Solving a time-fractional inverse heat conduction problem with an unknown nonlinear boundary condition

Afshin Babaei *

Department of Mathematics, University of Mazandaran, Babolsar, Iran
email: babaei@umz.ac.ir

Abstract. In this paper, we consider a time-fractional inverse heat conduction problem with an unknown function in the nonlinear boundary condition. First, ill-posedness of this problem is shown. Thus, we will apply the mollification regularization method with Gauss kernel to regularize the problem, then the space marching finite difference method is considered to solve numerically the mollified problem. The generalized cross-validation choice rule is used to find a suitable regularization parameter. The numerical scheme is completely described and the stability and convergence of the solutions are investigated. Finally, some numerical examples are presented to illustrate the validity and effectiveness of the proposed algorithm.

Keywords: Inverse problem, Caputo’s fractional derivative, Ill-posedness, Mollification, Convergence analysis.

AMS Subject Classification: 65M32, 35R11, 35R30.

1 Introduction

In recent decades, fractional calculus and fractional differential equations have been used widely to model a range of phenomena in different fields of sciences, such as physics, chemistry, biology and engineering [3, 4, 9, 12, 18, 22, 26]. The main reason for this occurrence is the memory and hereditary properties of the fractional operators. Thus, fractional calculus has been
introduced as an efficient tool for modeling physical problems. For example, fractional models have been successfully used to describe anomalous diffusion processes such as contaminant transport in soil, oil flow in porous media, and groundwater flow [6, 23, 24]. The fractional-order dynamics of brainstem vestibulo-oculomotor neurons [1], dispersive transport of charge carriers in disordered nanostructured materials [25], the total fractional-order variation regularizer in non-rigid image registration in image inverse problems [31], fractional order HIV immune system with memory in the study of infectious diseases [5] are some other applications of fractional order systems.

In forward time-fractional heat conduction problems, the unknown temperature is derived from the known appropriate initial distribution and boundary conditions. However, in many practical situations some parameters such as initial or boundary functions, source term, a coefficient of the heat conduction equation or fractional order of the equation cannot be directly specified. Therefore, some inverse problems have to be solved to infer these unknown parameters indirectly from additional measured data. In recent ten years many authors worked on these types of problems. In [2, 15, 19, 20, 27, 28] authors considered some inverse problems related to the heat conduction equation to find the inaccessible boundary data through interior measurements. In [29] authors proposed an optimal regularization method to solve a fractional order backward heat conduction problem. In [11] and [17] some inverse source problems of the heat equation involving fractional derivative were investigated. Also, in [32] the optimal order of Caputo’s fractional derivative for time-fractional heat conduction in a composite medium is estimated with Levenberg-Marquardt method. These inverse problems are generally ill-posed, namely, in present of noisy data, the solution of problems are not continuously dependent on the input data [10, 13]. Thus, the researchers combine their proposed algorithms with appropriate regularization methods to find stable numerical solutions. In this work, we investigate an inverse problem related to a time-fractional heat conduction equation with an unknown boundary function. Firstly, the problem is stabilized by using the mollification regularization technique. Afterwards, a numerical scheme based on the space marching finite difference method will be introduced to approximate the solution of the problem.

Bearing these ideas in mind, the manuscript is organized as follows. In Section 2, description of the inverse problem is considered and its ill-posedness is indicated. In Section 3, the mollification method is described and applied to the inverse problem in order to find a regularized problem.
Afterwards, the space marching algorithm for the numerical solution of the mollified problem is given. The stability and convergence of the proposed numerical scheme are studied in Section 4. Finally, some numerical test problems are investigated in Section 5.

2 Formulation of the problem

In this work, the one-dimensional time-fractional inverse heat conduction equation

$$D_t^{(\alpha)}u(x, t) = u_{xx}(x, t), \quad 0 < x < 1, \quad 0 < t < 1,$$

with the initial condition

$$u(x, 0) = u_0(x), \quad 0 \leq x \leq 1,$$

and the boundary conditions

$$u_x(0, t) = \gamma(t), \quad 0 \leq t \leq 1,$$

$$-u_x(1, t) = f(u(1, t)) + \eta(t), \quad 0 \leq t \leq 1,$$

is considered, where $u_0(x) \in L^2([0, 1])$ is the given initial temperature, $\eta(t)$ and $\gamma(t)$ are considered to be known continuous functions and $f(u)$ is the unknown boundary function. It is assumed that $f$ is a Lipschitz continuous function and satisfies the compatibility condition $-u_0'(1) = f(u_0(1)) + \eta(0)$.

In Eq. (1) $D_t^{(\alpha)}(.)$ is the Caputo time-fractional derivative of order $0 < \alpha < 1$ defined as [16]

$$D_t^{(\alpha)}\xi(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\xi'(s)}{(t-s)^\alpha} ds, \quad 0 \leq t \leq 1,$$

where $\xi$ is a differentiable function and $\Gamma(.)$ is the Gamma function.

To determine the set of functions $(u, f)$ in the problem (1)-(4) we need a additional condition. Here, the condition

$$u(0, t) = p(t), \quad 0 \leq t \leq 1,$$

is used. This problem with $\eta(t) = 0$ has been considered in [21] and the existence and uniqueness of the solution for this inverse problem is proved by applying the fixed point theorem. In our numerical approach that follows, we will assume that the solution of the problem (1)-(6) is a sufficiently smooth function.
In practice, the input functions \( u_0(x), \gamma(t), \eta(t) \) and \( p(t) \) are not exact, but some perturbed versions of them are in hand. On the other hand, the problem (1)-(6) is ill-posed, i.e. small errors in the input functions can blow up in the solution. To show the ill-posedness of this problem, we solve it in the frequency domain. For this purpose, all the related functions are extended to the whole real line \( \mathbb{R} \) by defining them to be zero for \( (-\infty, 0) \cup (1, +\infty) \).

Taking the Fourier transform of Eqs. (1), (3) and (6) with respect to \( t \) [18], we get

\[
(i\omega)^\alpha \hat{u}(x, \omega) = \hat{u}_{xx}(x, \omega),
\]

\[
\hat{u}_x(0, \omega) = \hat{\gamma}(\omega),
\]

\[
\hat{u}(0, \omega) = \hat{p}(\omega),
\]

where

\[
(i\omega)^\alpha = |\omega|^\alpha \left( \cos \frac{\alpha \pi}{2} + isgn(\omega) \sin \frac{\alpha \pi}{2} \right),
\]

and \( sgn(\omega) \) is the sign function. The second-order ordinary differential equation (7) under the conditions (8) and (9) has the following solution:

\[
\hat{u}(x, \omega) = \cosh \left( (i\omega)^{\frac{\alpha}{2}} x \right) \hat{p}(\omega) + (i\omega)^{-\frac{\alpha}{2}} \sinh \left( (i\omega)^{\frac{\alpha}{2}} x \right) \hat{\gamma}(\omega).
\]

Now, suppose that the input data functions \( u_0(x), \gamma(t) \) and \( \eta(t) \) be exact and we have only noisy data \( p(t) \). Let the measured data \( p^\varepsilon(t) = p(t) + \varepsilon(t) \), where \( \varepsilon \in L^2(\mathbb{R}) \) is the measured error. Hence, the solution to the problem (7)-(9) has the form

\[
\hat{\vartheta}(x, \omega) = \hat{u}(x, \omega) + \cosh \left( (i\omega)^{\frac{\alpha}{2}} x \right) \hat{\delta}(\omega).
\]

Let the solution \( \hat{u}(x, \omega) \) belongs to \( L^2(\mathbb{R}) \) with respect to \( \omega \). For a fixed value \( 0 < x < 1 \), \( \cosh \left( |\omega|^{\frac{\alpha}{2}} \cos \frac{\alpha \pi}{2} x \right) \) is unbounded as \( \omega \to \infty \). Therefore, the exact input data function \( \hat{p}(\omega) \) must decay rapidly. But, we have the noisy data \( \hat{p}^\varepsilon(t) \in L^2(\mathbb{R}) \) and cannot expect the measurement data \( \hat{p}^\varepsilon(\omega) \) to have the same decay in the frequency domain as the exact data \( \hat{p}(\omega) \). Hence, in general, the solution \( \hat{\vartheta}(x, \omega) \) will not belongs to \( L^2(\mathbb{R}) \). So, the numerical solution will be destroyed by magnified high frequency components of the error \( \varepsilon \). Thus, to solve the problem (1)-(4), we require a regularization method. The mollification technique utilizes a convolution of the input data and a smooth function with a parameter, to filter the high-frequency components of the noisy data. Therefore, the problem becomes well-posed. This method is being widely used as regularization method in many ill-posed problems [7, 8, 13, 30]. The simplicity of implementation and capability in the presence of high noise levels are principal advantages of the mollification technique.
3 Numerical scheme

3.1 Mollification method

Let $\delta > 0$, $p > 0$, and

$$A_p = \left( \int_{-p}^{p} \exp(-s^2) ds \right)^{-1}.$$  

The $\delta$-mollification of an integrable function is based on a convolution with the Gaussian kernel

$$\rho_{\delta,p}(t) = \begin{cases} A_p \delta^{-1} \exp\left(-\frac{t^2}{\delta^2}\right), & |t| \leq p \delta, \\ 0, & |t| > p \delta. \end{cases}$$  

The $\delta$-mollifier $\rho_{\delta,p}$ is a non-negative function on space of infinitely differentiable functions in $(-p\delta, p\delta)$ and satisfying $\int_{-p\delta}^{p\delta} \rho_{\delta,p}(t) dt = 1$.

Suppose $\zeta(t)$ is a locally integrable function in $\mathbb{R}$. The $\delta$-mollification of $\zeta(t)$ is defined by the convolution

$$\mathcal{J}_{\delta} \zeta(t) = (\rho_{\delta} \ast \zeta)(t) = \int_{t-p\delta}^{t+p\delta} \rho_{\delta}(t-s) \zeta(s) ds,$$

where the $p$-dependency on the kernel has been dropped for simplicity of notation. The radius of mollification $\delta$ is determined automatically by the Generalized Cross Validation (GCV) criteria [13]. In order to define the mollification of a discrete function, let

$$K = \{ t_j : j \in \mathbb{Z} \},$$

and $\Delta t = \sup_{j \in \mathbb{Z}} (t_{j+1} - t_j)$, satisfy

$$t_{j+1} - t_j > d > 0, \quad j \in \mathbb{Z},$$

where $\mathbb{Z}$ is a set of integers, and $d$ is a positive constant.

Let $G = \{ \zeta(t_j) = \zeta_j : j \in \mathbb{Z} \}$ be a discrete function defined on $K$. We set

$$s_j = \frac{1}{2}(t_j + t_{j+1}), \quad j \in \mathbb{Z}.$$

The discrete $\delta$-mollification of $G$ is defined as follows:

$$\mathcal{J}_{\delta} G(t) = \sum_{j=-\infty}^{\infty} \left( \int_{s_{j-1}}^{s_j} \rho_{\delta}(t-s) ds \right) \zeta_j,$$
where

\[ \sum_{j=-\infty}^{\infty} \left( \int_{s_j}^{s_{j-1}} \rho_\delta(t-s)ds \right) = \int_{-p\delta}^{p\delta} \rho_\delta(s)ds = 1. \]

We assume that instead of the function \( \zeta \), we know some noisy data function \( \zeta^\varepsilon \) such that \( \| \zeta - \zeta^\varepsilon \|_\infty \leq \varepsilon \).

**Theorem 1.** Let the functions \( \zeta \) and \( \zeta' \) are uniformly Lipschitz on \( \mathbb{R} \). Also let \( G = \{ \zeta_j : j \in Z \} \) and \( G^\varepsilon = \{ \zeta_j^\varepsilon : j \in Z \} \) be the discrete version of \( \zeta \) and \( \zeta^\varepsilon \), which are defined on \( K \), satisfying \( \| \zeta - \zeta^\varepsilon \|_\infty \leq \varepsilon \) and \( \| G - G^\varepsilon \|_\infty \leq \varepsilon \). Then

i) there exists a constant \( C \), independent of \( \delta \), such that

\[ \| J_\delta G - \zeta \|_\infty \leq C(\delta + \Delta t), \tag{11} \]

and

\[ \| (J_\delta G)' - \zeta' \|_\infty \leq C \left( \delta + \frac{\varepsilon + \Delta t}{\delta} \right). \tag{12} \]

ii) if \( \| G - G^\varepsilon \|_\infty \leq \varepsilon \), then

\[ \| J_\delta G - J_\delta G^\varepsilon \|_\infty \leq \varepsilon. \tag{13} \]

Also, there exists a constant \( C \), independent of \( \delta \), such that

\[ \| (J_\delta G)' - (J_\delta G^\varepsilon)' \|_\infty \leq \frac{C \varepsilon}{\delta}. \tag{14} \]

**Theorem 2.** Let a differentiation operator \( D_0^\delta \) be defined by the following rule:

\[ D_0^\delta(G) = D_0(J_\delta G)(t) \big|_K, \]

in which \( D_0 \) is the central difference operator

\[ D_0 \zeta(t) = \frac{\zeta(t + \Delta t) - \zeta(t - \Delta t)}{2\Delta t}. \]

Then

\[ \| D_0^\delta(G) \|_\infty \leq \frac{4A_p}{\delta} \| G \|_\infty. \]

The proofs of these theorems can be found in the reference [13].
3.2 Mollified fractional derivative

The process of numerical fractional differentiation is well known to be an ill-posed problem, because a small error in measurement data can induce a large error in the approximate derivative. Suppose \( \zeta^\varepsilon(t) \) is a perturbed version of the exact function \( \zeta(t) \). To approximate \( J_\delta(D_t^{(\alpha)} \zeta^\varepsilon) \) on a uniform partition \( K \) of the unit interval, we follow the mollification technique proposed in [14].

Let \( D_+ \) be forward finite difference operator. The discrete computed solution \( (D_t^{(\alpha)} G^\varepsilon)_{\delta} \) of the mollified function \( J_\delta(D_t^{(\alpha)} \zeta^\varepsilon) \) in the grid points, will be as

\[
\begin{align*}
(D_t^{(\alpha)} G^\varepsilon)_{\delta}(t_1) &= D_+(J_\delta G^\varepsilon)(t_1)W_1 \\
(D_t^{(\alpha)} G^\varepsilon)_{\delta}(t_2) &= D_+(J_\delta G^\varepsilon)(t_1)W_2 + D_+(J_\delta G^\varepsilon)(t_2)W_1, \\
(D_t^{(\alpha)} G^\varepsilon)_{\delta}(t_j) &= D_+(J_\delta G^\varepsilon)(t_1)W_j \\
&\quad + \sum_{i=2}^{j-1} D_0(J_\delta G^\varepsilon)(t_i)W_{j-i+1} + D_+(J_\delta G^\varepsilon)(t_j)W_1,
\end{align*}
\]

where \( j = 3, 4, \ldots, n \) and the weights \( W_j \) are as follows:

\[
\begin{align*}
W_1 &= \frac{1}{\Gamma(2-\alpha)} \left( \frac{\Delta t}{2} \right)^{1-\alpha}, \\
W_i &= \frac{1}{\Gamma(2-\alpha)} \left[ \left( \frac{(2i+1)\Delta t}{2} \right)^{1-\alpha} - \left( \frac{(2i-1)\Delta t}{2} \right)^{1-\alpha} \right], \\
W_j &= \frac{1}{\Gamma(2-\alpha)} \left[ j\Delta t - \left( \frac{j-1}{2} \right)\Delta t \right]^{1-\alpha},
\end{align*}
\]

where \( i = 2, 3, \ldots, j-1 \), and

\[
\text{Theorem 3. Let the functions } \zeta^t \text{ and } \zeta^\varepsilon \text{ are uniformly Lipschitz on } \mathbb{R} \text{ as } \| \zeta - \zeta^\varepsilon \|_\infty \leq \varepsilon. \text{ Also, suppose } G \text{ and } G^\varepsilon, \text{ the discrete versions of } \zeta \text{ and } \zeta^\varepsilon, \text{ satisfy } \| G - G^\varepsilon \|_\infty \leq \varepsilon. \text{ Then}
\]

\[
\left\| (D_t^{(\alpha)} G^\varepsilon)_{\delta} - D_t^{(\alpha)} \zeta^t \right\|_\infty \leq \frac{C}{\Gamma(2-\alpha)} \left( \delta + \frac{\varepsilon}{\delta} + \Delta t \right),
\]

\[
\text{where } C \text{ is a constant independent of } \delta.
\]

Proof. Refer to [14].
3.3 The $\delta$-mollified space marching algorithm

Suppose $v = \mathcal{J}_0 u$ is the mollified version of $u$. The regularized problem is formulated as follows

$$
\begin{align*}
D_t^{(\alpha)} v(x, t) &= v_{xx}(x, t), & 0 < x < 1, \ 0 < t < 1, \\
v(x, 0) &= \mathcal{J}_\delta u_0^\varepsilon(x), & 0 \leq x \leq 1, \\
v(0, t) &= \mathcal{J}_\delta p^\varepsilon(t), & 0 \leq t \leq 1, \\
v_x(0, t) &= \mathcal{J}_\delta' g^\varepsilon(t), & 0 \leq t \leq 1,
\end{align*}
$$

(16)

where $\delta$, $\delta_0$ and $\delta'_0$ are the radii of mollification, and will be chosen using the GCV criteria. After solving (16), the mollified function $F(v)$ is obtained from

$$
- v_x(1, t) = F(v(1, t)) + \mathcal{J}_{\delta'} \eta^\varepsilon(t), \quad 0 \leq t \leq 1,
$$

(17)

Let $M$ and $N$ be two positive integers. We consider a uniform grid in the unit interval $I = [0, 1] \times [0, 1]$ as

$$
\{(x_i = ih, t_n = nk), \ i = 0, 1, \ldots, M; \ n = 0, 1, \ldots, N\},
$$

in which $Mh = 1$ and $Nk = 1$. Let the value of $v(x, t)$ at $(x_i, t_n)$ is indicated by $R^n_i$. In addition, suppose

$$
W^n_i = v_x(ih, nk), \quad Q^n_i = D_t^{(\alpha)} v(ih, nk), \quad \eta_n = \mathcal{J}_{\delta'} \eta^\varepsilon(nk).
$$

Let

$$
R^0_0 = \mathcal{J}_\delta p^\varepsilon(nk), \quad W^0_0 = \mathcal{J}_\delta' g^\varepsilon(nk), \quad Q^0_0 = D_t^{(\alpha)}(\mathcal{J}_\delta p^\varepsilon(nk)), \quad n \in \{1, \ldots, N\},
$$

and

$$
R^0_i = \mathcal{J}_\delta u_0^\varepsilon(ih), \quad i \in \{0, 1, \ldots, M\}.
$$

We approximate the partial differential equation in system (16) by the finite difference schemes

$$
\begin{align*}
R_{i+1}^n &= R_i^n + hW_i^n, \\
W_{i+1}^n &= W_i^n + hQ_i^n, \\
Q_{i+1}^n &= D_t^{(\alpha)}(\mathcal{J}_{\delta_i+1} R_i^n).
\end{align*}
$$

(18)(19)(20)

where $i = 0, \ldots, M - 1$ and $n = 1, \ldots, N$. To approximate $D_t^{(\alpha)}(\mathcal{J}_\delta R_i^n)$, the quadrature formula (15) will be used. After applying these schemes, the values of $u(x, t)$ and $u_x(x, t)$ are calculated on the boundary $x = 1$. By substituting $R_M^n$ and $W_M^n$ in equation (17), we get

$$
-W^n_M = F(R^n_M) + \eta_n.
$$

(21)

Finally the unknown function $F$ is reconstructed by interpolating these values.
4 Stability and convergence analysis

Suppose the discrete function $G = \{\zeta_j = \zeta(t_j) : j \in \mathbb{Z}\}$ is defined on $K$. After applying Theorem 2, we have

$$|D_t^{(\alpha)} J_\delta G(t)| = \left| \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{(J_\delta G(s))^\prime}{(t-s)^\alpha} ds \right| = \left| \frac{1}{\Gamma(1-\alpha)} \int_0^t D_0(J_\delta G(s)) ds \right| \leq \frac{1}{\Gamma(1-\alpha)} \int_0^t 4A_p \|G\|_\infty ds = \frac{4A_p \|G\|_\infty t^{1-\alpha}}{\delta \Gamma(2-\alpha)}.$$  \hfill (22)

**Theorem 4.** Suppose $|R_i|, |W_i|, |Q_i|$ are maximum values of $|R^n_i|, |W^n_i|, |Q^n_i|$, where $n \in \{0, 1, \ldots, N\}$. For the marching scheme, there exists a constant $\theta$, such that

$$\max\{|R_M|, |W_M|, |Q_M|\} \leq \theta \max\{|R_0|, |W_0|, |Q_0|\}.$$ 

**Proof.** By using (18) and (19), we have

$$|R^n_{i+1}| \leq (1 + h) \max\{|R^n_i|, |W^n_i|\}, \quad (23)$$

$$|W^n_{i+1}| \leq (1 + h) \max\{|W^n_i|, |Q^n_i|\}. \quad (24)$$

Let $\delta^\prime = \min\{\delta_i\}$, from (20), (22) and (23), we have

$$|Q^n_{i+1}| \leq \frac{4A_p (nk)^{1-\alpha} (1 + h)}{\delta^\prime \Gamma(2-\alpha)} \max\{|R^n_i|, |W^n_i|\}. \quad (25)$$

Let

$$\hat{C} = \max\left\{1, \frac{4A_p (nk)^{1-\alpha}}{\delta^\prime \Gamma(2-\alpha)}\right\},$$

from (23)-(25), we obtain

$$\max\{|R_{i+1}|, |W_{i+1}|, |Q_{i+1}|\} \leq (\hat{C} + \hat{C} h) \max\{|R_i|, |W_i|, |Q_i|\}. \quad (26)$$

By iterating (26), $M$ times, we get

$$\max\{|R_M|, |W_M|, |Q_M|\} \leq (\hat{C} + \hat{C} h) \max\{|R_{M-1}|, |W_{M-1}|, |Q_{M-1}|\} \leq (\hat{C} + \hat{C} h)^2 \max\{|R_{M-2}|, |W_{M-2}|, |Q_{M-2}|\} \leq \cdots \leq (\hat{C} + \hat{C} h)^M \max\{|R_0|, |W_0|, |Q_0|\},$$

which implies

$$\max\{|R_M|, |W_M|, |Q_M|\} \leq \hat{C}^M \exp(1) \max\{|R_0|, |W_0|, |Q_0|\}. \quad (27)$$

Letting $\theta = \hat{C}^M \exp(1)$, completes the proof of this statement. \hfill \Box
**Remark 1.** For the boundary function $f$, there exists a constant $\Phi$, such that

$$\max\{\vert F(R_M)\vert\} \leq \Phi \max\{\vert R_0\vert, \vert W_0\vert, \vert Q_0\vert, \vert \eta \vert\}.$$ 

**Proof.** By using (21), we have

$$\vert F(R_M^n)\vert \leq \max\vert W_{M}^n\vert + \max\vert \eta_n\vert,$$

and from Theorem 4, we have

$$\max\vert F(R_M)\vert \leq \theta \max\{\vert R_0\vert, \vert W_0\vert, \vert Q_0\vert\} + \max\vert \eta_n\vert.$$ 

Letting $\Phi = \max\{1, \theta\}$, we obtain

$$\max\{\vert F(R_M)\vert\} \leq \Phi \max\{\vert R_0\vert, \vert W_0\vert, \vert Q_0\vert, \vert \eta \vert\}.$$ 

**Theorem 5.** For the marching schemes, when $\varepsilon, h$ and $k$ tend towards 0, by choosing $\delta' = \delta'(\varepsilon)$, the discrete mollified solution converges to the mollified exact solution.

**Proof.** Suppose $i \in \{0, 1, \ldots, M\}$ and $n \in \{0, 1, \ldots, N\}$. First, we define the discrete error functions $\Delta R_i^n = R_i^n - \nu(ih, nk)$ and $\Delta W_i^n = W_i^n - v(ih, nk)$. By applying Theorem 1, we have

$$\vert Q_i^n - D_t^{(\alpha)} \nu(ih, nk)\vert = \vert D_t^{(\alpha)} \int_0^n \frac{C \varepsilon}{\delta(nk - s)^{1-\alpha}} ds + O(k)\vert$$

$$\leq \frac{1}{\Gamma(1-\alpha)} \int_0^n \frac{C \varepsilon}{\delta(nk - s)^{1-\alpha}} ds + O(k)$$

$$= \frac{C \varepsilon (nk)^{1-\alpha}}{\delta \Gamma(2-\alpha)} + O(k)$$

$$\leq \frac{C \varepsilon}{\delta \Gamma(2-\alpha)} + O(k)$$

$$= C \alpha \frac{\varepsilon}{\delta} + O(k),$$

where $C = \frac{\varepsilon}{\delta \Gamma(2-\alpha)}$.

Expanding the mollified solution $\nu(x, t)$ by the Taylor series, we obtain

$$\nu((i+1)h, nk) = \nu(ih, nk) + h \nu_x(ih, nk) + O(h^2), \quad (27)$$

$$\nu_x((i+1)h, nk) = \nu_x(ih, nk) + h D_t^{(\alpha)} \nu(ih, nk) + O(h^2). \quad (28)$$
From (18) and (27), we have

\[ \Delta R_{i+1}^n = R_{i+1}^n - v((i + 1)h, nk) = R_i^n + hW_i^n - v((i + 1)h, nk) = R_i^n + hW_i^n - v(ih, nk) - hv_x(ih, nk) + O(h^2) = \Delta R_i^n + h\Delta W_i^n + O(h^2), \] (29)

and by using (19) and (28), we have

\[ \Delta W_{i+1}^n = W_{i+1}^n - v_x((i + 1)h, nk) = W_i^n + hQ_i^n - v_x((i + 1)h, nk) = W_i^n + hQ_i^n - v_x(ih, nk) - hD_i^{(\alpha)}v(ih, nk) + O(h^2) = \Delta W_i^n + hC_\alpha \frac{\varepsilon}{\delta} + O(hk) + O(h^2). \] (30)

Let \( |\Delta R_i| = \max_{0 \leq n \leq N} |\Delta R_i^n| \) and \( |\Delta W_i| = \max_{0 \leq n \leq N} |\Delta W_i^n| \). Thus, from (29) and (30), we obtain

\[ |\Delta R_{i+1}^n| \leq |\Delta R_i^n| + h|\Delta W_i^n| + O(h^2) \leq \max_{0 \leq n \leq N} |\Delta R_i^n| + h \max_{0 \leq n \leq N} |\Delta W_i^n| + O(h^2) = |\Delta R_i| + h|\Delta W_i| + O(h^2), \]

and

\[ |\Delta W_{i+1}^n| \leq |\Delta W_i^n| + hC_\alpha \frac{\varepsilon}{\delta} + O(hk) + O(h^2) \leq \max_{0 \leq n \leq N} |\Delta W_i^n| + hC_\alpha \frac{\varepsilon}{\delta} + O(hk) + O(h^2) = |\Delta W_i| + hC_\alpha \frac{\varepsilon}{\delta} + O(hk) + O(h^2). \]

Hence

\[ |\Delta R_{i+1}| = \max_{0 \leq n \leq N} |\Delta R_{i+1}^n| \leq |\Delta R_i| + h|\Delta W_i| + O(h^2), \]

\[ \leq (1 + h) \max \{|\Delta R_i|, |\Delta W_i|\} + O(h^2), \]

\[ |\Delta W_{i+1}| = \max_{0 \leq n \leq N} |\Delta W_{i+1}^n| \leq |\Delta W_i| + hC_\alpha \frac{\varepsilon}{\delta} + O(hk) + O(h^2), \]

\[ \leq \max \{|\Delta W_i|\} + hC_\alpha \frac{\varepsilon}{\delta} + O(hk) + O(h^2). \]

Then, we have

\[ \max \{|\Delta R_{i+1}|, |\Delta W_{i+1}|\} \leq (1 + h) \max \{|\Delta W_i|, |\Delta R_i|\} + \Lambda, \] (31)
where $\Lambda = hC_{\alpha, \delta} + O(hk) + O(h^2)$. Now, let $\Delta_i = \max\{|\Delta W_i|, |\Delta R_i|\}$. From (31), we have

$$\Delta_{i+1} \leq (1+h)\Delta_i + \Lambda,$$

and

$$\Delta M \leq (1+h)\Delta_{M-1} + \Lambda \leq (1+h)^2\Delta_{M-2} + (1+h)\Lambda + \Lambda \leq \cdots \leq (1+h)^M \Delta_0 + \tau \Lambda,$$

where $\tau = \sum_{i=0}^{M-1} (1+h)^i$. Now by using Theorem 1, for $n \in \{0, 1, \ldots, N\}$, there exists constant $C_n$ and $D_n$, such that

$$|\Delta R_0^n| = |R_0^n - v(0,nk)| \leq C_n(\delta_0 + k),$$

$$|\Delta W_0^n| = |W_0^n - v_x(0,nk)| \leq D_n(\delta_0 + k),$$

let $\delta' = \max\{\delta_0, \delta_0'\}$ and $C' = \max\{C_n, D_n\}$, then we have

$$\Delta_0 = \max\{|\Delta R_0^n|, |\Delta W_0^n|\} \leq C'(\delta' + k),$$

and

$$\Delta_M \leq \exp(1)C'(\delta' + k) + \tau \Lambda.$$

Thus when $\varepsilon$, $h$ and $k$ tend towards 0, by choosing $\delta' = \delta'(\varepsilon)$, $\Delta_M$ will tend to 0. It completes the proof.

**Remark 2.** For the marching schemes, when $\varepsilon$, $h$ and $k$ tend towards 0, the discrete mollified boundary function $\mathcal{F}$ converges to the mollified exact boundary function $f$.

**Proof.** Suppose $\Delta \mathcal{F}_n = \mathcal{F}(v(1,nk)) - \mathcal{F}(R^n_M)$. By using (17) and (21), we have

$$\Delta \mathcal{F}_n = (-v(1,nk) - J_{\delta'}^n \eta^f(nk)) - (-W^n_M - \eta_n)$$

$$= (W^n_M - v(1,nk)) + (\eta_n - J_{\delta'}^n \eta^f(nk))$$

$$= \Delta W^n_M + (\eta_n - J_{\delta'}^n \eta^f(nk)),$$

by applying Theorem 1, we have

$$|\Delta \mathcal{F}_n| \leq |\Delta W^n_M| + |(\eta_n - J_{\delta'}^n \eta^f(nk))| \leq \Delta_M + \varepsilon,$$

consequently by using (34), we have

$$|\Delta \mathcal{F}_n| \leq \exp(1)C'(\delta' + k) + \tau \Lambda + \varepsilon.$$

This implies that when $\varepsilon$, $h$ and $k$ tend to zero, $|\Delta \mathcal{F}_n|$ tend towards zero too. It completes the proof. \qed
5 Numerical Implementation

Suppose the number of space and time divisions are $M$ and $N$. Let $h = \frac{1}{M}$ and $k = \frac{1}{N}$ and $\varepsilon$ be the maximum level of noises in the data. We use the following formula to generate the noisy data

$$s^\varepsilon_n = s(t_n) + \varepsilon_n, \quad n = 0, \ldots, N,$$

where $s(t_n)$ is the exact data and $\varepsilon_n$ is a Gaussian random variable with variance $\sigma^2 = \varepsilon^2$ [13].

The temperature and heat flux errors are measured by the relative weighted $l_2$-norm defined as

$$\sigma_N(u) = \left(\frac{1}{N+1} \sum_{n=0}^{N} |R_M^n - u(1, nk)|^2 \right)^{\frac{1}{2}},$$

and the boundary function $f(u)$ errors are measured by

$$\sigma_N(f) = \left(\frac{1}{N+1} \sum_{n=0}^{N} |F(R_M^n) - f(u(1, nk))|^2 \right)^{\frac{1}{2}}.$$

Example 1. Consider the following problem:

$$\begin{cases} D_t^{(\alpha)} u(x, t) = u_{xx}(x, t), & (x, t) \in [0, 1] \times [0, 1], \\ u(x, 0) = u_0(x) = \begin{cases} x, & x \in [0, \frac{1}{2}], \\ 1 - x, & x \in [\frac{1}{2}, 1], \end{cases} \\ u(0, t) = 0, \quad t \in [0, 1], \\ u_x(0, t) = \sum_{n=1}^{\infty} nd_n E_\alpha(-n^2 t^\alpha), & t \in [0, 1], \end{cases}$$

where

$$d_n = \frac{2}{\pi} \int_0^\pi u_0(x) \sin(nx) dx = \frac{2(1 + (-1)^{n+1})}{\pi n^2} \left( \sin\left(\frac{n\pi}{2}\right) - \frac{n\pi}{2} \cos\left(\frac{n\pi}{2}\right) \right),$$

for $n = 1, 2, \ldots$. Also, suppose

$$\eta(t) = -1 - \sum_{n=1}^{\infty} nd_n E_\alpha(-n^2 t^\alpha) \cos(n) - 2 \sin\left(\sum_{n=1}^{\infty} d_n E_\alpha(-n^2 t^\alpha) \sin(n)\right).$$
The exact solution is

\[ u(x, t) = \sum_{n=1}^{\infty} d_n E_{\alpha}(-n^2 t^\alpha) \sin(nx), \]

where \( E_{\alpha}(\cdot) \) is the Mittag-Leffler function and

\[ f(u) = 1 + 2 \sin(4\pi u). \]

After adding noise to the data, to generate the noisy functions \( u^\varepsilon(0, t) \), \( u_x^\varepsilon(0, t) \) and \( u_x^\varepsilon(x, 0) \), the mollified problem (16) is solved with the various values of the parameters \( h \), \( k \) and the time fractional order \( \alpha \).

Figure 1 shows the exact solution of \( u(x, t) \) and Figure 2 and Figure 3 show the numerical approximation and absolute error of \( u(x, t) \) when \( \alpha = 0.75, M = 150, N = 150, \varepsilon = 0.01 \) and \( \varepsilon = 0.05 \).

Table 1 displays the errors of the approximated values of temperature and heat flux at the boundary \( x = 1 \) and \( f(u) \). Furthermore, Figure 4 shows the numerical approximations of \( f(u) \) with regularization and without regularization when \( M = 100, N = 128, \alpha = 0.8 \) and \( \varepsilon = 0.1 \). Also, Figure 5 shows the exact and numerical approximations to \( f(u) \) for several values of \( \varepsilon = 0.001,0.01,0.05,0.1 \), when \( M = 50, N = 128, \alpha = 0.5 \). Finally, Figure 6 indicates the absolute error of numerical approximations for \( f(u) \).

**Example 2.** Consider the problem (1)-(4) with \( u_0(x) = e^{-x} \), \( \gamma(t) = -E_{\alpha}(t^\alpha) \) and \( p(t) = E_{\alpha}(t^\alpha) \). The exact solution of this problem is

\[ u(x, t) = E_{\alpha}(t^\alpha)e^{-x}, \]
Solving a time-fractional inverse heat conduction problem

Numerical solution with $\alpha = 0.75$ and $\varepsilon = 0.01$

Figure 2: The numerical solution and absolute error of $u(x, t)$ for Example 1 when $\alpha = 0.75$ and $\varepsilon = 0.01$.

Numerical solution with $\alpha = 0.75$ and $\varepsilon = 0.05$

Figure 3: The numerical solution and absolute error of $u(x, t)$ for Example 1 when $\alpha = 0.75$ and $\varepsilon = 0.05$.

where $E_\alpha(.)$ is the Mittag-Leffler function and $f(u) = u^3 - \alpha u + 1$.

Figure 7, Figure 8 and Figure 9 show the exact and numerical solutions of $u(x, t)$ when $M = 150, N = 150, \alpha = 0.25, 0.5, 0.75$ and $\varepsilon = 0.05$.

Table 2 indicates the errors of the approximated values of temperature and heat flux at the boundary $x = 1$ and $f(u)$. Also, Figure 10 shows the exact and numerical solutions (left) and absolute error (right) of $f(u)$ when $N = 50, \alpha = 0.5, \varepsilon = 0.01$ and $M = 50, 100, 150$. 
Figure 4: The numerical approximations of $f(u)$ with regularization and without regularization in Example 1 when $\alpha = 0.8$ and $\varepsilon = 0.1$.

Figure 5: The exact and numerical approximations of $f(u)$ in Example 1 when $\alpha = 0.5$.

Figure 6: Absolute error of numerical approximations $f(u)$ in Example 1 when $\alpha = 0.5$. 
Table 1: The relative weighted $l_2$-norm errors between the computed and the exact values of $u(1,t)$, $u_x(1,t)$ and $f(u)$ in Example 1 for various values of $h$, $k$ and $\varepsilon$.

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6 Conclusion

In this article, a numerical algorithm for solving an inverse problem of a time-fractional inverse heat conduction equation with an unknown nonlinear boundary function is proposed. According to the ill-posedness of the problem, first the mollification regularization method is applied. Afterwards, a marching finite difference scheme is used to solve the stabilized problem. Stability and convergence analysis are theoretically proven. Finally, two numerical examples are provided to demonstrate the stability and convergence behaviours of the presented method.
Figure 7: The exact and numerical solutions of $u(x, t)$ for Example 2 when $\alpha = 0.25$ and $\varepsilon = 0.05$.

Figure 8: The exact and numerical solutions of $u(x, t)$ for Example 2 when $\alpha = 0.5$ and $\varepsilon = 0.05$.

References


Table 2: The relative weighted $l_2$-norm errors between the computed and the exact values of $u(1,t)$, $u_x(1,t)$ and $f(u)$ in Example 2 for various values of $h$, $k$ and $\varepsilon$.

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Figure 9: The exact and numerical solutions of $u(x,t)$ for Example 2 when $\alpha = 0.75$ and $\varepsilon = 0.05$.

Figure 10: The exact and numerical solutions (left) and absolute error (right) of $f(u)$ for Example 2 when $\alpha = 0.5$ and $\varepsilon = 0.01$.


Solving a time-fractional inverse heat conduction problem


