overset{\mathfrak{p}_{i_n}}{\subset} M'_n = M$ of M. In particular, if $\mathfrak{p}$ is a minimal element in $\text{Ass}(M)$, with $\text{pe.d}_A(M) = r$, then we can have a RPE filtration

$$0 \subset M_1 \subset \cdots \subset M_{n-r} \overset{\mathfrak{p}_{n-r+1}}{\subset} M_{n-r+1} \subset \cdots \overset{\mathfrak{p}_n}{\subset} M_n = M$$

where $\mathfrak{p}_i = \mathfrak{p}$ for $i = n - r + 1, \dots, n$.

Remark 2.6. PE dimension of a module measures how far the module is from torsion-freeness. Note that $\text{pe.d}_A(M) = 1$ if and only if $\text{ann}(M)$ is a prime ideal and M is torsion free $A/\text{ann}(M)$-module.

For $\text{pe.d}_A(M) = 1 \Leftrightarrow 0 \subset M$ is a RPE filtration of $M \Leftrightarrow 0$ is prime submodule of M with $(0 : M)$ is a prime ideal $\Leftrightarrow M$ is torsion free $A/\text{ann}(M)$-module. In particular, a ring A is integral domain if and only if $\text{pe.d}_A(A) = 1$.

Example 2.7. Consider the $\mathbb{Z}$-module $\mathbb{Z}/p^r\mathbb{Z}$ for some prime p and positive integer r. Then

$$0 \overset{p\mathbb{Z}}{\subset} \frac{p^{r-1}\mathbb{Z}}{p^r\mathbb{Z}} \subset \cdots \subset \frac{p\mathbb{Z}}{p^r\mathbb{Z}} \overset{p\mathbb{Z}}{\subset} \frac{\mathbb{Z}}{p^r\mathbb{Z}}$$

is a RPE filtration of $\mathbb{Z}/p^r\mathbb{Z}$ and hence $\text{pe.d}_{\mathbb{Z}}(\mathbb{Z}/p^r\mathbb{Z}) = r$. More generally, if $n = p_1^{r_1} \cdots p_k^{r_k}$ is the prime factorization of an integer n, then $\text{pe.d}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}) = r_1 + \cdots + r_k$.

Next we show PE dimension of a submodule of M is less than or equal to PE dimension of M. We need the following lemma.

Lemma 2.8. *Let N be a proper submodule of an A-module M. If K is a $\mathfrak{p}$-prime extension of N in M, then for any submodule L of M, either $L \cap N = L \cap K$ or $L \cap K$ is a $\mathfrak{p}$-prime extension of $L \cap N$ in L. Further, if K is regular $\mathfrak{p}$-prime extension of N, then $L \cap K$ is regular $\mathfrak{p}$-prime extension of $L \cap N$ in L, when $L \cap N \neq L \cap K$.*

Proof. Assume $L \cap N \neq L \cap K$. Then $(N : K) = (L \cap N : L \cap K)$. For, if $a \in (N : K)$, then $a(L \cap K) \subseteq aL \cap aK \subseteq L \cap N$ this implies $a \in (L \cap N : L \cap K)$. Conversely, let $a \in (L \cap N : L \cap K)$. For $x \in L \cap K \setminus L \cap N$, $ax \in L \cap N \subset N$. Since K is a $\mathfrak{p}$-prime extension of N and $x \notin N$, $a \in (N : K)$. Hence $(L \cap N : L \cap K) = (N : K) = \mathfrak{p}$.

Suppose $ax \in N \cap L$ for some $x \in K \cap L$ and $a \in A$. If $x \notin N \cap L$, then $x \in L$ implies $x \notin N$. Since $ax \in N$, $a \in \mathfrak{p} = (N \cap L : K \cap L)$. So $K \cap L$ is a $\mathfrak{p}$-prime extension of $N \cap L$ in L.

Now we assume K is regular $\mathfrak{p}$-prime extension of N and $L \cap N \neq L \cap K$. Since $L/(N \cap L) \cong (N + L)/N \subseteq M/N$, $\mathfrak{p}$ is a maximal element in $\text{Ass}(L/N \cap L)$. Suppose $K \cap L$ is not a maximal $\mathfrak{p}$-prime extension of $N \cap L$. Then there exists a submodule L' of L, such that L' is the unique maximal $\mathfrak{p}$-prime extension of $N \cap L$ in L. Let $x \in L'$. Then $\mathfrak{p}x \subseteq N \cap L$, this implies $\mathfrak{p} \subseteq (N : x)$. Since $\mathfrak{p}$ is a maximal element in $\text{Ass}(M/N)$ and by [2, Theorem 11], $x \in K$. That is $x \in K \cap L$ and therefore $K \cap L$ is a maximal $\mathfrak{p}$-prime extension of $N \cap L$ in L. □

Theorem 2.9. *Let L be any submodule of an A-module M and $0 \subset M_1 \subset \cdots \subset M_n = M$ be an RPE filtration of M. Then the filtration obtained from the chain $0 \subseteq M_1 \cap L \subseteq \cdots \subseteq M_n \cap L = L$ by removing the equalities is an RPE filtration of L and therefore $\text{pe.d}_A(L) \leq \text{pe.d}_A(M)$.*

Proof. Let $0 \subset M_1 \subset \cdots \subset M_n = M$ be a RPE filtration of M with length n. Now intersecting L with this filtration we have a chain $0 \subseteq L \cap M_1 \subseteq \cdots \subseteq L \cap M_n = L$ of submodules of L. After removing the equalities in the above chain we get a filtration of L in which each extension is regular prime extension by Lemma 2.8. That is, we have a RPE filtration of L of length less than or equal to n. Hence $\mathfrak{p}\text{-pe.d}_A(L) \leq \mathfrak{p}\text{-pe.d}_A(M)$ for every prime ideal $\mathfrak{p}$ and $\text{pe.d}_A(L) \leq \text{pe.d}_A(M)$. □

3. Prime Extension Dimension of a Quotient Module

Next we give two important lemma which is useful to prove the subsequent results.

Lemma 3.1. *Let* $N = M_0 \overset{\mathfrak{p}_1}{\subset} M_1 \subset \cdots \subset M_{i-1} \overset{\mathfrak{p}_i}{\subset} M_i \subset \cdots \overset{\mathfrak{p}_n}{\subset} M_n = M$ *be a RPE filtration of* M *over* N*. Then* $M_i = \{x \in M \mid \mathfrak{p}_1 \cdots \mathfrak{p}_i x \subseteq N\}$.

Proof. Proof by induction on i. For $i = 1$, the result is true by [2, Theorem 11]. Now assume this result is true for $i-1$, that is $M_{i-1} = \{x \in M \mid \mathfrak{p}_1 \cdots \mathfrak{p}_{i-1} x \subseteq N\}$. Let $x \in M_i$. Then $\mathfrak{p}_i x \subseteq M_{i-1}$ and by induction assumption $\mathfrak{p}_1 \cdots \mathfrak{p}_{i-1}(\mathfrak{p}_i x) \subseteq N$. Conversely, suppose $x \notin M_i$. Since M_i is a regular $\mathfrak{p}_i$-prime extension of M_{i-1}, there exists an element $a \in \mathfrak{p}_i$, such that $ax \notin M_{i-1}$. This gives $\mathfrak{p}_1 \cdots \mathfrak{p}_{i-1}(ax) \nsubseteq N$, and so $\mathfrak{p}_1 \cdots \mathfrak{p}_i x \nsubseteq N$. □

Lemma 3.2. *Let* $0 \subset M_1 \subset \cdots \subset M_{i-1} \overset{\mathfrak{p}_i}{\subset} M_i \subset \cdots \subset M_n = M$ *be a RPE filtration of* M *and* N *be a any submodule of* M*. If for some* i, $N \cap M_{i-1} = N \cap M_i$*, then* $(N : x) \subseteq \mathfrak{p}_i$*, for every* $x \in M_i \setminus M_{i-1}$.

Proof. Let $x \in M_i \setminus M_{i-1}$. Then $(M_{i-1} : x) = \mathfrak{p}_i$. If $x \in N$, then $x \in N \cap M_i = N \cap M_{i-1}$, this implies $x \in M_{i-1}$ which is a contradiction, so $x \notin N$. Let $a \in (N : x)$. Then $ax \in N$ implies $ax \in N \cap M_i = N \cap M_{i-1}$, that is $ax \in M_{i-1}$, this implies $a \in (M_{i-1} : x) = \mathfrak{p}_i$. Hence $(N : x) \subseteq \mathfrak{p}_i$. □

Proposition 3.3. *Let* M *be an* A*-module and* $N \overset{\mathfrak{p}}{\subset} K$ *be a* $\mathfrak{p}$*-prime extension in* M*. Then* $\text{pe.d}_A(K) \leq \text{pe.d}_A(N) + 1$.

Proof. Let

$$0 \overset{\mathfrak{p}_1}{\subset} K_1 \subset \cdots \subset K_{i-1} \overset{\mathfrak{p}_i}{\subset} K_i \subset \cdots \subset K_{n-1} \overset{\mathfrak{p}_n}{\subset} K_n = K \tag{3.1}$$

be a RPE filtration of K. Suppose $N \cap K_{i-1} = N \cap K_i$. We claim $\mathfrak{p}_i = \mathfrak{p}$. Let $x \in K_i \setminus K_{i-1}$. Then clearly $x \notin N$ and since K is a $\mathfrak{p}$-prime extension of N, $(N : x) = \mathfrak{p}$. Suppose the prime ideal $\mathfrak{p}_i$ occurs l times in the sequence $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_i$. Since (3.1) is RPE filtration, if $j < i$ and $\mathfrak{p}_j \neq \mathfrak{p}_i$, then $\mathfrak{p}_j \nsubseteq \mathfrak{p}_i$ and therefore we can choose a $p_j \in \mathfrak{p}_j \setminus \mathfrak{p}_i$ and p denotes their product, that is $p = \prod_{j<i} p_j$. Since $p \notin \mathfrak{p}_i$ and $x \notin K_{i-1}$, we have that $px \notin K_{i-1}$. Note $px \notin N$, otherwise $px \in K_i \cap N = K_{i-1} \cap N \subset K_{i-1}$, a contradiction. So $(N : px) = \mathfrak{p}$. Now by Lemma 3.1 $\mathfrak{p}_i^l(px) = 0 \in N$, this implies $\mathfrak{p}_i^l \subseteq (N : px) = \mathfrak{p}$, that is $\mathfrak{p}_i \subseteq \mathfrak{p}$. Also, by Lemma 3.2, $\mathfrak{p} = (N : x) \subseteq \mathfrak{p}_i$. Hence $\mathfrak{p}_i = \mathfrak{p}$. Next we show that two equalities can not occur on

$$0 \subseteq K_1 \cap N \subseteq \cdots \subseteq K_n \cap N = N \tag{3.2}$$

Suppose an equality occur at the ith place and the next equality occur at jth place in (3.2). That is $N\cap K_{i-1} = N\cap K_i$ and $N\cap K_{j-1} = N\cap K_j$ for $i < j$. Then by above claim $\mathfrak{p} = \mathfrak{p}_i = \mathfrak{p}_j$. Since for $i \leq r < j$, $N \cap K_{r-1} \neq N \cap K_r$, $\mathfrak{p}_r \neq \mathfrak{p}$ and because (3.2) is a RPE filtration, $\mathfrak{p}_r \nsubseteq \mathfrak{p}$. Let $y \in K_j \setminus K_{j-1}$. If $j = i+1$, choose $p' = 1$. Suppose $j > i+1$, let $p' = p_{i+1}\cdots p_{j-1}$, for some $p_r \in \mathfrak{p}_r \setminus \mathfrak{p}$, $i < r < j$. In either case $p'y \notin K_{j-1}$. Consider the RPE filtration

$$K_i \overset{\mathfrak{p}_{i+1}}{\subset} K_{i+1} \subset \cdots \overset{\mathfrak{p}_j}{\subset} K_j \subset \cdots \subset K_n = K$$

of K over K_i. Applying Lemma 3.1 to this filtration we have that $\mathfrak{p}_{i+1}\mathfrak{p}_{i+2}\cdots\mathfrak{p}_{j-1}\mathfrak{p}_j y \subseteq K_i$, since $y \in K_j$. Then $p' \in \mathfrak{p}_{i+1}\mathfrak{p}_{i+2}\cdots\mathfrak{p}_{j-1}$ implies that $\mathfrak{p}p'y \subseteq K_i$. Suppose $\mathfrak{p}p'y \subseteq N$, then $\mathfrak{p}p'y \subseteq K_i \cap N = K_{i-1} \cap N \subseteq K_{i-1}$. Therefore $\mathfrak{p} \subseteq (K_{i-1} : p'y)$ and by [2, Theorem 11], $p'y \in K_i \subseteq K_{j-1}$, which is not the case. Hence $\mathfrak{p} \nsubseteq (N : p'y)$. This is a contradiction to the fact that K is a $\mathfrak{p}$-prime extension of N. Therefore two equalities can not occur on $0 \subset N \cap K_1 \subset \cdots \subset N \cap K_n = N$. Hence by Theorem 2.9, $\text{pe.d}_A(K) \leq \text{pe.d}_A(N) + 1$. ☐

Proposition 3.4. *If N is a submodule of M which occurs in a RPE filtration of M, then* $\text{pe.d}_A(M/N) + \text{pe.d}_A(N) = \text{pe.d}_A(M)$.

Proof. Let $0 \subset M_1 \subset \cdots \subset M_n = M$ be a RPE filtration of M with $\text{pe.d}_A(M) = n$ and $N = M_i$ is a submodule which occur in RPE filtration of M for some i. Then clearly $\text{pe.d}_A(N) = i$ and $\text{pe.d}_A(M/N) = n - i$. ☐

Example 3.5. Converse of the Proposition 3.4 is not necessarily true.

Let $A = \mathbb{Z}$ and $M = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$, $N = 2\mathbb{Z}/4\mathbb{Z} \oplus 2\mathbb{Z}/6\mathbb{Z}$. Then clearly $\text{pe.d}_A(M/N) + \text{pe.d}_A(N) = \text{pe.d}_A(M)$, but N does not occur on any RPE filtration of M.

Theorem 3.6. *Let M be an A-module and N be any submodule of M. Then* $\text{pe.d}_A(N) + \text{pe.d}_A(M/N) \geq \text{pe.d}_A(M)$.

Proof. The proof is by induction on $\text{pe.d}_A(M/N)$. If $\text{pe.d}_A(M/N) = 1$, then $N \subset M$ is a RPE filtration of M over N. By Proposition 3.3,

$$\text{pe.d}_A(M) \leq \text{pe.d}_A(N) + 1 = \text{pe.d}_A(N) + \text{pe.d}_A(M/N)$$

Now assume that this result is true for $\text{pe.d}_A(M/N) \leq n$. Suppose $\text{pe.d}_A(M/N) = n+1$, then we have a RPE filtration $N \subset M_1 \subset \cdots \subset M_{n+1} = M$ of M over N. Since $\text{pe.d}_A(M/M_1) = n = \text{pe.d}_A(M/N) - 1$

and $N \subset M_1$ is a prime extension, by induction assumption we have

$$\begin{aligned} \text{pe.d}_A(M/N) - 1 &= \text{pe.d}_A(M/M_1) \geq \text{pe.d}_A(M) - \text{pe.d}_A(M_1) \\ \text{pe.d}_A(M/N) - 1 &\geq \text{pe.d}_A(M) - (\text{pe.d}_A(N) + 1) \text{ (by Proposition 3.3)} \\ \text{pe.d}_A(M/N) &\geq \text{pe.d}_A(M) - \text{pe.d}_A(N). \end{aligned}$$

Hence $\text{pe.d}_A(N) + \text{pe.d}_A(M/N) \geq \text{pe.d}_A(M)$. □

Next corollary shows that PE dimension of a module is less than or equal to the length of any prime extension filtration of that module.

Corollary 3.7. *Let M be an A-module and $0 \subset M_1 \subset \cdots \subset M_n = M$ be a prime extension filtration of M. Then $\text{pe.d}_A(M) \leq n$.*

Proof. Proof by induction on n. For $n = 1$, it is trivial. Now assume this result is true for any A-module having prime extension filtration of length $n-1$. Then by induction assumption $\text{pe.d}_A(M/M_1) \leq n-1$. Since $\text{pe.d}_A(M_1) = 1$ and by Theorem 3.6, $\text{pe.d}_A(M) \leq \text{pe.d}_A(M/M_1) + \text{pe.d}_A(M_1) \leq n$. □

Remark 3.8. By the above corollary and the fact that RPE filtrations are prime extension filtrations, PE dimension of a module is infimum of all length of prime extension filtrations of a module.

Corollary 3.9. *PE dimension of an A-module M is less than or equal to the length of the module M.*

Proof. It is trivial if length of M is not finite. So we assume that M is of finite length. Since a composition series of M is a prime filtration of M and prime filtrations are prime extension filtrations, by Corollary 3.7, length of M is greater than or equal to PE dimension of M. □

4. Prime Extension Dimension of Primary Submodules

Next we show that we can compute the PE dimension of a module using the PE dimension of primary submodule of M which occurs in a minimal primary decomposition of 0 in M.

Theorem 4.1. *Let N be a $\mathfrak{p}$-primary submodule of an A-module M which occur in a minimal primary decomposition of 0 in M. Then $\text{pe.d}_A(N) = \text{pe.d}_A(M) - \mathfrak{p}\text{-pe.d}_A(M)$.*

Proof. Let $N_1 \cap \cdots \cap N_r = 0$ be a minimal primary decomposition of 0 in M and let $N = N_1$. Since we have the injective homomorphism $N \to \dfrac{M}{N_2 \cap \cdots \cap N_r}$, $\text{Ass}(N) \subseteq \text{Ass}\left(\frac{M}{N_2 \cap \cdots \cap N_r}\right) = \{\mathfrak{p}_2, \ldots, \mathfrak{p}_r\}$. That is

$\mathfrak{p}_1 \notin \operatorname{Ass}(N)$. Now consider a RPE filtration $0 \overset{\mathfrak{p}_1}{\subset} M_1 \subset \cdots \subset M_{i-1} \overset{\mathfrak{p}_i}{\subset} M_i \subset \cdots \overset{\mathfrak{p}_n}{\subset} M_n = M$ of M. Then by intersecting this filtration with N we have a chain of submodules of N,

$$0 \subseteq M_1 \cap N \subseteq \cdots \subseteq M_{i-1} \cap N \subseteq M_i \cap N \subseteq \cdots \subseteq M_n \cap N = N \quad (4.1)$$

Now we claim $\mathfrak{p} = \mathfrak{p}_i$ if and only if $M_{i-1} \cap N = M_i \cap N$. We have that $\mathfrak{p} \notin \operatorname{Ass}(N)$ if and only if $\mathfrak{p}$ does not occur on (4.1) by [2, Corollary 15]. By Theorem 2.9, after removing the equalities in the chain (4.1) we have a RPE filtration of N. Therefore, whenever $\mathfrak{p} = \mathfrak{p}_i$, $M_{i-1} \cap N = M_i \cap N$. Conversely, assume $\mathfrak{p} \neq \mathfrak{p}_i$ and we show $M_{i-1} \cap N \neq M_i \cap N$.

Case(i) $\mathfrak{p} \not\subseteq \mathfrak{p}_i$. Let $p \in \mathfrak{p} \setminus \mathfrak{p}_i$ and $x \in M_i \setminus M_{i-1}$. Since $p \in \mathfrak{p} = \sqrt{(N : M)}$, there exists a positive integer m such that $p^m M \subseteq N$. In particular $p^m x \in N$ and so $p^m x \in M_i \cap N$. Since $p^m \notin \mathfrak{p}_i$ and $x \notin M_{i-1}$, we have that $p^m x \notin M_{i-1}$, that is $p^m x \notin M_{i-1} \cap N$. Hence $M_{i-1} \cap N \neq M_i \cap N$.

Case(ii) $\mathfrak{p} \subset \mathfrak{p}_i$. By definition of RPE filtration, $\mathfrak{p}_t \not\subseteq \mathfrak{p}$, for $1 \leq t \leq i$. Let $p_t \in \mathfrak{p}_t \setminus \mathfrak{p}$ for $t = 1, \ldots, i$ and let $p' = p_1 \cdots p_i$. Let $x \in M_i$. Then by Lemma 3.1, $p'x = 0 \in N$. Since N is $\mathfrak{p}$-primary submodule of M and $p' \notin \mathfrak{p}$, we have that $x \in N$. That is $M_i \subseteq N$. If $M_{i-1} \cap N = M_i \cap N$, then $M_{i-1} = M_i$ a contradiction. Hence $M_{i-1} \cap N \neq M_i \cap N$.

So we prove that $M_{i-1} \cap N = M_i \cap N$ if and only if $\mathfrak{p} = \mathfrak{p}_i$. That is the number of equalities in (4.1) is exactly equal to the number of times $\mathfrak{p}$ occurs in the RPE filtration of M, that is $\mathfrak{p}\text{-pe.d}_A(M)$. So, the RPE filtration of N obtained from (4.1) by removing equalities has length $n - \mathfrak{p}\text{-pe.d}_A(M)$. That is $\text{pe.d}_A(N) = \text{pe.d}_A(M) - \mathfrak{p}\text{-pe.d}_A(M)$. □

Corollary 4.2. *Let $N_1 \cap \cdots \cap N_r = 0$ be a minimal primary decomposition of 0 in M. Then* $(r-1)\,\text{pe.d}_A(M) = \sum_{i=1}^{r} \text{pe.d}_A(N_i)$.

Proof. Let N_i be a $\mathfrak{p}_i$-primary submodule of M and $\operatorname{Ass}(M) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$. Then by Theorem 4.1,

$$\begin{aligned} \sum_{i=1}^{r} \text{pe.d}_A(N_i) &= \sum_{i=1}^{r} (\text{pe.d}_A(M) - \mathfrak{p}_i\text{-pe.d}_A(M)) \\ &= r \cdot \text{pe.d}_A(M) - \sum_{i=1}^{r} \mathfrak{p}_i\text{-pe.d}_A(M). \end{aligned}$$

This implies $(r-1)\,\text{pe.d}_A(M) = \sum_{i=1}^{r} \text{pe.d}_A(N_i)$ by Proposition 2.4. □

Theorem 4.3. *Let* $\mathfrak{p}$ *be a minimal element in* $\mathrm{Ass}(M)$*. Suppose* N *is a* $\mathfrak{p}$*-primary submodule of* M *which occur in a minimal primary decomposition of* 0 *in* M*, then* $\mathrm{pe.d}_A(M/N) = \mathfrak{p}\text{-}\mathrm{pe.d}_A(M)$*.*

Proof. Let $\mathfrak{p}$ be a minimal element in $\mathrm{Ass}(M)$ with $\mathfrak{p}\text{-}\mathrm{pe.d}_A(M) = r$. Then by Remark 2.5, there exists an RPE filtration

$$0 \subset M_1 \subset \cdots \subset M_{n-r} \overset{\mathfrak{p}_{n-r+1}}{\subset} M_{n-r+1} \subset \cdots \subset M_{n-1} \overset{\mathfrak{p}_n}{\subset} M_n = M \quad (4.2)$$

of M, with $\mathfrak{p}_i = \mathfrak{p}$ for $i = n-r+1, \ldots, n$. Applying the argument in the proof of the Theorem 4.1, to the chain $0 \subset M_1 \cap N \subset \cdots \subset M_n \cap N = N$ of N, we have that $M_{n-r} \cap N = \cdots = M_n \cap N = N$. This implies $M_{n-r} \supseteq N$. Next we show $M_{n-r} \subset N$. By the assumption on RPE filtration (4.2), $\mathfrak{p}_i \nsubseteq \mathfrak{p}$ for $i = 1, \ldots, n-r$. For each $1 \leq i \leq n-r$ choose $p_i \in \mathfrak{p}_i \setminus \mathfrak{p}$. If $x \in M_{n-r}$, then by Lemma 3.1, $p_1 \cdots p_{n-r} x = 0 \in N$. Since N is $\mathfrak{p}$-primary submodule of M and $p_1 \cdots p_{n-r} \notin \mathfrak{p}$, we have $x \in N$. Therefore $M_{n-r} \subseteq N$. So $M_{n-r} = N$ and by Proposition 3.4 and Theorem 4.1, $\mathrm{pe.d}_A(M/N) = r = \mathfrak{p}\text{-}\mathrm{pe.d}_A(M)$. □

Corollary 4.4. *Let* $N_1 \cap \cdots \cap N_r = 0$ *be a minimal primary decomposition of* 0 *in* M*. Suppose all the* $\mathfrak{p}_i$*-primary component of* 0 *are minimal (that is, all the elements of* $\mathrm{Ass}(M)$ *are minimal). Then*
$$\mathrm{pe.d}_A(M) = \sum_{i=1}^{r} \mathrm{pe.d}_A(M/N_i).$$

Proof. $$\mathrm{pe.d}_A(M) = \sum_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}\text{-}\mathrm{pe.d}_A(M) = \sum_{i=1}^{r} \mathrm{pe.d}_A(M/N_i),$$
by Theorem 4.3. □

Acknowledgments

The authors would like to thank the referee for the valuable suggestions and comments.

References

1. A. Dress, *A new algebraic criterion for shellability*, Beitrage zur Alg. und Geom, (1) **34** (1993), 45–55.
2. T. Duraivel, S. Mangayarcarassy and K. Premkumar, *Prime extension filtration of a module*, International Journal of Pure and Applied Mathematics, (2) **98** (2015), 211–220.
3. J. Herzog and D. Popescu, *Finite filtrations of modules and shellable multicomplexes*, Manuscripta Math, **121** (2006), 385–410.
4. A. Li, *Associated prime filtrations of finitely generated modules over noetherian rings*, Comm. Algebra, (4) **23** (1995), 1511–1526.
5. H. Matsumura, *Commutative Ring Theory*, Cambridge University Press, 1989.

6. K. Premkumar, *Prime extension filtration of a module*, (Ph.D thesis, Pondicherry University, India.), 2017.

T. Duraivel
Department of Mathematics, Pondicherry University, Kalapet-605014, Puducherry, India.
Email: tduraivel@gmail.com

S. Mangayarcarassy
Department of Mathematics, Pondicherry Engineering College, Kalapet-605014, Puducherry, India.
Email: dmangay@pec.edu

K. Premkumar
Department of Mathematics, Indira Gandhi Institute of Technology, Sarang-759146, Odisha, India.
Email: prem.pondiuni@gmail.com